Algebraic cycle complexes

Marc Levine

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Outline

- Algebraic cycles and algebraic K-theory
- The Beilinson-Lichtenbaum conjectures
- Bloch's cycle complexes
- Suslin's cycle complexes

Algebraic cycles

and

algebraic K-theory

Algebraic cycles

X: an algebraic variety over a field k.

An algebraic cycle on X is a finite $\mathbb{Z}$-linear combination of irreducible reduced closed subschemes of X:

$$Z := \sum_{i=1}^{r} n_i Z_i.$$

Say Z has (co)dimension q if each Z_i has (co)dimension q.

$z^q(X) :=$ the group of codimenison q algebraic cycles on X.

$z_q(X) :=$ the group of dimenison q algebraic cycles on X.

$X \rightsquigarrow z^q(X)$ is contravariantly functorial for flat maps:

$$f^*(Z) := \sum_i \mathrm{mult}_i \cdot W_i.$$

W_i the irreducible components of $f^{-1}(Z)$.

$X \rightsquigarrow z_q(X)$ is covariantly functorial for proper maps:

$$f_*(Z) := \begin{cases} 0 & \text{if } \dim f(Z) < \dim Z \\ [k(Z) : k(f(Z))] \cdot f(Z) & \text{if } \dim f(Z) = \dim Z. \end{cases}$$

$f : Y \to X$ a morphism of smooth varieties, $Z \subset X$ a subvariety.

Suppose Z and $f^{-1}(Z)$ both have pure codimension q.

Define

$$f^*(Z) := \sum_i \mathrm{mult}_i \cdot W_i.$$

W_i the irreducible components of $f^{-1}(Z)$,

mult_i = Serre's intersection multiplicity:

$$\mathrm{mult}_i = \sum_j (-1)^j \ell_{\mathcal{O}_{Y,W_i}} \left[\mathrm{Tor}_j^{\mathcal{O}_{X,Z}}(\mathcal{O}_{Y,W_i}, k(Z)) \right].$$

Given subvarieties $Z \subset X$, $W \subset Y$, the product $Z \times_k W$ is a union of subvarieties of $X \times_k Y$ (with multiplicities in char. $p > 0$):

$$Z \times_k W := \sum_j m_j T_j$$

This extends to the external product

$$\times : z_q(X) \otimes z_p(Y) \to z_{p+q}(X \times_k Y).$$

External product is compatible with proper push-forward and with pull-back.

To make the groups of algebraic cycles look more like a (co)homology theory, we introduce an adequate equivalence relation, analogous to homology of singular chains.

An adequate equivalence relation $\sim$ is given by a subgroup $R^q_\sim(X) \subset z^q(X)$ for each smooth X, satisfying:

1. For $f : Y \to X$ proper of codimension d, $f_*(R^q_\sim(Y)) \subset R^{q+d}_\sim(X)$.
2. For $f : Y \to X$ flat, $f^*(R^q_\sim(X)) \subset R^q_\sim(Y)$.
3. For $Z \in R^q_\sim(X)$, $W \in z^p(X)$, $Z \cdot W$ is in $R^{p+q}_\sim(X)$ if defined.
4. Given $Z \in z^q(X)$, $W \in z^p(Y \times X)$, there is a $Z' \in z^q(X)$ with $Z - Z' \in R^q_\sim(X)$ and with $Y \times Z' \cdot W$ defined.
5. $[0] - [\infty] \in R^1_\sim(\mathbb{P}^1)$.

Note. One often restricts to smooth projective varieties.

Set $A^q_\sim(X) := z^q(X)/R^q_\sim(X)$. Then

1. $A^*_\sim(X) := \oplus_q A^q_\sim(X)$ is a commutative, graded ring with unit $[X] \in A^0_\sim(X)$.
2. $f^* : A^q_\sim(X) \to A^q_\sim(Y)$ is well-defined for all $f : Y \to X$ in $\mathbf{Sm}/k$: $f^*(Z) :=$ intersect $Y \times Z$ with the graph of f.
3. $f^*(Z \cdot W) = f^*(Z) \cdot f^*(W)$
4. $f_*(f^*(Z) \cdot W) = Z \cdot f_*(W)$ for f proper.

Examples.

1. Rational equivalence
2. Algebraic equivalence
3. Numerical equivalence
4. Homological equivalence

Rational equivalence is the finest adequate equivalence relation.

Definition

Let $i_0, i_1 : X \to X \times \mathbb{A}^1$ be the 0, 1 sections. Z is in $R^q_{\mathrm{rat}}(X)$ if there is a $W \in z^q(X \times \mathbb{A}^1)_{0,1}$ with

$$Z = i_0^*(W) - i_1^*(W).$$

It is not at all obvious that $\sim_{\mathrm{rat}}$ is an adequate relation, this is Chow's moving lemma.

A^*_{rat} is usually written CH^*.

Example. $\mathrm{CH}^n(\mathbb{P}^N) = \mathbb{Z}$ for $0 \le n \le N$, with generator a $\mathbb{P}^{N-n} \subset \mathbb{P}^N$.

For $Z \in z_n(\mathbb{P}^N)$ we have

$$[Z] \cdot [\mathbb{P}^{N-n}] = n_Z \cdot [pt],$$

defining the degree $n_Z := \deg(Z)$ of Z. Note that

$$[Z] = \deg(Z)[\mathbb{P}^n]$$

since $[\mathbb{P}^n] \cdot [\mathbb{P}^{N-n}] = 1 \cdot [pt]$.

This gives us Bezout's theorem: For $Z \in z_n(\mathbb{P}^N)$, $W \in z_{N-n}(\mathbb{P}^N)$,

$$[Z] \cdot [W] = \deg(Z) \deg(W)[pt]$$

Definition

Z is in $R^q_{\mathrm{alg}}(X)$ if there is a smooth curve C, k-points a, b in C and $W \in z^q(X \times C)_{a,b}$ with

$$Z = i_a^*(W) - i_b^*(W).$$

Note. If $X \subset \mathbb{P}^N$ is projective, there are projective schemes $C_{r,d,X}$ parametrizing all effective dimension r, degree d cycles on X. For $k = \bar{k}$, this gives the description of $\sim_{\mathrm{alg}}$ on X as

$$Z \sim_{\mathrm{alg}} W \Leftrightarrow \exists \text{ a cycle } T \text{ on } X \text{ with } Z + T, W + T$$
$$\text{in the same connected component of } C_{r,d,X}.$$

Thus $\sim_{\mathrm{alg}}$ is "topological" in nature: $A_r^{\mathrm{alg}}(X) = \pi_0([\amalg_d C_{r,d,X}]^+)$.

We have the degree map

$$\deg : z_0(X) \to \mathbb{Z}; \quad \deg(\sum_j n_j z_j) = \sum_j n_j [k(z_j) : k].$$

Since $z_0(\mathrm{Spec}\, k) = \mathbb{Z}[pt]$, $\deg = p_{X*}$.
For X projective, p_{X*} passes to

$$\deg = p_{X*} : \mathrm{CH}_0(X) \to \mathbb{Z} = \mathrm{CH}_0(pt).$$

Definition

Let X be smooth and projective over k. $Z \in z^q(X)$ is $\sim_{\mathrm{num}} 0$ if for all $W \in z_q(X)$,

$$\deg(Z \cdot W) = 0.$$

We have surjections

$$\mathrm{CH}^* \to A^*_{\mathrm{alg}} \to A^*_{\mathsf{num}}$$

In fact, $\sim_{\mathsf{num}}$ is the coarsest (non-zero) adequate equivalence relation on smooth projective varieties.

Example

$$\mathrm{CH}^*(\mathbb{P}^N) = A^*_{\mathrm{alg}}(\mathbb{P}^N) = A^*_{\mathsf{num}}(\mathbb{P}^N)$$

and similarly for other "cellular" varieties, such as Grassmann varieties or flag varieties.

This follows from the right-exact localization sequence for CH_*:

Theorem
Let $i : W \to X$ be a closed immersion, $j : U = X \setminus W \to X$ the open complement. Then the sequence

$$\mathrm{CH}_*(W) \xrightarrow{i_*} \mathrm{CH}_*(X) \xrightarrow{j^*} \mathrm{CH}_*(U) \to 0$$

is exact,

and the homotopy property for CH_*:

Theorem
The projection $p : X \times \mathbb{A}^1 \to X$ induces an isomorphism

$$p^* : \mathrm{CH}_*(X) \to \mathrm{CH}_{*+1}(X \times \mathbb{A}^1).$$

For instance, the known structure of the Schubert varieties give a stratification of a Grassmann variety $X := \mathrm{Gr}(n, N)$

$$\emptyset = F_{-1}X \subset F_0X \subset \ldots \subset F_DX = X$$

with $F_jX \setminus F_{j-1}X = \amalg_i \mathbb{A}^j$.

Localization and homotopy imply that the closures of the $\mathbb{A}^j$'s give generators for $\mathrm{CH}_j(X)$; the classical Schubert calculus says that these classes are independent modulo $\sim_{\mathrm{num}}$.

Weil cohomology

The fourth type of equivalence relation, homological equivalence, requires the notion of a Weil cohomology, a formal version of singular cohomology for smooth projective varieties. We'll discuss this further in the second lecture.

Algebraic K-theory

Besides using algebraic cycles as an algebraic substitute for singular homology, one can use algebraic vector bundles (locally free coherent sheaves) to give an algebraic version of topological K-theory.

Definition

Let X be a scheme, $\mathcal{P}_X$ the category of locally free coherent sheaves on X. The Grothendieck group of algebraic vector bundles on X, $K_0(X)$, is the free abelian group on the isomorphism classes in $\mathcal{P}_X$, modulo relations

$$[E] = [E'] + [E'']$$

for each exact sequence

$$0 \to E' \to E \to E'' \to 0$$

in $\mathcal{P}_X$.

Replacing $\mathcal{P}_X$ with the entire category $\mathcal{M}_X$ of coherent sheaves on X gives the Grothendieck group of coherent sheaves $G_0(X)$ with the map

$$K_0(X) \to G_0(X)$$

Theorem (Regularity)

If X is regular, $K_0(X) \to G_0(X)$ is an isomorphism.

Proof.

Resolve a coherent sheaf $\mathcal{F}$ by vector bundles

$$0 \to E_n \to \ldots \to E_0 \to \mathcal{F} \to 0$$

Sending $[\mathcal{F}] \in G_0(X)$ to $\sum_i (-1)^i [E_i] \in K_0(X)$ gives the inverse. $\square$

1. $\otimes$ of sheaves makes $K_0(X)$ a commutative ring and $G_0(X)$ a $K_0(X)$-module.
2. For $f : Y \to X$, sending $[E] \in K_0(X)$ to $[f^*E] \in K_0(Y)$ makes K_0 a contravariant functor.
3. For $f : Y \to X$ flat, sending $[\mathcal{F}] \in G_0(X)$ to $[f^*\mathcal{F}] \in G_0(Y)$ makes G_0 a contravariant functor for flat maps.
4. For $f : Y \to X$ projective, sending $[\mathcal{F}] \in G_0(Y)$ to $\sum_i (-1)^i [R^i f_* \mathcal{F}]$ makes G_0 a functor for projective maps.
5. Using regularity, we have f_* on K_0 for projective maps of smooth varieties, and f^* on G_0 for all maps of smooth varieties.

Grothendieck defined the Chern character

$$ch : K_0(X) \to \mathrm{CH}^*(X)_{\mathbb{Q}}$$

a natural ring homomorphism (for X smooth).

Theorem (Grothendieck-Riemann-Roch)

1. *$ch : K_0(X)_{\mathbb{Q}} \to \mathrm{CH}^*(X)_{\mathbb{Q}}$ is an isomorphism.*

2. *For $f : Y \to X$ projective, $x \in K_0(Y)$,*

$$f_*(ch(x)) = ch(f_*(x \cup Td(T_f)))$$

Here Td is the Todd class and T_f is the formal vertical tangent bundle,

$$T_f := [T_Y] - f^*[T_X] \in K_0(Y).$$

Quillen's construction of the K-theory of an exact category, applied to $\mathcal{P}_X$, extended the Grothendieck group construction to give higher algebraic K-theory:

$$X \rightsquigarrow K_n(X) := K_n(\mathcal{P}_X); \quad n = 0, 1, \ldots$$

and theory built on coherent sheaves $\mathcal{M}_X$:

$$X \rightsquigarrow G_n(X) := K_n(\mathcal{M}_X); \quad n = 0, 1, \ldots$$

The basic structures: product and pull-back on $K_*(X)$, flat pull-back and projective push-forward for $G_*(X)$, natural map $K_*(X) \to G_*(X)$ and the regularity theorem, all extend.

In addition, one has

Theorem (Homotopy invariance)

$p^* : G_n(X) \to G_n(X \times \mathbb{A}^1)$ is an isomorphism.

Theorem (Localization)

Let $i : W \to X$ be a closed immersion, $j : U := X \setminus W \to X$ to open complement. There is a long exact sequence

$$\ldots \to G_n(W) \xrightarrow{i_*} G_n(X) \xrightarrow{j^*} G_n(U) \xrightarrow{\partial} G_{n-1}(W) \to \ldots \to G_0(X) \xrightarrow{j^*} G_0(U) \to 0$$

This yields e.g. a Mayer-Vietoris sequence for G-theory. For regular schemes, the regularity theorem gives analogous properties for K-theory.

Cycles revisited

Since we have the Chern character isomorphism

$$ch : K_{0\mathbb{Q}} \to \mathrm{CH}^*_{\mathbb{Q}}$$

and a right-exact localization sequence

$$\mathrm{CH}_*(W) \to \mathrm{CH}_*(X) \to \mathrm{CH}_*(U) \to 0$$

it is natural to ask: Can one extend the Chow groups to a larger theory, that after $\otimes\mathbb{Q}$ gives all of algebraic K-theory, and extends the right-exact sequence for CH_* to a long exact sequence?

In fact, much more was conjectured and turned out to be true.

The Beilinson-Lichtenbaum conjectures

In the early '80's Beilinson and Lichtenbaum gave conjectures for versions of universal cohomology which would arise as hypercohomology (in the Zariski, resp. étale topology) of certain complexes of sheaves. The conjectures describe sought-after properties of these representing complexes.

The complexes are supposed to explain values of L-functions and at the same time incorporate Milnor K-theory into the picture.

Theorem (Quillen)

Let F be a number field with ring of integers $\mathcal{O}_F$. Then the K-groups $K_n(\mathcal{O}_F)$ are finitely generated abelian groups for all $n \geq 0$.

Let F be a number field. By the theorem of Bass-Milnor-Serre, $K_1(\mathcal{O}_F) = \mathcal{O}_F^*$, and thus by Dirichlet's unit theorem, $K_1(\mathcal{O}_F)$ is a finitely generated abelian group of rank $r_1 + r_2 - 1$. This computation is generalized in Borel's theorem:

Theorem

Let F be a number field. Then $K_{2m}(\mathcal{O}_F)\otimes\mathbb{Q}=0$ for $m\ge 1$, and $K_{2m-1}(\mathcal{O}_F)\otimes\mathbb{Q}\cong\mathbb{Q}^{r(m)}$, where

$$r(m)=\begin{cases} r_1+r_2-1 & \text{for } m=1\\ r_1+r_2 & \text{for } m>1 \text{ odd,}\\ r_2 & \text{for } m>1 \text{ even.}\end{cases}$$

By comparing these ranks with the known orders of vanishing of the zeta function, we have

$$\mathrm{ord}_{s=1-m}\zeta_F(s) = \mathrm{rank} K_{2m-1}(\mathcal{O}_F). \tag{1}$$

Furthermore, by comparing two different $\mathbb{Q}$ structures on the real cohomology of $\mathrm{SL}(\mathcal{O}_F \otimes_{\mathbb{Q}} \mathbb{R})$, Borel defined "regulators" $R_m(F)$. These are also related to the zeta function, by

$$\zeta_F(1-m)^* \equiv R_m(F) \mod \mathbb{Q}^*,$$

where $\zeta_F(1-m)^*$ is the leading term

$$\zeta_F(1-m)^* := \lim_{s\to 1-m} (s+m-1)^{-r(m)}\zeta_F(s).$$

What about an integral statement?

Conjecture (Lichtenbaum)

For F totally real and for m even,

$$\zeta_F(1-m)_l = \frac{\# H^2_{\acute{e}t}(\mathcal{O}_F, j_*\mathbb{Z}_l(m))}{\# H^1_{\acute{e}t}(\mathcal{O}_F, j_*\mathbb{Z}_l(m))}$$

This follows (at least for odd l) from the "main conjecture" of Iwasawa theory, proved by Mazur-Wiles for F abelian, and for F arbitrary by Wiles.
Note that, if F is totally real ($r_2 = 0$) and m is even, then $r(m) = 0$, and $R_m(F) = 1$.

Lichtenbaum extended his conjecture to cover the case of all $m \geq 1$ and all number fields, and at the same time used K-theory to remove the dependence on l:

Conjecture

Let F be a number field, $m \geq 1$ an integer. Then

$$\frac{\zeta_F(1-m)^*}{R_m(F)} = \pm \frac{\# K_{2m-2}(\mathcal{O}_F)}{K_{2m-1}(\mathcal{O}_F)_{tor}}$$

This is classical for $m = 1$, but turned out to be false in general: For $F = \mathbb{Q}$, $m = 2$, $K_2(\mathbb{Z}) = \mathbb{Z}/2$, $K_3(\mathbb{Z}) = \mathbb{Z}/48$, but $\zeta(-1) = -1/12$

To correct, Lichtenbaum suggested breaking up K-theory by "weights".

Adams operations act on K-theory, giving the decomposition

$$K_n(X)_{\mathbb{Q}} = \oplus_q K_n(X)^{(q)}; \quad K_n(X)^{(q)} := k^q\text{-eigenspace for } \psi_k.$$

Beilinson reindexed to conform with other cohomology theories and defined the universal rational cohomology of X as

$$H^p(X, \mathbb{Q}(q)) := K_{2q-p}(X)^{(q)}.$$

An integral version, $H^p(\mathcal{O}_F, \mathbb{Z}(q))$, could possibly replace $K_{2q-p}(\mathcal{O}_F)$ in a formula for $\zeta_F(1-q)$.

The second ingredient is Milnor K-theory.

Definition

Let F be a field. The graded-commutative ring $K_*^M(F)$ is defined as the quotient of the tensor algebra $T(F^\times) := \oplus_{n\geq 0}(F^\times)^{\otimes n}$ by the two-sided ideal generated by elements $a \otimes (1-a)$, $a \in F \setminus \{0,1\}$:

$$K_*^M(F) := \oplus_{n\geq 0}(F^\times)^{\otimes n}/\langle\{a \otimes (1-a)\}\rangle.$$

This was introduced by Milnor in his study of quadratic forms.

Since $F^\times = K_1(F)$, products in K-theory induce

$$(F^\times)^{\otimes n} \to K_n(F).$$

Elements $a \otimes (1-a)$ go to zero in $K_2(F)$ (the Steinberg relation), so we have

$$\rho_n : K_n^M(F) \to K_n(F).$$

Suslin showed that the kernel of ρ_n is killed by $(n-1)!$. For $n = 1, 2$, ρ_n is an isomorphism ($n = 2$ is Matsumoto's theorem), but in general ρ_n is far from surjective.

Example. For F a number field, $K_n^M(F)$ is 2-torsion for $n \geq 3$, but $K_3(F)$ has rank r_2.

Milnor K-theory is closely related to Galois cohomology.
To start, use the Kummer sequence (n prime to the characteristic)

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m \to 1$$

which gives (since $H^0_{\mathrm{Gal}}(F, \mathbb{G}_m) = F^\times$, $H^1_{\mathrm{Gal}}(F, \mathbb{G}_m) = 0$)

$$H^1_{\mathrm{Gal}}(F, \mu_n) \cong F^\times/(F^\times)^n = K^M_1(F)/n.$$

Extend by taking products and showing the Steinberg relation holds in Galois cohomology to give

$$\theta_{F,q,n} : K^M_q(F)/n \to H^q_{\mathrm{Gal}}(F, \mu_n^{\otimes q}).$$

Conjecture (Bloch-Kato)

$\theta_{F,q,n}$ is an isomorphism for all F, q and n prime to char(F).

$\theta_{F,q,n} : K^M_q(F)/n \to H^q_{\mathrm{Gal}}(F, \mu_n^{\otimes q})$ is an isomorphism for all F, q and n prime to char(F).

Case	Solved by	Date
$q = 1$	Kummer	
$q = 2$	Tate	(for number fields)1970's
$q = 2$	Merkurjev-Suslin	1983
$q = 3$	Rost	1990's
$q > 2, n = 2^\nu$	Voevodsky	1996
general	Voevodsky, Rost,...	2007

Beilinson-Lichtenbaum conjectures

The idea was that one should get the right values for zeta functions by replacing K-theory with "universal cohomology" $H^p(X, \mathbb{Z}(q))$, that would give an integral version of the eigenspaces of Adams operations on higher K-theory, with the weight q motivic cohomology contributing to $\zeta(1-q)$.

At the same time, the work of Merkurjev-Suslin on K_2 and Galois cohomology made clear the importance of the Bloch-Kato conjecture in giving the link between universal cohomology and values of zeta functions. This led Beilinson and Lichtenbaum (independently) to conjecture that universal cohomology should arise as hypercohomology of complexes of sheaves with certain properties.

Conjecture (Beilinson-letter to Soulé 1982)

For $X \in \mathbf{Sm}/k$ there are complexes $\Gamma_{\mathrm{Zar}}(q)$, $q \geq 0$, in the derived category of sheaves of abelian groups on X_{Zar}, (functorial in X) with functorial graded product, and

(0) $\Gamma_{\mathrm{Zar}}(0) \cong \mathbb{Z}$, $\Gamma_{\mathrm{Zar}}(1) \cong \mathbb{G}_m[-1]$

(1) $\Gamma_{\mathrm{Zar}}(q)$ *is acyclic outside* $[1, q]$ *for* $q \geq 1$.

(2) $\Gamma_{\mathrm{Zar}}(q) \otimes^L \mathbb{Z}/n \cong \tau_{\leq q} R\alpha\mu_n^{\otimes q}$ *if* n *is invertible on* X, *where* $\alpha : X_{\acute{e}t} \to X_{\mathrm{Zar}}$ *is the change of topology morphism.*

(3) $K_n(X)^{(q)} \cong \mathbb{H}^{2q-n}(X_{\mathrm{Zar}}, \Gamma_{\mathrm{Zar}}(q))_{\mathbb{Q}}$ *(or up to small primes)*

(4) $\mathcal{H}^q(\Gamma_{\mathrm{Zar}}(q)(F)) = \mathcal{K}_q^M$.

Conjecture (Lichtenbaum LNM 1068)

For $X \in \mathbf{Sm}/k$ there are complexes $\Gamma_{\acute{e}t}(r)$, $r \geq 0$, in the derived category of sheaves of abelian groups on $X_{\acute{e}t}$, (functorial in X) with functorial graded product, and

(0) $\Gamma_{\acute{e}t}(0) \cong \mathbb{Z}$, $\Gamma_{\acute{e}t}(1) \cong \mathbb{G}_m[-1]$

(1) $\Gamma_{\acute{e}t}(q)$ *is acyclic outside* $[1, q]$ *for* $q \geq 1$.

(2) $R^{q+1}\alpha_*\Gamma_{\acute{e}t}(q) = 0$ *(Hilbert Theorem 90)*

(3) $\Gamma_{\acute{e}t}(q) \otimes^L \mathbb{Z}/n \cong \mu_n^{\otimes q}$ *if* n *is invertible on* X.

(4) $\mathcal{K}_n^{(q)\acute{e}t} \cong \mathcal{H}^{2q-n}(\Gamma(q))$ *(up to small primes), where* $\mathcal{K}_n^{(q)\acute{e}t}$ *and* $\mathcal{H}^{2q-n}(\Gamma_{\acute{e}t}(q))$ *are the respective Zariski sheaves.*

(5) *For a field* F, $H^q(\Gamma_{\acute{e}t}(q)(F)) = K_q^M(F)$.

Beilinson-Lichtenbaum conjectures

The two constructions should be related by

$$\tau_{\leq q} R\alpha_* \Gamma_{\text{ét}}(q) = \Gamma_{\mathrm{Zar}}(q);\ \Gamma_{\text{ét}}(q) = \alpha^* \Gamma_{\mathrm{Zar}}(q).$$

These conjectures, somewhat reinterpreted for motivic cohomology, are now known as the Beilinson-Lichtenbaum conjectures. The translation is

$$H^p(X, \mathbb{Z}(q)) := \mathbb{H}^p(X_{\mathrm{Zar}}, \Gamma_{\mathrm{Zar}}(q)) = \mathbb{H}^p(X_{\mathrm{Zar}}, \tau_{\leq q} R\alpha_* \Gamma_{\text{ét}}(q)).$$

The relation with the Chow groups is

$$\mathrm{CH}^p(X) = H^{2p}(X, \mathbb{Z}(p)) \cong H^p(X, \mathcal{K}^M_p).$$

The relations (2) and (4) in Beilinson's conjectures and (2), (3) and (5) in Lichtenbaum's version imply the Bloch-Kato conjectures.

In fact, they say: There is a natural map

$$H^p(X, \mathbb{Z}/n(q)) \to H^p_{\text{ét}}(X, \mu_n^{\otimes q})$$

which is an isomorphism for $p \leq q$. For $p > q$, $H^p(F, \mathbb{Z}(q)) = 0$, so

$$H^q(F, \mathbb{Z}/n(q)) = K^M_q(F)/n.$$

Remarkably, the Beilinson-Lichtenbaum conjectures have all been verified, with the exception of the boundedness result (1). This is the Beilinson-Soulé vanishing conjecture:

$$H^p(X, \mathbb{Z}(q)) = 0 \text{ if } p \leq 0 \text{ and } q \neq 0.$$

The first successful candidate for the motivic complexes was constructed by Bloch.

Bloch's cycle complexes

Roughly speaking, Bloch starts with the presentation for $\mathrm{CH}^q(X)$:

$$z^q(X \times \mathbb{A}^1)_{0,1} \xrightarrow{i_0^* - i_1^*} z^q(X) \to \mathrm{CH}^q(X) \to 0$$

and extends to the left by using the algebraic n-simplices. The resulting complex is a good candidate for $\Gamma(q)(X)$.

The standard n-simplex is

$$\Delta_n := \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1, t_i \geq 0\}.$$

These fit together to form a cosimplicial space: Let **Ord** be the category with objects the ordered sets $\underline{n} := \{0, \ldots, n\}$ and maps the order-preserving maps of sets.

For $i \in \underline{n}$, we have the ith vertex v_i of Δ_n, with $t_i = 1$, $t_j = 0$ for $j \neq i$.

Given $g : \underline{n} \to \underline{m}$ define

$$\Delta(g) : \Delta_n \to \Delta_m$$

to be the convex-linear extension of the map $v_i \mapsto v_{g(i)}$.

Explicitly

$$\Delta(g)(t_0, \ldots, t_n) = (\Delta(g)(t)_0, \ldots, \Delta(g)(t)_m)$$

with

$$\Delta(g)(t)_j := \sum_{i \in g^{-1}(j)} t_i.$$

This gives us the functor

$$\Delta_* : \mathbf{Ord} \to \mathbf{Spaces}$$

i.e., a cosimpicial space.

If now T is a topological space, let $C_n^{\mathrm{sing}}(T)$ be the free abelian group on the set of continuous maps $\sigma : \Delta_n \to T$. Composition with the maps $\Delta(g)$ defines the simplicial abelian group

$$C_*^{\mathrm{sing}}(T) := \mathbb{Z}[Maps(\Delta_*, T)] : \mathbf{Ord}^{\mathrm{op}} \to \mathbf{Ab}.$$

As usual, we form the complex $(C_*^{\mathrm{sing}}(T), d)$, with

$$d_n := \sum_i (-1)^i \delta_n^{i*} : C_{n+1}^{\mathrm{sing}}(T) \to C_n^{\mathrm{sing}}(T)$$

where $\delta_n^i : \Delta_n \to \Delta_{n+1}$ is the coface map

$$\delta_n^i(t_0, \ldots, t_n) := (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n).$$

$(C_*^{\mathrm{sing}}(T), d)$ is the singular chain complex of T and

$$H_n(T, \mathbb{Z}) := H_n(C_*^{\mathrm{sing}}(T), d).$$

We now make an algebraic analog.

Definition

The algebraic n-simplex over k, Δ^n_k, is the hyperplane $\sum_i t_i = 1$ in $\mathbb{A}^{n+1}_k$. For $g : \underline{n} \to \underline{m}$ let

$$\Delta(g) : \Delta^n_k \to \Delta^m_k$$

be the affine linear map $t \mapsto (\Delta(g)(t)_0, \ldots, \Delta(g)_m)$ with

$$\Delta(g)(t)_j := \sum_{i \in g^{-1}(j)} t_i.$$

This defines the cosimplicial scheme $\Delta^*_k : \mathbf{Ord} \to \mathbf{Sm}/k$.

A face of Δ^n_k is a subscheme defined by $t_{i_1} = \ldots = t_{i_r} = 0$.

Noting that $(\mathbb{A}^1, 0, 1) = (\Delta^1, \delta_0^1(\Delta^0), \delta_0^0(\Delta^0))$, we extend our presentation of $\mathrm{CH}^q(X)$ by defining

$$z^q(X, n) := \mathbb{Z}\{\text{irreducible, codimension } q \text{ subvarieties} \\ W \subset X \times \Delta^n \text{ in good position}\}.$$

where "good position" means:

$$\mathrm{codim}_{X\times F} W \cap X \times F \geq q$$

for all faces $F \subset \Delta^n$.

The "good position" condition implies that we have a well-defined pull-back

$$\Delta(g)^* : z^q(X,n) \to z^q(X,m)$$

for every $g : \underline{n} \to \underline{m}$ in **Ord**, giving us the simplicial abelian group

$$n \mapsto z^q(X,n)$$

and the complex $(z^q(X,*),d)$. Explicitly, this is

$$\ldots \to z^q(X,n+1) \xrightarrow{d_n} z^q(X,n) \to \\ \ldots \to z^q(X,1) = z^q(X\times\mathbb{A}^1)_{0,1} \xrightarrow{i_0^*-i_1^*} z^q(X)$$

so

$$H_0(z^q(X,*),d) = \mathrm{CH}^q(X).$$

Definition
The complex $(z^q(X,*), d)$ is Bloch's cycle complex. The higher Chow groups of X are

$$\mathrm{CH}^q(X,n) := H_n(z^q(X,*), d).$$

The complexes $z^q(X,*)$ inherit the properties of $z^q(X)$ (more or less):

1. For $f : Y \to X$ flat, we have $f^* : z^q(X,*) \to z^q(Y,*)$
2. For $f : Y \to X$ proper, we have $f_* : z_q(Y,*) \to z_q(X,*)$ $(z_q(Y,*) := z^{\dim Y-q}(Y,*))$.
3. Cycle product induces a product
$$\cup : z^q(X,*) \otimes^L z^p(Y,*) \to z^{p+q}(X \times_k Y,*)$$
in $D^-(\mathbf{Ab})$.
4. For arbitrary $f : Y \to X$ in $\mathbf{Sm}/k$ there is $f^* : z^q(X,*) \to z^q(Y,*)$ in $D^-(\mathbf{Ab})$.

These induce corresponding operations on the homology $\mathrm{CH}^*(-,n)$.

Two important results:

Theorem (Homotopy invariance)

For any X

$$p^* : z_q(X,*) \to z_{q+1}(X,*)$$

is a quasi-isomorphism, i.e. $p^* : \mathrm{CH}_q(X,n) \to \mathrm{CH}_{q+1}(X\times\mathbb{A}^1,n)$ *is an isomorphism.*

Theorem (Localization)

Let $i : W \to X$ be a closed immersion, $j : U := X \setminus W \to X$ the open complement. Then the sequence

$$z_q(W,*) \xrightarrow{i_*} z_q(X,*) \xrightarrow{j^*} z_q(U,*)$$

extends canonically to a distinguished triangle in $D^-(\mathbf{Ab})$, i.e., there is a long exact localization sequence

$$\begin{aligned} \ldots \to \mathrm{CH}_q(W,n) \xrightarrow{i_*} \mathrm{CH}_q(X,n) \xrightarrow{j^*} \mathrm{CH}_q(U,n) \\ \xrightarrow{\partial} \mathrm{CH}_q(W,n-1) \to \ldots \to \mathrm{CH}_q(U,1) \\ \xrightarrow{\partial} \mathrm{CH}_q(W) \xrightarrow{i_*} \mathrm{CH}_q(X) \xrightarrow{j^*} \mathrm{CH}_q(U) \to 0. \end{aligned}$$

Corollary (Mayer-Vietoris)

If $X = U \cup V$, U, V open, we have a long exact Mayer-Vietoris sequence

$$\ldots \to \mathrm{CH}_q(X,n) \to \mathrm{CH}_q(U,n) \oplus \mathrm{CH}_q(V,n) \to \mathrm{CH}_q(U \cap V, n)$$
$$\xrightarrow{\partial} \mathrm{CH}_q(X, n-1) \to \ldots \to \mathrm{CH}_q(U \cap V, 1)$$
$$\xrightarrow{\partial} \mathrm{CH}_q(X) \to \mathrm{CH}_q(U) \oplus \mathrm{CH}_q(V) \to CH_q(U \cap V) \to 0.$$

Theorem (Totaro, Nestorenko-Suslin)

For a field F and integer $q \geq 0$, there is a natural isomorphism

$$K_q^M(F) \cong \mathrm{CH}^q(F, q).$$

Note. $\mathrm{CH}^q(F, n) = 0$ for $n > q$ for dimensional reasons.

Finally, we have a Chern character isomorphism:

$$ch_n : K_n(X)_{\mathbb{Q}} \to \oplus_{q \geq 0} \mathrm{CH}^q(X, n)_{\mathbb{Q}}$$

identifying $K_n(X)^{(q)}$ with $\mathrm{CH}^q(X, n)_{\mathbb{Q}}$.

Definition

1. Integral universal cohomology of $X \in \mathbf{Sm}/k$ is

$$H^p(X, \mathbb{Z}(q)) := \mathrm{CH}^q(X, 2q-p).$$

2. The sheaf of complexes $\Gamma_{Bl}(q)^*$ is

$$U \mapsto z^q(U, 2q-*)$$

Remark It follows from the Mayer-Vietoris property that the natural map

$$H^p(\Gamma_{Bl}(q)(X)) \to \mathbb{H}^p(X_{\mathrm{Zar}}, \Gamma_{Bl}(q))$$

is an isomorphism. Thus

$$H^p(X, \mathbb{Z}(q)) = H^p(\Gamma_{Bl}(q)(X)) = \mathbb{H}^p(X_{\mathrm{Zar}}, \Gamma_{Bl}(q)).$$

From our comments, the complexes $\Gamma_{Bl}(q)$ satisfy all requirements of Beilinson, except

1. The functoriality and products are only in $D^{-}(\mathbf{Ab})$, not as complexes (technical issue).
2. $\Gamma_{Bl}(q)$ is not known to be cohomologically supported in $[1, q]$, although $\Gamma_{Bl}(q)$ is supported in $(-\infty, q]$.
3. Note that the mod n properties rely on the Bloch-Kato conjecture.

Suslin's cycle complexes

Suslin's construction of cycle complex is more closely related to constructions in topology than Bloch's version.

We have seen that the singular chain complex of a space T can be written as

$$C_*^{\text{sing}}(T) = \mathbb{Z}[\text{Maps}(\Delta_*, T)]$$

with the differential induced by the cosimplicial structure in Δ_*.

Why not try the same, replacing Δ_* with Δ_k^* and T with a variety X?

Answer. It fails miserably! There are usually very few maps $\Delta_k^n \to X$.

Instead, use the Dold-Thom theorem as starting point:

Theorem (Dold-Thom)

Let $(T, *)$ be a pointed CW complex. There is a natural isomorphism

$$H_n(T, *) \cong \pi_n(\mathrm{Sym}^\infty T).$$

Here

$$\mathrm{Sym}^\infty T = \varinjlim[T \to \mathrm{Sym}^2 T \to \ldots \to \mathrm{Sym}^n T \to \ldots]$$

with $\mathrm{Sym}^n T \to \mathrm{Sym}^{n+1} T$ the map "add $*$ to the sum".

What is the algebraic analog? A simple answer is given by "finite cycles"

Definition

For X, Y varieties, X smooth and irreducible, set

$$z_{\mathrm{fin}}(Y)(X) := \mathbb{Z}[\{\text{irreducible, reduced } W \subset X \times_k Y \text{ with } W \to X \text{ finite and surjective}\}].$$

Think of such a $W \subset X \times Y$ as a map from X to $\mathrm{Sym}^d Y$, $d = \deg W/X$:

$$x \in X \mapsto (x \times Y) \cdot W = \sum_i n_i y_i \in \mathrm{Sym}^d Y.$$

Note that $z_{\mathrm{fin}}(Y)$ is a presheaf on $\mathbf{Sm}/k$: Given $f : X' \to X$ and $W \subset X \times Y$ finite over X, the pull-back

$$(f \times \mathrm{id}_Y)^{-1}(W) \subset X' \times Y$$

is still finite over X'. Thus the cycle pull-back is defined, giving

$$f^* : z_{\mathrm{fin}}(Y)(X) \to z_{\mathrm{fin}}(Y)(X').$$

In particular, we can evaluate $z_{\mathrm{fin}}(Y)$ on Δ^*_k, giving us the simplicial abelian group

$$n \mapsto z_{\mathrm{fin}}(Y)(\Delta^n_k).$$

Definition

For a k-scheme Y, the Suslin complex of Y, $C_*^{\mathrm{Sus}}(Y)$, is the complex associated to the simplicial abelian group

$$n \mapsto z_{\mathrm{fin}}(Y)(\Delta_k^n).$$

The Suslin homology of Y is

$$H_n^{\mathrm{Sus}}(Y, A) := H_n(C_*^{\mathrm{Sus}}(Y) \otimes A).$$

Since $z_{\mathrm{fin}}(Y)(X)$ is covariantly functorial in Y (for arbitrary maps), Suslin homology is covariantly functorial, as it should be.

One beautiful result on Suslin homology (proved before Bloch-Kato) is

Theorem (Suslin-Voevodsky)

Let Y be a finite type scheme over $\mathbb{C}$. Then there is a natural isomorphism

$$H_*^{\mathrm{Sus}}(Y, \mathbb{Z}/n) \cong H_*(Y(\mathbb{C}), \mathbb{Z}/n).$$

Since a cycle in $z_{\mathrm{fin}}(Y)(\Delta^n)$ is a cycle on $Y \times \Delta^n$, obviously in good position, we have a natural inclusion

$$C_*^{\mathsf{Sus}}(Y) \hookrightarrow z^d(Y,*); \quad d = \dim_k Y,$$

Theorem (Suslin-Voevodsky)

For Y smooth and projective, $C_*^{\mathsf{Sus}}(Y) \hookrightarrow z^d(Y,*)$ is a quasi-isomorphism.

Thus

$$H_n^{\mathsf{Sus}}(Y,\mathbb{Z}) \cong H^{2d-n}(Y,\mathbb{Z}(d))$$

for Y smooth and projective of dimension d (Poincaré duality).

One can recover all the universal cohomology groups from the Suslin homology construction, properly modified. For this, we recall how the Dold-Thom theorem gives a model for cohomology.

Since S^n has only one non-trivial reduced homology group, $H_n(S^n, \mathbb{Z}) = \mathbb{Z}$, the Dold-Thom theorem tells us that $\mathrm{Sym}^\infty S^n$ is a $K(\mathbb{Z}, n)$, i.e.

$$\pi_m(\mathrm{Sym}^\infty S^n) = \begin{cases} 0 & \text{for } m \neq n \\ \mathbb{Z} & \text{for } m = n. \end{cases}$$

Obstruction theory tells us that

$$H^m(X, \mathbb{Z}) = \pi_{m-n}(Maps(X, Sym^\infty S^n)).$$

for $m \leq n$.

To rephrase this in the algebraic setting, we need a good replacement for the n-spheres. It turns out we get a very good replacement for the 2-sphere by taking $\mathbb{P}^1$, pointed by ∞, and thus we use $(\mathbb{P}^1)^{\wedge n}$ for S^{2n}.

The wedge product doesn't make much sense, but since we are going to apply this to finite cycles, we just take a quotient by the cycles "at infinity":

$$z_{\mathrm{fin}}((\mathbb{P}^1)^{\wedge n})(X) := z_{\mathrm{fin}}((\mathbb{P}^1)^n)(X)/\sum_{j=1}^{n} i_{\infty,j*}(z_{\mathrm{fin}}((\mathbb{P}^1)^{n-1})(X)),$$

where $i_{\infty,j} : (\mathbb{P}^1)^{n-1} \to (\mathbb{P}^1)^n$ inserts ∞ in the jth spot.
This leads to

Definition

The Friedlander-Suslin weight q cycle complex of X is

$$\Gamma_{FS}(q)^*(X) := z_{\mathrm{fin}}((\mathbb{P}^1)^{\wedge q})(X \times \Delta^{2q-*}).$$

This gives us the complex of sheaves $U \mapsto \Gamma_{FS}(q)(U)^*$.

Restriction from $X \times \Delta^n \times (\mathbb{P}^1)^q \to X \times \Delta^n \times \mathbb{A}^q$ defines the inclusion

$$\Gamma_{FS}(q)^*(X) \hookrightarrow z^q(X \times \mathbb{A}^q, 2q - *) = \Gamma_{Bl}(q)^*(X \times \mathbb{A}^q)$$

Theorem (Friedlander-Suslin-Voevodsky)

For X smooth and quasi-projective, the maps

$$\Gamma_{FS}(q)^*(X) \to \Gamma_{Bl}(q)^*(X \times \mathbb{A}^q) \xleftarrow{p^*} \Gamma_{Bl}(q)^*(X)$$

are quasi-isomorphisms. In particular, we have natural isomorphisms

$$\mathbb{H}^p(X_{\mathrm{Zar}}, \Gamma_{FS}(q)) \cong H^p(\Gamma_{FS}(q)(X)^*) \cong H^p(X, \mathbb{Z}(q)).$$

Since

$$X \mapsto \Gamma_{FS}(q)^*(X) := z_{\mathrm{fin}}((\mathbb{P}^1)^{\wedge q})(X \times \Delta^{2q-*})$$

is functorial in X, the Friedlander-Suslin complex gives a functorial model for Bloch's cycle complex.

Products for $\Gamma_{FS}(q)$ are similarly defined on the level of complexes.

This completes the Beilinson-Lichtenbaum program, with the exception of the vanishing conjectures.

Concluding remarks

Computations

One can make some computations.

Proposition

Let X be a smooth quasi-projective variety. Then

1. $H^n(X, \mathbb{Z}(q)) = 0$ *for* $n > 2q$.
2. $H^{2q}(X, \mathbb{Z}(q)) = \mathrm{CH}^q(X)$
3. $H^1(X, \mathbb{Z}(1)) = \Gamma(X, \mathcal{O}_X^*)$; $H^n(X, \mathbb{Z}(1)) = 0$ *for* $n \neq 1, 2$

For a field F:

1. $H^n(F, \mathbb{Z}(q)) = 0$ *for* $n > q$
2. $H^q(F, \mathbb{Z}(q)) = K_q^M(F)$
3. $H^1(F, \mathbb{Z}(2)) = K_3(F)/K_3^M(F)$.

Atiyah-Hirzebruch spectral sequence

Theorem (Bloch-Lichtenbaum/Friedlander-Suslin)

Let X be smooth. There is a strongly convergent spectral sequence

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Longrightarrow K_{-p-q}(X)$$

Adams operations act on the spectral sequence, with ψ_k acting by $\times k^q$ on $E_2^{p,-q}$, so one can refine the $\mathbb{Q}$-isomorphism of universal cohomology with K-theory, limiting the primes one needs to invert.

Atiyah-Hirzebruch spectral sequence

Combining with the mod n information furnished by the Beilinson-Lichtenbaum conjectures, the known computations of K-theory of finite fields and of number rings yield similar computations for universal cohomology.

Conversely, the Beilinson-Lichtenbaum conjectures plus the AH spectral sequence allow one to compare mod n algebraic K-theory with mod n topological (étale) K-theory, giving a proof of the Quillen-Lichtenbaum conjecture:

Theorem

The natural map

$$K_p(X;\mathbb{Z}/n) \to K_p^{ét}(X,\mathbb{Z}/n)$$

is an isomorphism for $p \geq cd_n(X) - 1$ *and injective for* $p = cd_n(X) - 2$.

An advertisement

In the next lecture, we show how the theory of universal cohomology acquires a categorical foundation, becoming motivic cohomology via Voevodsky's triangulated category of motives.

Thank you!