K-THEORY AND MOTIVIC COHOMOLOGY OF SCHEMES, I

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ABSTRACT. We examine the basic properties satisfied by Bloch's cycle complexes for quasi-projective varieties over a field, and extend most of them to the cycle complex of a scheme of finite type over a regular dimension one base. We also extend these properties to the simplicial spectra in the homotopy niveau tower of the cosimplicial scheme Δ^*_X. As applications, we show that the homotopy coniveau spectral sequence from motivic cohomology to K-theory is functorial for smooth schemes, admits a multiplicative structure and has lambda operations. We also show that the homotopy coniveau filtration on algebraic K-theory agrees with the gamma-filtration, up to small primes.

0. INTRODUCTION

In [20], we have described an extension of the cycle complexes $z_q(X,*)$ of Bloch to schemes X of finite type over a regular one-dimensional base B. We also considered the *homotopy niveau tower*

(0.1)

$$\ldots \to G_{(q-1)}(X,-) \to G_{(q)}(X,-) \to \ldots \to G_{(d)}(X,-) \sim G(X); \quad \dim X \leq d,$$

where $G_{(q)}(X,-)$ is the simplicial spectrum $p \mapsto G_{(q)}(X,p)$, and $G_{(q)}(X,p)$, roughly speaking, is defined by taking the G-theory spectra of $X \times \Delta^p$ with support in closed subsets W of dimension $\leq p+q$ such that $W \cap (X \times F)$ has dimension $\leq q+r$ for each face $F \cong \Delta^r$ of Δ^p (see §2 below for a precise definition). By extending the localization techniques developed by Bloch in [3], we have shown that, for $B = \operatorname{Spec} A$, A a semi-local PID, the complexes $z_q(X,*)$, as well as the simplicial spectra $G_{(q)}(X,-)$, satisfy a localization property with respect to closed subschemes of X. Defining the motivic Borel-Moore homology of X as the shifted homology

$$H_p^{\mathrm{B.M.}}(X,\mathbb{Z}(q)) := H_{p-2q}(z_q(X,*)),$$

this gives the motivic Borel-Moore homology the formal properties of classical Borel-Moore homology. Additionally, combining the localization properties of the simplicial spectra $G_{(q)}(X,-)$ with the fundamental interpretation of the Bloch-Lichtenbaum spectral sequence [4] given by Friedlander-Suslin [6] allows one to extend the spectral sequence of Bloch and Lichtenbaum to a spectral sequence from motivic Borel-Moore homology of X to the G-theory of X, for X a scheme of finite type over $\operatorname{Spec} A$. For a general one-dimensional regular base B, sheafifying these constructions over B gives similar results (see (2.4) below). Extending the

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definition of motivic Borel-Moore homology suitably gives us the *homotopy niveau spectral sequence*

$$E^1_{p,q} = H^{\mathrm{B.M.}}_{q-p}(X, \mathbb{Z}(p)) \Longrightarrow G_{p+q}(X). \tag{0.2}$$

If X is regular, relabelling using codimension and changing G-theory to K-theory gives us the *homotopy coniveau tower*

$$\ldots \to K^{(q+1)}(X,-) \to K^{(q)}(X,-) \to \ldots \to K^{(0)}(X,-) \sim K(X); \tag{0.3}$$

Similarly, reindexing (0.2) to form an E_2 spectral sequence gives us the *homotopy coniveau spectral sequence*

$$E_2^{p,q} = H^p(X, \mathbb{Z}(-q/2)) \Longrightarrow K_{-p-q}(X). \tag{0.4}$$

All these spectral sequences are strongly convergent.

In this paper, we consider the other important properties of Bloch's cycle complexes, as established in [2], [3], [19, Chap. II, §3.5] and [31]: homotopy, functorialities and products, and show how these extend to the generalized complexes $z_q(X,*)$, and the simplicial spectra $G_{(q)}(X,-)$. We are not entirely successful in extending the theory of cycle complexes over a field to a one-dimensional base; there is a mixed characteristic version of the classical Chow's moving lemma which is missing at present. This causes some technical annoyance in the mixed characteristic case, but substantial portions of the theory still go through, at least for schemes smooth over the base B. In case $B = \operatorname{Spec} F$, F a field, the entire theory of the cycle complexes extends to the simplicial spectra $G_{(q)}(X,-)$, giving the homotopy property, contravariant functoriality for smooth X, products and an associated étale theory. In terms of the extended Bloch-Lichtenbaum spectral sequence, this gives us the homotopy property, a product structure, functoriality for smooth schemes and a comparison with a spectral sequence from étale cohomology to étale K-theory. Much of this theory extends to the case of a one-dimensional base, but there are some restrictions in the functoriality and product structure. In any case, we are able to directly relate the Beilinson-Lichtenbaum conjectures for mod n motivic cohomology to the Quillen-Lichtenbaum conjectures for mod n algebraic K-theory.

We also define functorial λ-operations on the homotopy groups of the $G_{(q)}(X,-)$ (for X regular), which give Adams operations for the homotopy niveau sequence. This implies the rational degeneration of this spectral sequence (even for singular subschemes of a regular scheme), giving an isomorphism of rational G-theory with rational motivic Borel-Moore homology; for regular X, this gives an isomorphism of the weight-graded pieces of K-theory with motivic cohomology after inverting small primes. We have have a similar comparison of the filtration on $K_*(X)$ induced by the tower (0.3) and the γ-filtration, generalizing the Grothendieck comparison of the topological filtration and the γ-filtration on K_0.

A different construction of a tower giving an interesting filtration on K-theory has been given by Grayson [11], building on ideas of Goodwillie and Lichtenbaum. In a series of papers, Walker ([40], [41], [42] and [43]) has studied Grayson's construction, and has been able to relate the weight one portion of Grayson's tower to motivic cohomology. He has also shown that the filtration on K-theory given by Grayson's tower agrees with the γ-filtration, up to torsion. Recently, Suslin has constructed an isomorphism of the Grayson spectral sequence with the one considered in this paper, in the case of schemes of finite type over a field.

An outline of the paper is as follows: In §1 and §2, we recall the basic definitions of the cycle complexes $z_q(X,*)$, the simplicial spectra $G_{(q)}(X,-)$, and various versions of the extended Bloch-Lichtenbaum spectral sequence. We also give a brief discussion of equi-dimension cycles.

In §3 we discuss a K-theoretic version $K^{(q)}(X,-)$ of the simplicial spectrum $G_{(q)}(X,-)$. In §4, we discuss the covariant functoriality of the homotopy niveau tower and the homotopy niveau spectral sequence (0.2). In §5 we prove the homotopy property for $z_q(X,*)$ and $G_{(q)}(X,-)$, and in §6 we briefly recall the well-known connection of localization and the Mayer-Vietoris property; we also check the compatibility of the spectral sequences with localization.

We formulate the fundamental "moving lemma" (Theorem 7.3) for the spectra $K^{(q)}(X,-)$ in §7. In §8 we show how Theorem 7.3 gives the contravariant functoriality for morphisms of smooth B-schemes, and in §9 we prove Theorem 7.3; §10 is a recapitulation of the results of §8 and §9 in the equi-dimensional setting. We give the construction of a product structure for the spectral sequence (0.4) in §11.

In §12 we construct λ-operations for the spectra $K^{(q)}(X,-)$, and discuss the Adams operations on the spectral sequence (0.4) for regular schemes. In §13 we use the constructions of Friedlander-Suslin [6] to show that the γ-filtration on $K_*(X)$ is finer than the filtration $F^*_{\mathrm{HC}}K_*(X)$ induced by the tower (0.3). We make some explicit computations of motivic Borel-Moore homology and motivic cohomology in §14, and use the degeneration of (0.4) (after inverting small primes) to compare motivic cohomology with K-theory (for regular schemes) and motivic Borel-Moore homology with G-theory (see Theorem 14.7 and Theorem 14.8). This extends the results of [2], [3], [17] and [18] to schemes of finite type over a regular one-dimensional base. In addition, this shows that the homotopy coniveau tower (3.3) gives the "Adams weight filtration" on the spectrum $K(X)$, for X regular and essentially of finite type over a regular one-dimensional base. In Theorem 14.7 we show as well that the filtrations $F^*_{\mathrm{HC}}K_*(X)$ and $F^*_\gamma K_*(X)$ agree up to groups of explicit finite exponent.

In part II of this work, we will discuss the associated étale theory, giving a version of the homotopy coniveau spectral sequence for mod n étale K-theory. Using the comparison of the mod n version of homotopy coniveau spectral sequence with the étale spectral sequence, we give a number of applications. We show that the Beilinson-Lichtenbaum conjectures for motivic cohomology implies the Quillen-Lichtenbaum conjectures for algebraic K-theory; we add in the reduction steps of [33] and [9] to reduce the Quillen-Lichtenbaum conjectures to the Bloch-Kato conjectures. This shows that Voevodsky's verification of the Milnor Conjecture [38] yields the sharp version of the 2-primary part of the Quillen-Lichtenbaum conjectures, at least for schemes essentially of finite type over a one-dimensional regular base; as an example, we recover the results of Rognes-Weibel [26] relating the 2-adic algebraic K-theory and étale K-theory of rings of S-integers in a totally imaginary number field. We give some applications to computations in various arithmetic settings, including the 2-primary motivic cohomology of finite fields, and rings of S-integers in a number field. The multiplicativity of the Bloch-Lichtenbaum spectral sequence, as described in §11, fills in a gap in some arguments of B. Kahn [15] computing the 2-localized K-theory of rings of S-integers in a number field (the computation was made by a different method in [25]). We also extend some results of Kahn [15] on the map of Milnor K-theory to Quillen K-theory.

We conclude this paper with a four appendices. In the first, we fix notations and recall some results on the category of presheaves of spectra. In the second, for lack of a suitable reference, we give extensions of many of the basic constructions of algebraic K-theory to the setting of cosimplicial schemes. In the third, we recall the construction of products in K-theory, and in the fourth, we discuss the spectral sequence associated to a tower of spectra, as well as multiplicative structure on such a spectral sequence.

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1. Higher Chow groups

1.1. **Bloch's higher Chow groups.** We recall Bloch's definition of the higher Chow groups [2]. Fix a base field k. Let Δ^N_k denote the standard "algebraic N-simplex"

$$\Delta^N_k := \operatorname{Spec} k[t_0, \ldots, t_N]/\sum_i t_i - 1,$$

let X be a quasi-projective scheme over k, and let Δ^*_X be the cosimplicial scheme

$$N \mapsto X \times_k \Delta^N_k.$$

A *face* of Δ^N_X is a subscheme defined by equations of the form $t_{i_1} = \ldots = t_{i_r} = 0$. Let $X_{(p,q)}$ be the set of dimension $q+p$ irreducible closed subschemes W of Δ^p_X such that W intersects each dimension r face F in dimension $\leq q+r$. We have Bloch's simplicial group

$$p \mapsto z_q(X, p),$$

with $z_q(X,p)$ the subgroup of the dimension $q+p$ cycles on $X \times \Delta^p$ generated by $X_{(p,q)}$, and the associated complex $z_q(X,*)$. The *higher Chow groups* of X are defined by

$$\mathrm{CH}_q(X,p) := H_p(z_q(X,*)).$$

If X is equi-dimensional over k, we may label these complexes by codimension, and define

$$\mathrm{CH}^q(X,p) := H_p(z^q(X,*)),$$

where $z^q(X,p) = z_{d-p}(X,p)$ if X has dimension d over k. We extend the definition of $z^q(X,*)$ to arbitrary smooth X by taking the direct sum of the $z^q(X_i,*)$ over the irreducible components X_i of X.

These groups compute the motivic Borel-Moore homology of X and, for X smooth over k, the motivic cohomology of X by

Theorem 1.2. *We have the natural isomorphism*

$$H^{\mathrm{B.M.}}_p(X, \mathbb{Z}(q)) \cong \mathrm{CH}_q(X, p-2q),$$

where $H_*^{\mathrm{B.M.}}$ *is the motivic Borel-Moore homology. Suppose* X *is smooth over* k*. There is a natural isomorphism*

$$H^p(X, \mathbb{Z}(q)) \cong \mathrm{CH}^q(X, 2q-p).$$

Here the motivic cohomology is that defined by the construction of [37], [12] or [19].

The complexes $z_q(X,*)$ are covariantly functorial for proper maps, and contravariantly functorial (with the appropriate shift in q) for flat equi-dimensional maps. In particular, we may sheafify $z_q(X,*)$ for either the Zariski or the étale topology; we let $\mathcal{Z}_q(X,*)$ be the Zariski sheafification of $z_q(X,*)$, and $\mathcal{Z}_q^{\text{ét}}(X,*)$ the étale sheafification. We define $\mathcal{Z}^q(X,*)$ and $\mathcal{Z}^q_{\text{ét}}(X,*)$ similarly.

Here is a list of the important properties of the cycle complexes for quasi-projective schemes over k:

(1.1)

(1) *Homotopy.* Let $p : \mathbb{A}^1 \times X \to X$ be the projection. Then the map

$$p^* : z_q(X,*) \to z_{q+1}(\mathbb{A}^1 \times X, *)$$

is a quasi-isomorphism.

(2) *Localization.* Let $i : Z \to X$ be the inclusion of a closed subscheme, with complement $j : U \to X$. Then the exact sequence

$$0 \to z_q(Z,*) \xrightarrow{i_*} z_q(X,*) \xrightarrow{j^*} z_q(U,*)$$

is a distinguished triangle, i.e., the map i_* induces a quasi-isomorphism

$$z_q(Z,*) \to \mathrm{cone}(j^*)[-1].$$

(3) *Mayer-Vietoris.* Let $X = U \cup V$ be a Zariski open cover. Then the Mayer-Vietoris sequence

$$z_q(X,*) \xrightarrow{(j_U^*, j_V^*)} z_q(U,*) \oplus z_q(V,*) \xrightarrow{j^{V*}_{U\cap V} - j^{U*}_{U\cap V}} z_q(U \cap V, *)$$

is a distinguished triangle.

(4) *Functoriality.* Suppose that X is smooth over k, and let $f : Y \to X$ be a morphism of quasi-projective k-schemes. For each p, let $z^q(X,p)_f$ be the subgroup of $z^q(X,p)$ generated by those codimension q $W \subset X \times \Delta^p$ such that W intersects $X \times F$ properly for each face F of Δ^p, and each component of $(f \times \mathrm{id})^{-1}(W)$ has codimension q on $Y \times \Delta^p$ and intersects $Y \times F$ properly, for each face F of Δ^p. The $z_q(X,p)_f$ form a subcomplex $z_q(X,*)_f$ of $z_q(X,*)$. Then, in case X is affine, the inclusion $z_q(X,*)_f \to z_q(X,*)$ is a quasi-isomorphism. Using the Mayer-Vietoris property, this gives rise to functorial pull-back morphisms

$$f^* : z^q(X,*) \to z^q(Y,*)$$

in the derived category, for each k-morphism $f : Y \to X$, with X smooth over k.

(5) *Products.* The operation of taking products of cycles extends to give natural external products (in the derived category)

$$\boxtimes_{X,Y} : z^q(X,*) \otimes z^{q'}(Y,*) \to z^{q+q'}(X \times_k Y, *).$$

Taking $X = Y$ and pulling back by the diagonal gives the cup product map (in the derived category)

$$\cup_X : z^q(X,*) \otimes z^{q'}(X,*) \to z^{q+q'}(X,*).$$

(6) Let n be prime to the characteristic of k. There is a natural quasi-isomorphism $\mathcal{Z}^q_{\text{ét}}(X,*)/n \to \mu_n^{\otimes q}$, where μ_n is the étale sheaf (on X) of nth roots of unity.

Remarks 1.3. (1) The Mayer-Vietoris property (1.1)(3) follows from localization (1.1)(2). Indeed, the localization property implies excision, i.e., the natural map

$$\operatorname{cone}(j_U^* : z_q(X,*) \to z_q(U,*)) \to \operatorname{cone}(j_{U\cap V}^{V*} : z_q(V,*) \to z_q(U \cap V,*))$$

is a quasi-isomorphism, since both cones are quasi-isomorphic to $z_q(W,*)[1]$, with $W = X \setminus U = V \setminus U \cap V$.

(2) The functoriality (1.1)(4) for a map $f : Y \to X$, with X smooth over k (but not necessarily affine) is accomplished as follows: Let $p_X : X \to \operatorname{Spec} k$ be the projection. It follows from Mayer-Vietoris (1.1)(3) that the natural map $z^q(X,*) \to Rp_{X*}\mathcal{Z}^q(X,*)$ is a quasi-isomorphism. From (1.1)(4), the natural map $\mathcal{Z}^q(X,*)_f \to \mathcal{Z}(X,*)$ is a quasi-isomorphism, where $\mathcal{Z}^q(X,*)_f$ is the complex of sheaves associated to the complex of presheaves

$$U \mapsto z^q(U,*)_{f_{|f^{-1}(U)}}.$$

We let $f^* : z^q(X,*) \to z^q(Y,*)$ be the composition (in $\mathbf{D}^-(\mathbf{Ab})$)

$$z^q(X,*) \cong Rp_{X*}\mathcal{Z}^q(X,*) \cong Rp_{X*}\mathcal{Z}^q(X,*)_f \xrightarrow{f^*} Rp_{Y*}\mathcal{Z}^q(Y,*) \cong z^q(Y,*).$$

(3) The functoriality for affine X, as stated in (1.1)(4), is a consequence of the following "moving lemma", which is a version of the classical Chow's moving lemma:

Proposition 1.4 (Chow's moving lemma for $z^q(X,*)$)**.** *Let X be smooth over a field k, and let $\mathcal{C}$ be a finite collection of irreducible locally closed subsets of X, with $\mathcal{C}$ containing each irreducible component of X. Let $z^q_{\mathcal{C}}(X,p)$ be the subgroup of $z^q(X,p)$ generated by irreducible $W \subset X \times \Delta^p$ such that, for each $C \in \mathcal{C}$, each face F of Δ^p, and each irreducible component W' of $W \cap (C \times \Delta^p)$, we have $\operatorname{codim}_{C\times F}(W') \geq q$. Let $z^q_{\mathcal{C}}(X,*)$ be the subcomplex of $z^q(X,*)$ formed by the $z^q_{\mathcal{C}}(X,p)$, and suppose that X is affine. Then the inclusion $z^q_{\mathcal{C}}(X,*) \to z^q(X,*)$ is a quasi-isomorphism.*

For a proof of this result, we refer the reader to [19, Chap. II, §3.5].

(4) The list of properties (1.1) shows that the pair

$$(\oplus_{p,q} H_p^{\mathrm{B.M.}}(-,\mathbb{Z}(q)), \oplus_{p,q} H^p(-,\mathbb{Z}(q)))$$

satisfy the Bloch-Ogus axioms for a twisted duality theory [5].

1.5. **Cycle complexes in mixed characteristic.** In [20], we have extended the localization property (1.1)(2) to the case of a finite type scheme X over a regular noetherian scheme B of dimension at most one. Before describing this, we first recall the definition of the cycle complexes in this setting. For $p : X \to B$ an irreducible B-scheme of finite type, the *dimension* of X is defined as follows: Let $\eta \in B$ be the image of the generic point of X, X_η the fiber of X over η. If η is a closed point of B, then X is a scheme over the residue field $k(\eta)$, and we set $\dim X := \dim_{k(\eta)} X$. If η is not a closed point of B, we set $\dim X := \dim_{k(\eta)} X_\eta + 1$.

If $X \to B$ is proper, then $\dim X$ is the Krull dimension of X, but in general $\dim X$ is only greater than or equal to the Krull dimension.

We let $\Delta^n = \mathrm{Spec}\,_B(\mathcal{O}_B[t_0, \ldots, t_n]/\sum_i t_i - 1)$, giving the cosimplicial B-scheme Δ^*. We have for each B-scheme X the cosimplicial scheme $\Delta^*_X := X \times_B \Delta^*$, and for each (p,q) the set $X_{(p,q)}$ of irreducible closed subsets C of Δ^p_X of dimension $p+q$, such that, for each face F of Δ^p of dimension r over B, we have

$$\dim(C \cap X \times F) \leq r + q.$$

If U is an open subscheme of X, we let $U^X_{(p,q)}$ be the subset of $U_{(p,q)}$ consisting of those irreducible closed subsets whose closure in Δ^p_X are in $X_{(p,q)}$.

The complexes $z_q(X,*)$ are covariant for proper morphisms, and contravariant for flat equi-dimensional morphisms (if $f : Y \to X$ is flat of relative dimension d, we have $f^* : z_q(X,*) \to z_{q+d}(Y,*)$). We may therefore form the complex of presheaves $\mathcal{Z}_q(X,*)$ on X_{Zar}, with $\mathcal{Z}_q(X,*)(U) = \mathcal{Z}_q(U,*)$ (this is already a complex of sheaves). In particular, we have the complex of presheaves $p_*\mathcal{Z}_q(X,*)$ on B_{Zar}. These complexes of presheaves have the same functoriality as the complexes $z_q(-,*)$.

Here is our extension of Bloch's localization result:

Theorem 1.6 ([20], Theorem 0.6)**.** *Let $i : Z \to X$ be a closed subscheme of a finite-type B-scheme X, $j : U \to X$ the complement. Then the (exact) sequence of sheaves on B*

$$0 \to (p \circ i)_*\mathcal{Z}_q(Z,*) \xrightarrow{i_*} p_*\mathcal{Z}_q(X,*) \xrightarrow{j^*} (p \circ j)_*\mathcal{Z}_q(U,*)$$

is stalk-wise a distinguished triangle. If B is semi-local, then $z_q(U,*)/j^*z_q(X,*)$ is acyclic, hence the exact sequence of complexes

$$0 \to z_q(Z,*) \xrightarrow{i_*} z_q(X,*) \xrightarrow{j^*} z_q(U,*)$$

is a distinguished triangle.

If we set $\mathrm{CH}_q(X,p) := \mathbb{H}^{-p}(B_{\mathrm{Zar}}, p_*\mathcal{Z}_q(X,*)) =: H^{\mathrm{B.M.}}_{p-2q}(X, \mathbb{Z}(q))$, then Theorem 1.6 gives a long exact localization sequence for the higher Chow groups/motivic Borel-Moore homology. We also have the identity

$$\mathbb{H}^{-p}(B, p_*\mathcal{Z}_q(X,*)) = H_p(z_q(X,*))$$

for B semi-local. In addition, the Mayer-Vietoris property implies that the natural map $\mathbb{H}^{-p}(B, p_*\mathcal{Z}_q(X,*)) \to \mathbb{H}^{-p}(X, \mathcal{Z}_q(X,*))$ is an isomorphism, for arbitrary regular B of dimension at most one (see Remark A.3).

2. The G-theory spectral sequence

We have given in [20] a globalization of the Bloch-Lichtenbaum spectral sequence [4]

$$E_2^{p,q} = H^p(F, \mathbb{Z}(-q/2)) \Longrightarrow K_{-p-q}(F),$$

F a field, to a spectral sequence (of homological type) for $X \to B$ of finite type, B as above a regular one-dimensional noetherian scheme,

$$E^2_{p,q}(X) = H^{\mathrm{B.M.}}_p(X, \mathbb{Z}(-q/2)) \Longrightarrow G_{p+q}(X).$$

We recall the rough outline of the construction of this spectral sequence.

2.1. **The homotopy niveau tower spectral sequence.** Applying the techniques of Appendix B and Appendix D to the "topological filtration" on the cosimplicial scheme $X\times\Delta^*$ yields the homotopy niveau tower and the resulting spectral sequence converging to the G-theory of X. In this section, we recall some details of this construction.

2.2. **The homotopy niveau tower.** Let X be a finite-type B-scheme. We have the exact category $\mathcal{M}_X$ of coherent sheaves on X, and the corresponding K-theory spectrum $G(X) := K(\mathcal{M}_X)$.

Let U be an open subscheme of $X \times \Delta^p$. From Appendix B, we have the full subcategory $\mathcal{M}_U(\partial)$ of $\mathcal{M}_U$ with objects the coherent sheaves $\mathcal{F}$ such that $\mathrm{Tor}_q^{\mathcal{O}_U}(\mathcal{F}, \mathcal{O}_{U\cap(X\times F)}) = 0$ for all faces F of Δ^p and all $q > 0$. We write $\mathcal{M}_X(p)$ for $\mathcal{M}_{X\times\Delta^p}(\partial\Delta^p)$ and let $G(X,p)$ denote the K-theory spectrum $K(\mathcal{M}_X(p))$.

Let $X_{(p,\le q)}$ be the set of irreducible closed subsets W of $X \times \Delta^p$ such that, for each face F of Δ^p (including $F = \Delta^p$), and each irreducible component W' of $W \cap (X \times F)$, we have

$$\dim(W') \le q + \dim_B(F).$$

For a closed subset W of $X\times\Delta^p$, we have the spectrum with supports $G_W(X,p)$, defined as the homotopy fiber of the map of spectra

$$j^* : K(\mathcal{M}_X(p)) \to K(\mathcal{M}_U(\partial\Delta^p)),$$

where U is the complement $X \times \Delta^p \setminus W$ and $j : U \to X \times \Delta^p$ is the inclusion. We let $G_{(q)}(X,p)$ denote the direct limit of the $G_W(X,p)$, as W runs over finite unions of irreducible closed subsets $C \in X_{(p,\le q)}$.

The assignment $p \mapsto \mathcal{M}_X(p)$ extends to a simplicial exact category $\mathcal{M}_X(-)$; we let $G(X,-) := K(\mathcal{M}_X(-))$ be the corresponding simplicial spectrum. Similarly, the assignments $p \mapsto G_{(q)}(X,p)$ extends to a simplicial spectrum $G_{(q)}(X,-)$. The augmentation $\Delta^*_X \to X$ induces a weak equivalence $G(X) \to G(X,-)$.

We let $\dim X$ denote the maximum of $\dim X_i$ over the irreducible components X_i of X. We note that $G_{(q)}(X,p) = G(X,p)$ for all $q \ge \dim X$. The evident maps

$$G_{(q-1)}(X,p) \to G_{(q)}(X,p) \to$$

give the tower of simplicial spectra

$$\ldots \to G_{(q-1)}(X,-) \to G_{(q)}(X,-) \to \ldots \to G_{(\dim X)}(X,-) \xleftarrow{\epsilon} G(X), \tag{2.1}$$

Remark 2.3. Using Lemma B.6, one can also define $G_{(q)}(X,-)$ as the limit of the simplicial spectra $G_W(X,-)$, as W runs over all cosimplicial closed subsets W of $X \times \Delta^*$, such that $W^p \subset X \times \Delta^p$ is a finite union of elements of $X_{(q,p)}$ for each p.

Let $f : X' \to X$ be a flat morphism of B-schemes of finite type of relative dimension d. For each open subscheme $U \subset X \times \Delta^p$, we have the exact functor $(f\times\mathrm{id})^* : \mathcal{M}_U(\partial) \to \mathcal{M}_{U'}(\partial)$, where $U' = (f\times\mathrm{id})^{-1}(U) \subset X'\times\Delta^p$. Similarly, for $W \in X_{(p,\le q)}$, each irreducible component of $(f\times\mathrm{id})^{-1}(W)$ is in $X'_{(p,\le q+d)}$. Thus, the functors $(f\times\mathrm{id})^*$ define the map of simplicial spectra

$$f^* : G_{(q)}(X,-) \to G_{(q+d)}(X',-),$$

with the functoriality $(f\circ g)^* = g^*\circ f^*$ for composable flat morphisms f and g (of pure relative dimension). In particular, the assignment $V \mapsto G_{(p/q)}(V,-)$ defines a presheaf of simplicial spectra $\mathcal{G}_{(p/q)}(X,-)$ on X_{Zar}. Similarly, we denote the presheaf $U \mapsto G(U)$ by $\mathcal{G}(X)$.

The tower (2.1) is natural with respect to flat equi-dimesnional pull-back; in particular, we have the tower of presheaves

$$\ldots \to \mathcal{G}_{(q-1)}(X,-) \to \mathcal{G}_{(q)}(X,-) \to \ldots \to \mathcal{G}_{(\dim X)}(X,-) \xleftarrow{\epsilon} \mathcal{G}(X), \tag{2.2}$$

which we call the *homotopy niveau tower* for X.

2.4. **The spectral sequence.** We let $G_{(p/q)}(X,-)$ denote the homotopy cofiber of the map $G_{(q)}(X,-) \to G_{(p)}(X,-)$, for $p \geq q$. The homotopy property of G-theory, together with Lemma B.8, implies that the augmentation $\epsilon : G(X) \to G_{\dim X}(X,-)$ is a weak equivalence. Following the constructions of Appendix D, the tower (2.1) gives rise to a spectral sequence (of homological type)

$$E^1_{p,q} = \pi_{p+q}(G_{(p/p-1)}(X,-)) \Longrightarrow G_{p+q}(X). \tag{2.3}$$

Since $G_{(p)}(X,n)$ is the one-point spectrum for $p+n<0$, it follows that $\pi_N(G_{(p)}(X,-)) = 0$ for $p < -N$. Thus, the spectral sequence (2.3) is strongly convergent.

Similarly, the augmentation $\mathcal{G}(X) \to \mathcal{G}(X,-)$ is a weak equivalence of presheaves of spectra. The Mayer-Vieotoris property of G-theory implies that the natural map

$$G_n(X) \to \pi_n(X, \mathcal{G}(X))$$

is an isomorphism for each n. Taking the spectral sequence associated to the tower of presheaves (2.2) thus gives us the spectral sequence

$$E^1_{p,q} = \pi_{p+q}(X; \mathcal{G}_{(p/p-1)}(X,-)) \Longrightarrow G_{p+q}(X), \tag{2.4}$$

which we call the *homotopy niveau spectral sequence.* Since X has finite Zariski cohomological dimension, say D, the vanishing of $\pi_N(G_{(p)}(X,-))$ for $p < -N$ (and all X) implies that $\pi_N(X; \mathcal{G}_{(p)}(X,-)) = 0$ for $p < -N - D$. Therefore, (2.4) is strongly convergent. We will denote this spectral sequence by $\mathrm{AHG}(X)$.

2.5. **Identifying the E^1-term.** Taking the cycle-class of a coherent sheaf defines the map

$$\mathrm{cl}_q : G_{(q/q-1)}(X,-) \to z_q(X,-). \tag{2.5}$$

In case $X = \operatorname{Spec} F$ for a field F, Friedlander and Suslin [6, Theorem 3.3] have shown that this map is a weak equivalence.

The arguments used to prove Theorem 1.6 can be modified to show that the simplicial spectra $G_{(q)}(X,-)$ satisfy a similar localization property:

Theorem 2.6 ([20, Corollary 7.10])**.** *Suppose that $B = \operatorname{Spec} A$, A a semi-local principal ideal ring, and let $i : Z \to X$ be a closed subscheme of X with complement $j : U \to X$. Then the sequence*

$$G_{(q)}(Z,-) \xrightarrow{i_*} G_{(q)}(X,-) \xrightarrow{j^*} G_{(q)}(U,-) \tag{2.6}$$

is a homotopy fiber sequence.

Using this localization property together with the Friedlander-Suslin theorem mentioned above, we show in [20, Corollary 7.6] that the cycle map cl_q a weak equivalence, for $B = \operatorname{Spec} A$ a semi-local principal ideal ring. This identifies the E^1-terms in (2.3) as

$$E^1_{p,q} = \mathrm{CH}_p(X, p+q) = H^{\mathrm{B.M.}}_{q-p}(X, \mathbb{Z}(p)),$$

in case $B = \operatorname{Spec} A$, A a principal ideal ring. The localization property also implies that the evident map of spectral sequence (2.3) to (2.4) is an isomorphism on the E_1-terms, hence the two spectral sequences are isomorphic in this case.

For $f : X \to B$ of finite type over a general base B (regular and dimension at most one), the isomorphism (2.5) in the semi-local case gives the stalk-wise weak equivalence of presheaves

$$\mathrm{cl}_q : f_*\mathcal{G}_{(q/q-1)}(X, -) \to \mathcal{Z}_q(X, -), \tag{2.7}$$

which leads to the analogous identification of the E^1-term in (2.4) as

$$E^1_{p,q} = \mathbb{H}^{-p-q}(B, f_*\mathcal{Z}_q(X, *)) = H^{\mathrm{B.M.}}_{q-p}(X, \mathbb{Z}(p)).$$

After reindexing (2.4) to give an E^2-spectral sequence, we arrive at the strongly convergent homological spectral sequence

$$E^2_{p,q} = H^{\mathrm{B.M.}}_p(X, \mathbb{Z}(-q/2)) \Longrightarrow G_{p+q}(X). \tag{2.8}$$

In case X is regular, the natural map $G_*(X) \to K_*(X)$ is an isomorphism. If X is irreducible, $\dim X = d$, we define $H^p(X, \mathbb{Z}(q)) := \mathrm{CH}_{d-q}(X, 2d - 2q - p)$ and extend this definition to arbitrary regular X by taking the direct sum over the irreducible components. The spectral sequence (2.8) then becomes the strongly convergent cohomological spectral sequence

$$E_2^{p,q} = H^p(X, \mathbb{Z}(-q/2)) \Longrightarrow K_{-p-q}(X). \tag{2.9}$$

The above constructions give similar spectral sequences with finite coefficients as well. For instance, define the complex of sheaves $\mathcal{Z}_q(X, *)/n$ as the cone of multiplication by n, $\times n : \mathcal{Z}_q(X, *) \to \mathcal{Z}_q(X, *)$, and set

$$H^{\mathrm{B.M.}}_p(X, \mathbb{Z}/n(q)) := \mathbb{H}^{2q-p}(B; f_*\mathcal{Z}_q(X, *)/n) = \mathbb{H}^{2q-p}(X; \mathcal{Z}_q(X, *)/n).$$

Replacing the homotopy groups $\pi_s(-)$ throughout with the homotopy groups with coefficients mod n, $\pi_s(-; \mathbb{Z}/n)$, gives the strongly convergent spectral sequence

$$E^2_{p,q} = H^{\mathrm{B.M.}}_p(X, \mathbb{Z}/n(-q/2)) \Longrightarrow G_{p+q}(X; \mathbb{Z}/n). \tag{2.10}$$

The other spectral sequences discussed above have their mod n counterparts as well.

2.7. **Cycles and G-theory with equi-dimensional supports.** The main purpose of this paper is to establish the fundamental properties of the spectral sequences (2.8) and (2.9), as well as the mod n versions. Along the way, we will need to examine some extensions of the properties (1.1) to mixed characteristic. Unfortunately, at present we are only able to prove a limited functoriality and product structure in this setting. We conclude this introduction with a statement of a conjecture which, if valid, would give the cycle complexes $z^q(X, *)$ the necessary functoriality and product structure in mixed characteristic. We give a quick outline of the theory of equi-dimensional cycles, in the case of base scheme of dimension at most one; for details and the general theory, we refer the reader to [32] and to [19, I, Appendix A].

Let B be a regular irreducible scheme of dimension at most one. An irreducible B-scheme $p : X \to B$ of finite type is *equi-dimenisonal* over B if p is dominant. If this is the case, we set

$$\dim_B X := \dim_{k(\eta)} X_\eta,$$

where X_η is the fiber of X over the generic point η of B. If X is not necessarily irreducible, we say that X is equi-dimensional over B of dimension d if each irreducible component X_i of X is equi-dimensional over B and $\dim_B X_i = d$ for all i.

If B is regular of dimension at most one, but not necessarily irreducible, we call $p : X \to B$ equi-dimensional over B of dimension d if the restriction of p to $X_i := p^{-1}(B_i) \to B_i$ is equi-dimenisonal of dimension d for each irreducible component B_i of B, or is empty.

For $X \to B$ of finite type, we let $(X/B)_{(p,q)}$ be the set of irreducible closed subsets $W \subset X \times \Delta^p$ such that, for each face F of Δ^p, and each irreducible component W' of $W \cap (X \times F)$, W' is equi-dimensional over B, and

$$\dim_B W' = \dim_B F + q. \tag{2.11}$$

We let $(X/B)_{(p,\le q)}$ be defined similarly, where we replace the condition (2.11) with

$$\dim_B W' \le \dim_B F + q.$$

Let $z_q(X/B,p)$ be the free abelian group on $(X/B)_{(p,q)}$, forming the simplicial abelian group $z_q(X/B,-)$ and the associated complex $z_q(X/B,*)$. Similarly, we let $G_{(q)}(X/B,p)$ be the limit of the spectra $G_W(X,p)$, where W runs over finite unions of elements of $(X/B)_{(p,\le q)}$, giving the simplicial spectrum $G_{(q)}(X/B,-)$.

If $X \to B$ is equi-dimensional of dimension d, we let $z^q(X/B,-) = z_{d-q}(X/B,-)$ and $G^{(q)}(X/B,-) = G_{(d-q)}(X/B,-)$; more generally, if X is a disjoint union of equi-dimensional $X_i \to B$, we set

$$z^q(X/B,-) := \oplus_i z^q(X_i/B,-), \quad G^{(q)}(X/B,-) := \prod_i G^{(q)}(X_i/B,-).$$

Set

$$\mathrm{CH}_q(X/B,p) := H_p(z_q(X/B,*)),\ \mathrm{CH}^q(X/B,p) := H_p(z^q(X/B,*)).$$

In case each point of B is closed, we have $z_q(X/B,-) = z_q(X,-)$, and in case each component of B has dimension one, $z_q(X/B,-)$ is a simplicial subgroup of $z_{q+1}(X,-)$. If $z^q(X/B,-)$ is defined, then so is $z^q(X,-)$, and $z^q(X/B,-)$ is a simplicial subgroup of $z^q(X,-)$. The analogous statements hold for $G_{(q)}(X/B,-)$ and $G^{(q)}(X/B,-)$.

The simplicial group $z_q(X/B,-)$ is covariantly functorial for proper maps, and contravariantly functorial for flat equi-dimenisional maps (with the appropriate shift in q). When defined, the simplicial group $z^q(X/B,-)$ is covariantly functorial for proper maps (with the appropriate shift in q), and contravariantly functorial for flat maps. Similarly for $G_{(q)}(X/B,-)$ and $G^{(q)}(X/B,-)$.

We let $\mathcal{Z}_q(X/B,p)$ denote the presheaf on X_{Zar}, $U \mapsto z_q(U/B,p)$, giving the simplicial abelian presheaf $\mathcal{Z}_q(X/B,-)$ and the associated complex of presheaves $\mathcal{Z}_q(X/B,*)$. We similarly have the simplicial abelian presheaf $\mathcal{Z}^q(X/B,-)$ and the complex of presheaves $\mathcal{Z}^q(X/B,*)$ in case each connected component of X is equi-dimensional over B. All these presheaves are sheaves.

We may also form presheaves of simplicial spectra on X_{Zar}, $\mathcal{G}_{(q)}(X/B,-)$ and $\mathcal{G}^{(q)}(X/B,-)$, by taking the functors $U \mapsto G_{(q)}(U/B,-)$ and $G^{(q)}(U/B,-)$. For $X \to B$ equi-dimensional with $d = \dim_B X$, set $G^{(q)}(X,-) := G_{(d-q+1)}(X,-)$ and $\mathcal{G}^{(q)}(X,-) = \mathcal{G}_{(d-q+1)}(X,-)$. We extend this notation to $X \to B$ such that each connected component of X is equi-dimensional over B as above.

We can now state our main conjecture on the equi-dimensional cycle complexes and spectra.

Conjecture 2.8. Suppose that $X \to B$ is a regular B-scheme of finite type. Then the natural maps

$$\mathcal{Z}^q(X/B, -) \to \mathcal{Z}^q(X, -),\ \mathcal{G}^{(q)}(X/B, -) \to \mathcal{G}^{(q)}(X, -)$$

are stalk-wise weak equivalences on X.

Remark 2.9. Conjecture 2.8 is trivially true in case B has pure dimension zero. If B has dimension one over a field k, then Conjecture 2.8 for the cycle complexes $\mathcal{Z}^q$ is a consequence of Proposition 1.4. Indeed, choose a point $x \in X$, replace X with an affine neighborhood of x, and take $\mathcal{C}$ to be the collection of the irreducible components of the fibers of X containing x; Proposition 1.4 implies that the stalks $\mathcal{Z}^q(X/B, *)_x \subset \mathcal{Z}^q(X, *)_x$ are quasi-isomorphic. In case X is smooth over B, this is pointed out in [19, Chap. II, Lemma 3.6.4]; the same proof works for X smooth over k. In case X is only assumed to be regular, the standard trick of replacing k with a perfect subfield k_0 and making a limit argument reduces us to the case of X smooth over k. Thus, the conjecture for the cycle complexes is open only in the mixed characteristic case.

Using the extension Theorem 7.3 of Proposition 1.4 proved below, the same argument proves Conjecture 2.8 for the simplicial spectra $\mathcal{G}^{(q)}$, hence the full conjecture is valid in the geometric case.

3. K-theory and G-theory

In this section the base-scheme B will be a regular scheme of dimension at most one.

3.1. **The functor $- \times \Delta^*$.** We have the functor from B-schemes to cosimplicial B-schemes $X \mapsto X \times \Delta^*$. We note some basic properties of this functor (for terminology, see Appendix B):

(1) $X \times \Delta^*$ is quasi-projective over $X \times \Delta^0 = X$.
(2) $X \times \Delta^*$ is a cosimplicial scheme of finite Tor-dimension.
(3) Let $f : Y \to X$ be a morphism of B-schemes. Then the morphism $f \times \mathrm{id} : Y \times \Delta^* \to X \times \Delta^*$ is Tor-independent.

The proofs of these results are elementary, and are left to the reader. These properties allow us to apply the results of Appendix B without making additional technical assumptions; we will do so in the sequel without explicitly referring to the above list of properties.

3.2. **K-theory spectral sequence.** Let X be a finite-type B-scheme such that each irreducible component has Krull dimension d (we call such an X *equi-dimensional of dimension* d). We set $X^{(p,q)} := X_{(p,d-q)}$, i.e., we index by codimension rather than dimension. We extend the definition of $X^{(p,q)}$ to disjoint unions of equi-dimensional B-schemes X by taking the disjoint union over the connected components of X. The group $z^q(X, p)$ is then the free abelian group on $X^{(p,q)}$. We similarly define $X^{(p,\geq q)} := X_{(p,\leq d-q)}$ in case each connected component of X is equi-dimensional, and extend to disjoint unions of equi-dimensional B-schemes as above.

For a scheme Y, let $\mathcal{P}_Y$ be the exact category of locally free coherent sheaves on Y. We let $K(X,p)$ be the K-theory spectrum $K(\mathcal{P}_{X\times\Delta^p})$, and for $W\subset X\times\Delta^p$ a closed subset, we let $K^W(X,p)$ denote the homotopy fiber of the restriction map $j^*: K(\mathcal{P}_{X\times\Delta^p})\to K(\mathcal{P}_U)$, where $j: U\to X\times\Delta^p$ is the complement of W.

Let X be a disjoint union of equi-dimensional B-schemes. Taking the limit of the $K^W(X,p)$ as W runs over finite unions of elements of $X^{(p,\geq q)}$ defines the spectrum $K^{(q)}(X,p)$; the assignment $p\mapsto K^{(q)}(X,p)$ clearly extends to a simplicial spectrum $K^{(q)}(X,-)$. We let $\mathcal{K}^{(q)}(X,-)$ be the simplicial presheaf $U\mapsto K^{(q)}(U,-)$ on X_{Zar}, and let $\mathcal{K}(X)$ denote the presheaf of spectra $U\mapsto K(U)$.

We define the simplicial spectrum $G^{(q)}(X,-)$ to be $G_{(d-q)}(X,-)$ in case $\dim X = d$; taking coproducts over the connected components of X defines $G^{(q)}(X,-)$ for X a disjoint union of equi-dimensional B-schemes. We define the presheaf $\mathcal{G}^{(q)}(X,-)$ on X_{Zar} similarly. These notations agree with those given in §2.7 in case each connected component of X is equi-dimensional over B. The natural inclusion $\mathcal{P}_-\to\mathcal{M}_-$ defines the map of simplicial spectra

$$K^{(q)}(X,-)\to G^{(q)}(X,-). \tag{3.1}$$

If W is a cosimplicial closed subset of a cosimplicial scheme Y with complement $U:=Y\setminus W$ of Y, the K-theory spectra with supports, $K^{W_p}(Y_p)$, form a simplicial spectrum $K^W(Y)$. Similarly, for W a cosimplicial closed subset of $X\times\Delta^*$, we may form the simplicial G-theory spectrum $G_W(X,-)$, giving the natural map

$$K^W(X,-)\to G_W(X,-) \tag{3.2}$$

Lemma 3.3. *If X is regular, the maps* (3.1) *and* (3.2) *are term-wise weak equivalences.*

Proof. For (3.2), this is a special case of Remark B.9(1); for (3.1), this follows from the result for (3.2), together with Remark 2.3. □

Remark 3.4. Suppose that X is regular. Lemma 3.3 and the homotopy property for G-theory of schemes imply that the augmentation $\mathcal{K}(X)\xrightarrow{\epsilon}\mathcal{K}^{(0)}(X,-)$ is a weak equivalence of presheaves. Similarly, the Mayer-Vietoris property for G-theory yields the Mayer-Vietoris property for K-theory of regular scheme, which in turn implies that the natural maps $K_n(X)\to\mathbb{H}^{-n}(B,f_*\mathcal{K}(X))\to\mathbb{H}^{-n}(X,\mathcal{K}(X))$ are isomorphisms (see Remark A.3(1)).

Changing G-theory to K-theory and indexing by codimension rather than dimension, the method for constructing the homotopy niveau tower (2.2) gives us the *homotopy coniveau tower*

$$\ldots\to\mathcal{K}^{(q+1)}(X,-)\to\mathcal{K}^{(q)}(X,-)\to\ldots\to\mathcal{K}^{(0)}(X,-)\xleftarrow{\epsilon_X}\mathcal{K}(X), \tag{3.3}$$

with ϵ_X a weak equivalence for X regular. The maps (3.1) for various q induce the map of towers

(3.4)
$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{K}^{(q+1)}(X,-) \arrow[r]\arrow[d] & \mathcal{K}^{(q)}(X,-) \arrow[r]\arrow[d] & \cdots \arrow[r] & \mathcal{K}^{(0)}(X,-)\arrow[d] & \mathcal{K}(X) \arrow[l]\arrow[d]\\
\cdots \arrow[r] & \mathcal{G}^{(q+1)}(X,-) \arrow[r] & \mathcal{G}^{(q)}(X,-) \arrow[r] & \cdots \arrow[r] & \mathcal{G}^{(0)}(X,-) & \mathcal{G}(X) \arrow[l]
\end{tikzcd}$$

Let $\mathcal{K}^{(q/q+1)}(X,-)$ denote the cofiber of the map $\mathcal{K}^{(q+1)}(X,-)\to\mathcal{K}^{(q)}(X,-)$, and define $\mathcal{G}^{(q/q+1)}(X,-)$ similarly.

Using Remark 3.4, these two towers give us the spectral sequences (for X regular)

$$E^1_{p,q}(K)=\mathbb{H}^{-p-q}(X,\mathcal{K}^{(-p/-p+1)}(X,-))\Longrightarrow K_{p+q}(X) \tag{3.5}$$

$$E^1_{p,q}(G)=\mathbb{H}^{-p-q}(X,\mathcal{G}^{(-p/-p+1)}(X,-))\Longrightarrow G_{p+q}(X). \tag{3.6}$$

Proposition 3.5. *Suppose that X is regular. Then the maps* (3.1) *induce weak equivalences of presheaves $\mathcal{K}^{(q)}(X,-)\to\mathcal{G}^{(q)}(X,-)$, and an isomorphism of spectral sequences $E(K)\to E(G)$.*

Proof. By Lemma 3.3, the map $\mathcal{K}^{(q)}(X,-)(U)\to\mathcal{G}^{(q)}(X,-)(U)$ is a weak equivalence for each open $U\subset X$. □

4. Projective pushforward

We have already mentioned that the presheaves of simplicial spectra $\mathcal{G}_{(q)}(X,-)$ are contravariantly functorial in X for flat morphisms of pure relative dimension; evidently the same is true for the presheaves $\mathcal{G}_{(q)}(X/B,-)$ and $\mathcal{K}^{(q)}(X,-)$. We will now discuss the covariant functoriality for projective morphisms. This is essentially an application of the general techiniques and results discussed in §B.10 and §B.15

Let $f:X\to X'$ be a projective morphism of finite type B-schemes. Following the construction of f_* from §B.10, we have, for each cosimplicial open subscheme U of $X\times\Delta^*$, an exact simplicial subcategory $\mathcal{M}_U(\partial)_f$ of $\mathcal{M}_U(\partial)$, and the weak equivalence of K-theory spectra $K(\mathcal{M}_U(\partial)_f)\to K(\mathcal{M}_U(\partial))$. In addition, on $\mathcal{M}_U(\partial)_f$, f_* is an exact functor. More precisely, letting W be the complement of U, and setting $V=X'\setminus f(W)$, the functor $f_*:\mathcal{M}_U\to\mathcal{M}_V$ defines an exact functor $f_*:\mathcal{M}_U(\partial)_f\to\mathcal{M}_V$. Thus, defining $G(U,-)_f$ as the simplicial spectrum $p\mapsto K(\mathcal{M}_U(\partial)_f)$, and letting $G_W(X,-)_f$ be the homotopy fiber of $G(X,-)_f\to G(U,-)_f$, we have the map of spectra $f_*:G_W(X,-)_f\to G_{f(W)}(X',-)$.

In fact, if W is a cosimplicial closed subset of $X\times\Delta^*$, the methods of §B.10 yield the map $f_*:f_*\mathcal{G}_W(X,-)\to\mathcal{G}_{f\times\mathrm{id}(W)}(X',-)$ in $\mathbf{Hot}(X')$, defined as the composition of maps of presheaves of spectra:

$$\mathcal{G}_W(X,-)\xleftarrow{\sim}\mathcal{G}_W(X,-)_f\xrightarrow{(f\times\mathrm{id})_*}\mathcal{G}_{f\times\mathrm{id}(W)}(X',-).$$

Taking the limit over all simplicial closed subsets W with $W^p\subset X\times\Delta^p$ a union of elements of $X_{(p,q)}$, we have thus constructed functorial push-forward maps $f_*:f_*\mathcal{G}_{(q)}(X,-)\to\mathcal{G}_{(q)}(X',-)$ in $\mathbf{Hot}(X')$. Taking homotopy cofibers yields the maps $f_*:f_*\mathcal{G}_{(q/r)}(X,-)\to\mathcal{G}_{(q/r)}(X',-)$ for $r\le q$.

Proposition 4.1. *Let $f:X\to X'$ be a projective morphism of finite type B-schemes.*

(1) *Let* $\mathrm{AHG}(X,f)$ *denote the spectral sequence*

$$E^1_{p,q}(X,f)=\mathbb{H}^{-p-q}(X,\mathcal{G}_{(p/p-1)}(X,-)_f)\Longrightarrow\mathbb{H}^{-p-q}(X,\mathcal{G}(X)_f)$$

arising from the tower of preseheaves

$$\ldots\to\mathcal{G}_{(p-1)}(X,-)_f\to\mathcal{G}_{(p)}(X,-)_f\to\ldots\to\mathcal{G}_{(\dim X)}(X,-)_f\sim\mathcal{G}(X)_f.$$

Then the weak equivalences $G_{(q)}(X,-)_f\to G_{(q)}(X,-)$ and $\mathcal{G}(X)_f\to\mathcal{G}(X)$ induce an isomorphism of spectral sequences $\mathrm{AHG}(X,f)\to\mathrm{AHG}(X)$*, as well as an isomorphism $\mathbb{H}^{-p-q}(X,\mathcal{G}(X)_f)\cong G_{p+q}(X)$.*

(2) *The maps* $f_*: f_*\mathcal{G}_{(p/q)}(X,-)_f \to \mathcal{G}_{(p/q)}(X',-)$ *and* $f_*: f_*\mathcal{G}(X)_f \to \mathcal{G}(X')$ *induce a map of spectral sequences*

$$f_*: \mathrm{AHG}(X,f) \to \mathrm{AHG}(X').$$

Proof. It is evident from the construction of the spectral sequence of a tower of presheaves of spectra, that the collection of maps $\mathcal{G}_{(p)}(X,-)_f \to \mathcal{G}_{(p)}(X,-)$ gives rise to a map of spectral sequences $\mathrm{AHG}(X,f) \to \mathrm{AHG}(X)$. Since the maps $\mathcal{G}_{(p)}(X,-)_f \to \mathcal{G}_{(p)}(X,-)$ are weak equivalences of presheaves (Lemma B.12), they induce isomorphisms $\mathbb{H}^n(X,\mathcal{G}_{(p/q)}(X,-)_f) \to \mathbb{H}^n(X,\mathcal{G}_{(p/q)}(X,-))$ for all $p \geq q$, from which it follows that the map $\mathrm{AHG}(X,f) \to \mathrm{AHG}(X)$ is an isomorphism of spectral sequences. Similarly, the map $\mathcal{G}(X)_f \to \mathcal{G}(X)$ induces an isomorphism $\mathbb{H}^n(X,\mathcal{G}(X)_f) \to \mathbb{H}^n(X,\mathcal{G}(X))$, whence (1).

For (2), the map $f_*: f_*\mathcal{G}_{(p/q)}(X,-)_f \to \mathcal{G}_{(p/q)}(X',-)$ is the one induced on the presheaf cofiber by the maps $f_*\mathcal{G}_{(p)}(X,-)_f \to \mathcal{G}_{(p)}(X',-)$ and $f_*\mathcal{G}_{(q)}(X,-)_f \to \mathcal{G}_{(q)}(X',-)$, from which it immediately follows that the map on the E^1-term, $f_*: \mathbb{H}^n(X,\mathcal{G}_{(p/p-1)}(X,-)_f) \to \mathbb{H}^n(X',\mathcal{G}_{(p/p-1)}(X',-))$ extends to a map of spectral sequences. □

From Proposition 4.1, we see that a projective morphism $f: X \to X'$ induces a map of spectral sequences $f_*: \mathrm{AHG}(X) \to \mathrm{AHG}(X')$.

Lemma 4.2. *Let* $f: X \to X'$ *and* $g: X' \to X''$ *be projective morphisms of finite type* B*-schemes. Then* $(g\circ f)_* = g_* \circ f_*$*, as maps of spectral sequences* $\mathrm{AHG}(X) \to \mathrm{AHG}(X'')$*, and as maps in* $\mathbf{Hot}(X'')$*,* $(g\circ f)_*\mathcal{G}_{(p/q)}(X,-) \to \mathcal{G}_{(p/q)}(X'',-)$*.*

Proof. This follows from the general functoriality for projective pushforward given in Lemma B.13. □

We conclude this section with a compatibility result.

Lemma 4.3. *Let*

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"'] & X \end{tikzcd}$$

be a cartesian square of finite type B*-schemes, with* f *projective and* g *flat and of pure relative dimension* d*. Then* $g^*f_* = f'_*g'^*$*, as maps* $f_*\mathcal{G}_{(p/q)}(Y,-)) \to g_*\mathcal{G}_{(p+d/q+d)}(Z,-))$ *in* $\mathbf{Hot}(X)$*, and as maps of spectral sequences* $\mathrm{AHG}(Y) \to \mathrm{AHG}(Z)[d]$*.*

Proof. It suffices to prove the statement in $\mathbf{Hot}(X)$. This follows from Proposition B.17. □

5. Homotopy

Let $X \to B$ be a B-scheme of finite type, with $B = \operatorname{Spec} A$, A a semi-local principal ideal ring. In this section, we discuss the *homotopy property* for the simplicial spectra $G_{(q)}(X,-)$ and $z_q(X,-)$, i.e., that the pull-back maps

$$p^*: G_{(q)}(X,-) \to G_{(q+1)}(X\times\mathbb{A}^1,-)$$
$$p^*: z_q(X,-) \to z_{q+1}(X\times\mathbb{A}^1,-)$$

are weak equivalences. In case X is regular, this gives the homotopy property for the simplicial spectrum $K^{(q)}(X,-)$ (see §3.2 below). We state and prove a somewhat more general result.

Theorem 5.1. *Let $B = \operatorname{Spec} A$, where A is a semi-local principal ideal ring, and let $X \to B$ be a B-scheme of finite type. Let $p : E \to X$ be a flat morphism of finite type, such that, for each $x \in X$, the fiber $p^{-1}(x)$ is isomorphic to $\mathbb{A}^n_{k(x)}$. Then the pull-back maps*

$$p^* : G_{(q)}(X,-) \to G_{(q+n)}(E,-)$$
$$p^* : z_q(X,-) \to z_{q+n}(E,-)$$

are weak equivalences.

Proof. If we truncate the tower (2.1) at $G_{(q)}(X,-)$, we get the spectral sequence (of homological type)

$$E^1_{a,b}(X,q) = \pi_{a+b}(G_{(a/a-1)}(X,-)) \Longrightarrow \pi_{a+b}(G_{(q)}(X,-)).$$

The flat map $p : E \to X$ gives the map of towers

$$p^* : G_{(*\leq q)}(X,-) \to G_{(*\leq q+n)}(E,-),$$

and hence the map of spectral sequences

$$p^* : E(X,q) \to E(E,q+n).$$

The map on the E^1-terms is just the pull-back

$$p^* : z_q(X,-) \to z_{q+n}(E,-). \tag{5.1}$$

Thus, it suffices to show that the pull-back map (5.1) is a weak equivalence, i.e., that

$$p^* : z_q(X,*) \to z_{q+n}(E,*). \tag{5.2}$$

is a quasi-isomorphism.

Using Theorem 1.6, the standard limit process gives rise to the Quillen spectral sequence on X

$$E^1_{a,b}(X,q) = \oplus_{x\in X_{(a)}} H_{a+b}(z_q(\operatorname{Spec} k(a),*)) \Longrightarrow H_{a+b}(z_q(X,*)),$$

and similarly for E. The map p gives the map of convergent spectral sequences

$$p^* : E(X,q) \to E(E,q+n),$$

so we need only show that p^* induces an isomorphism on the E^1-terms. This reduces us to the case $X = \operatorname{Spec} k$, k a field, which is proved in [2, Theorem 2.1]. □

As an immediate consequence of Theorem 5.1 we have

Corollary 5.2. *Let B be a regular scheme of dimension at most one, and let $X \to B$ be a B-scheme of finite type. Let $p : E \to X$ be a flat morphism of finite type, such that, for each $x \in X$, the fiber $p^{-1}(x)$ is isomorphic to $\mathbb{A}^n_{k(x)}$. Then the pull-back maps*

$$p^* : \mathcal{G}_{(q)}(X,-) \to \mathcal{G}_{(q+n)}(E,-)$$
$$p^* : f_*\mathcal{G}_{(q)}(X,-) \to f_*\mathcal{G}_{(q+n)}(E,-)$$
$$p^* : \mathcal{Z}_q(X,-) \to \mathcal{Z}_{q+n}(E,-)$$
$$p^* : f_*\mathcal{Z}_q(X,-) \to f_*\mathcal{Z}_{q+n}(E,-)$$

are stalk-wise weak equivalences.

6. Localization and Mayer-Vietoris

We have already mentioned the fundamental localization property for the simplicial spectra $G_{(q)}(-,-)$, namely, that the sequence (2.6) is a homotopy fiber sequence. In this section, we list some immediate consequences of this property.

6.1. **Mayer-Vietoris.** We first consider the case of a semi-local base $B = \operatorname{Spec} A$, A a semi-local PID.

Lemma 6.2. *Let X be a finite type B-scheme, with B semi-local. Then the presheaf $\mathcal{G}^{(q)}(X,-)$ satisfies Mayer-Vietoris.*

Proof. Let U and V be Zariski open subschemes of X, let Z be the reduced closed subscheme $U \cup V \setminus V = U \setminus U \cap V$, and consider the commutative diagram

$$\begin{tikzcd}
\mathcal{G}_{(q)}(Z,-) \arrow[r, equal] \arrow[d, "i_*"'] & \mathcal{G}_{(q)}(Z,-) \arrow[d, "i_*"'] \\
\mathcal{G}_{(q)}(U \cup V,-) \arrow[r] \arrow[d] & \mathcal{G}_{(q)}(U,-) \arrow[d] \\
\mathcal{G}_{(q)}(V,-) \arrow[r] & \mathcal{G}^{(q)}(U \cap V,-).
\end{tikzcd}$$

By the localization property (Theorem 2.6) for $\mathcal{G}_{(q)}$, the columns in this diagram are homotopy fiber sequences. Thus, the diagram

$$\begin{tikzcd}
\mathcal{G}_{(q)}(Z,-) \arrow[r, equal] \arrow[d, "i_*"'] & \mathcal{G}_{(q)}(Z,-) \arrow[d, "i_*"'] \\
\mathcal{G}_{(q)}(U \cup V,-) \arrow[r] \arrow[d] & \mathcal{G}_{(q)}(U,-) \arrow[d] \\
\mathcal{G}_{(q)}(V,-) \arrow[r] & \mathcal{G}^{(q)}(U \cap V,-).
\end{tikzcd}$$

is homotopy cartesian. $\square$

6.3. **Čech complex.** Suppose as above that $B = \operatorname{Spec} A$ is semi-local. We have the category $\square_0^n$ of non-empty subsets of $\{1,\dots,n\}$, with maps the inclusions. Let $\mathcal{U} = \{U_1,\dots,U_n\}$ be an open cover of a finite type B-scheme X; $\mathcal{U}$ determines a functor

$$\mathcal{U}_* : (\square_0^n)^{\mathrm{op}} \to \mathbf{Sch}_X$$

by sending $I \subset \{1,\dots,n\}$ to $\mathcal{U}_I := \cap_{i\in I} U_i$. We let $j_I : \mathcal{U}_I \to X$ be the inclusion.

Applying the functor $G_{(q)}(-,-)$ gives us the functor

$$\begin{aligned} G_{(q)}(\mathcal{U}_*,-) : \square_0^n &\to \mathbf{Sp}^\Delta \\ I &\mapsto G_{(q)}(\mathcal{U}_I,-). \end{aligned}$$

We set

$$G_{(q)}(\mathcal{U},-) := \operatorname*{holim}_{\square_0^n} G_{(q)}(\mathcal{U}_*,-);$$

the maps j_I^* give the natural map of simplicial spectra

$$j_{\mathcal{U}}^* : G_{(q)}(X,-) \to G_{(q)}(\mathcal{U},-).$$

It follows from the Mayer-Vietoris property of §6.1 that $j_{\mathcal{U}}^*$ is a weak equivalence. If X is a disjoint union of equi-dimensional schemes, we may make a similar construction with $K^{(q)}$ replacing $G_{(q)}$, giving us the simplicial spectrum $K^{(q)}(\mathcal{U},-)$ and the map of simplicial spectra

$$j_{\mathcal{U}}^* : K^{(q)}(X,-) \to K^{(q)}(\mathcal{U},-).$$

If X is regular, Lemma 3.3 shows that the natural map $K^{(q)}(-,-) \to G^{(q)}(-,-)$ is a weak equivalence, so the map $j_{\mathcal{U}}^*$ for the K-theory spectra is a weak equivalence as well.

6.4. **Presheaves.** Now suppose only that B is regular and has dimension at most one, and X is a finite type B-scheme.

We have the natural maps

$$\pi_n(G_{(q)}(X,-)) \xrightarrow{\alpha} \mathbb{H}^{-n}(B_{\mathrm{Zar}}, f_*\mathcal{G}_{(q)}(X,-)) \xrightarrow{\beta} \mathbb{H}^{-n}(X_{\mathrm{Zar}}, \mathcal{G}_{(q)}(X,-)) \tag{6.1}$$

In case X is regular, we have similarly defined maps

$$K_n^{(q)}(X,-) \xrightarrow{\alpha'} \mathbb{H}^{-n}(B_{\mathrm{Zar}}, f_*\mathcal{K}^{(q)}(X,-)) \xrightarrow{\beta'} \mathbb{H}^{-n}(X_{\mathrm{Zar}}, \mathcal{K}^{(q)}(X,-))$$

Proposition 6.5. *Let B be regular and of dimension at most one, and let $f : X \to B$ be a finite type B scheme. Then β is an isomorphism. If B is semi-local, then α is an isomorphism as well.*

If X is regular, then β' is an isomorphism, and if in addition B is semi-local, α' is an isomorphism.

Proof. The result for G-theory follows from Mayer-Vietoris property for $G_{(q)}(-,-)$ over a semi-local base, and Remark A.3. The K-theory result follows from this and the comparison isomorphism Lemma 3.3. □

Remark 6.6. Let $j : U \to X$ be the inclusion of an open subscheme, with X a finite-type B-scheme. Then the natural map

$$\mathbb{H}^n(X, j_*\mathcal{G}_{(q)}(U,-)) \to \mathbb{H}^n(U, \mathcal{G}_{(q)}(U,-))$$

is an isomorphism for all n. Indeed, let $\mathcal{G}_{(q)}(U,-) \to \mathcal{G}_{(q)}(U,-)^*$ be a stalk-wise weak equivalence of $\mathcal{G}_{(q)}(U,-)$ with a globally fibrant presheaf $\mathcal{G}_{(q)}(U,-)^*$ on U. Then $j_*\mathcal{G}_{(q)}(U,-)^*$ is globally fibrant on X and

$$\begin{aligned}\mathbb{H}^n(U, \mathcal{G}_{(q)}(U,-)) &= \pi_{-n}(\Gamma(U, \mathcal{G}_{(q)}(U,-)^*)) \\ &= \pi_{-n}(\Gamma(X, j_*\mathcal{G}_{(q)}(U,-)^*)) = \mathbb{H}^n(X, j_*\mathcal{G}_{(q)}(U,-)^*).\end{aligned}$$

On the other hand, if X is local, then we may take B to be local. Thus $\mathcal{G}_{(q)}(U,-)$ satisfies Mayer-Vietoris on U, hence the natural map

$$\pi_{-n}(\Gamma(X, j_*\mathcal{G}_{(q)}(U,-))) = \pi_{-n}(\Gamma(U, \mathcal{G}_{(q)}(U,-))) \to \mathbb{H}^n(U, \mathcal{G}_{(q)}(U,-))$$

is an isomorphism for all n. Thus the map $j_*\mathcal{G}_{(q)}(U,-) \to j_*\mathcal{G}_{(q)}(U,-)^*$ is a stalk-wise weak equivalence for general X, hence

$$\mathbb{H}^n(X, j_*\mathcal{G}_{(q)}(U,-)) = \mathbb{H}^n(X, j_*\mathcal{G}_{(q)}(U,-)^*) = \mathbb{H}^n(U, \mathcal{G}_{(q)}(U,-)),$$

as claimed.

Similarly, the natural maps

$$\mathbb{H}^n(X, j_*\mathcal{G}_{(q/r)}(U,-)) \to \mathbb{H}^n(U, \mathcal{G}_{(q/r)}(U,-));$$
$$\mathbb{H}^n(X, j_*\mathcal{Z}_q(U,-)) \to \mathbb{H}^n(U, \mathcal{Z}_q(U,-))$$

are isomorphisms.

6.7. **Compatibility of localization with the spectral sequences.**

Proposition 6.8. *Let B be a regular scheme of Krull dimension at most one, X a finite type B-scheme, $i : Z \to X$ a closed subscheme with complement $j : U \to X$. Then:*

(1) *The boundary map ∂ in the localization sequence*

$$\ldots \to H_p^{\mathrm{B.M.}}(X, \mathbb{Z}(q)) \xrightarrow{j^*} H_p^{\mathrm{B.M.}}(U, \mathbb{Z}(q)) \xrightarrow{\partial} H_{p-1}^{\mathrm{B.M.}}(Z, \mathbb{Z}(q)) \to \ldots$$

induces maps $\partial_r^{p,-2q} : E_r^{p,-2q}(U) \to E_r^{p-1,-2q}(Z)$ for all $r \geq 2$.

(2) *The diagram*

$$\begin{tikzcd}
E_r^{p,-2q}(U) \arrow[r, "\partial_r^{p,-2q}"] \arrow[d, "d_r^{p,-2q}"'] & E_r^{p-1,-2q}(Z) \arrow[d, "d_r^{p-1,-2q}"] \\
E_r^{p+r,-2q-r+1}(U) \arrow[r, "\partial_r^{p+r,-2q-r+1}"'] & E_r^{p+r-1,-2q-r+1}(Z)
\end{tikzcd}$$

commutes.

Both (1) and (2) hold for the mod n *sequences as well.*

Proof. We have the commutative diagram with the rows stalk-wise homotopy fiber sequences

$$\begin{tikzcd}
i_*\mathcal{G}_{(q/q-1)}(Z,-) \arrow[r, "i_*"] \arrow[d, "\mathrm{cl}_q(Z)"'] & \mathcal{G}_{(q/q-1)}(X,-)) \arrow[r, "j^*"] \arrow[d, "\mathrm{cl}_q(X)"'] & j_*\mathcal{G}_{(q/q-1)}(U,-) \arrow[d, "\mathrm{cl}_q(U)"] \\
i_*\mathcal{Z}_q(Z,-) \arrow[r, "i_*"'] & \mathcal{Z}_q(X,-)) \arrow[r, "j^*"'] & j_*\mathcal{Z}_q(U,-),
\end{tikzcd}$$

and the vertical maps stalk-wise weak equivalences. Using Remark 6.6, we have the isomorphism of long exact hypercohomology sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{H}^n(Z, \mathcal{G}_{(\frac{q}{q-1})}(Z,-)) \arrow[r, "i_*"] \arrow[d, "\mathrm{cl}_q(Z)"'] & \mathbb{H}^n(X, \mathcal{G}_{(\frac{q}{q-1})}(X,-)) \arrow[r, "j^*"] \arrow[d, "\mathrm{cl}_q(X)"'] & \mathbb{H}^n(U, \mathcal{G}_{(\frac{q}{q-1})}(U,-)) \arrow[r] \arrow[d, "\mathrm{cl}_q(U)"] & \cdots \\
\cdots \arrow[r] & \mathbb{H}^n(Z, \mathcal{Z}_q(Z,-)) \arrow[r, "i_*"'] & \mathbb{H}^n(X, \mathcal{Z}_q(X,-)) \arrow[r, "j^*"'] & \mathbb{H}^n(U, \mathcal{Z}_q(U,-)) \arrow[r] & \cdots.
\end{tikzcd}$$

In other words, the localization sequences for $\mathcal{Z}_q$ and $\mathcal{G}_{(q/q-1)}$ are isomorphic via cl_q.

Take integers $r \leq q$. We have the commutative diagram of presheaves on X

$$\begin{array}{ccccc} i_*\mathcal{G}_{(r)}(Z,-) & \xrightarrow{i_*} & \mathcal{G}_{(r)}(X,-) & \xrightarrow{j^*} & j_*\mathcal{G}_{(r)}(U,-) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ i_*\mathcal{G}_{(q)}(Z,-) & \xrightarrow[i_*]{} & \mathcal{G}_{(q)}(X,-) & \xrightarrow[j^*]{} & j_*\mathcal{G}_{(q)}(U,-). \end{array}$$

It follows from the localization result Theorem 2.6 that the rows above are stalk-wise homotopy fiber sequences. Thus, for $a \geq b \geq c$, the diagram

$$\begin{array}{ccccc} i_*\mathcal{G}_{(b/c)}(Z,-) & \xrightarrow{i_*} & \mathcal{G}_{(b/c)}(X,-) & \xrightarrow{j^*} & j_*\mathcal{G}_{(b/c)}(U,-) \\ \downarrow & & \downarrow & & \downarrow \\ i_*\mathcal{G}_{(a/c)}(Z,-) & \xrightarrow[i_*]{} & \mathcal{G}_{(a/c)}(X,-) & \xrightarrow[j^*]{} & j_*\mathcal{G}_{(a/c)}(U,-) \\ \downarrow & & \downarrow & & \downarrow \\ i_*\mathcal{G}_{(a/b)}(Z,-) & \xrightarrow[i_*]{} & \mathcal{G}_{(a/b)}(X,-) & \xrightarrow[j^*]{} & j_*\mathcal{G}_{(a/b)}(U,-) \end{array}$$

is commutative and each row and each column is a stalk-wise homotopy fiber sequence.

Using Remark 6.6 again, this gives us the commutative diagram

$$\begin{array}{ccccc} \mathbb{H}^n(U,\mathcal{G}_{(a/c)}(U,-)) & \xrightarrow{\partial_h} & \mathbb{H}^{n+1}(Z,\mathcal{G}_{(a/c)}(Z,-)) & \xrightarrow{i_*} & \mathbb{H}^{n+1}(X,\mathcal{G}_{(a/c)}(X,-)) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{H}^n(U,\mathcal{G}_{(a/b)}(U,-)) & \xrightarrow{\partial_h} & \mathbb{H}^{n+1}(Z,\mathcal{G}_{(a/b)}(Z,-)) & \xrightarrow{i_*} & \mathbb{H}^{n+1}(X,\mathcal{G}_{(a/b)}(X,-)) \\ {\scriptstyle\partial_v}\downarrow & & {\scriptstyle\partial_v}\downarrow & & {\scriptstyle\partial_v}\downarrow \\ \mathbb{H}^{n+1}(U,\mathcal{G}_{(b/c)}(U,-)) & \xrightarrow{\partial_h} & \mathbb{H}^{n+2}(Z,\mathcal{G}_{(b/c)}(Z,-)) & \xrightarrow{i_*} & \mathbb{H}^{n+2}(X,\mathcal{G}_{(b/c)}(X,-)) \end{array}$$

where ∂_h, ∂_v are the boundary maps coming from the rows (resp. columns) of the diagram above. From the construction of the spectral sequence of a tower given in Appendix D, this, together with the isomorphisms cl_q discussed above, suffices to prove the result.

□

7. Moving lemmas for cycles, G-theory and K-theory

The main moving lemmas for the cycle complexes $z^q(X,*)$, X a variety over a field k, are first found in the article [2]. Unfortunately, there are several gaps in the proofs, so we will recall these results, together with some of the missing details, as well as giving the required extensions to schemes over a one-dimension regular bases. At the same time, we will prove the analogous results for the simplicial spectrum $G^{(q)}(X,-)$ and the related K-theory simplicial spectrum $K^{(q)}(X,-)$ (see §3.2 below).

The technique we use is to apply the program of Bloch for the contravariant functoriality of the cycle complexes $z^q(X,*)$ to the simplicial spectra $G^{(q)}(X,-)$. One first proves an "easy" moving lemma, in case X admits a transitive group action by a group-scheme such as $\mathbb{A}^n$, and then one uses the classical technique of the projecting cone (see for instance [27] for a detailed treatment) to prove a moving lemma for smooth affine (or projective) X. The general case is handled using the Mayer-Vietoris properties discussed in Section 6. We will adapt the various geometric results proved in [19, Chap. II, §3.5] to this purpose.

7.1. **Formulation of the moving lemma.** Let X be B-scheme of finite type. An irreducible subset C of X has *pure codimension* d if $\dim X_i - \dim C = d$ for each irreducible component X_i of X containing C. Let $\mathcal{C}$ be a finite collection of irreducible locally closed subsets of X such that each $C \in \mathcal{C}$ has pure codimension in X. We let $X^{\mathcal{C}}_{(p,q)}$ be the subset of $X_{(p,q)}$ defined by the following condition: An element W of $X_{(p,q)}$ is in $X^{\mathcal{C}}_{(p,q)}$ if, for each $C \in \mathcal{C}$, each irreducible component W' of $W \cap (C \times \Delta^p)$ is in $C_{(q-d,p)}$, where $d = \mathrm{codim}_X C$.

Similarly, we let $X^{\mathcal{C}}_{(p,\leq q)}$ be the subset of $X_{(p,\leq q)}$ consisting of those W such that for each $C \in \mathcal{C}$, each irreducible component W' of $W \cap (C \times \Delta^p)$ is in $C_{(p,\leq q-d)}$, $d = \mathrm{codim}_X C$.

We let $z^{\mathcal{C}}_q(X,p)$ be the subgroup of $z_q(X,p)$ generated by $X^{\mathcal{C}}_{(p,q)}$, giving the simplicial subgroup $z^{\mathcal{C}}_q(X,-)$ of $z_q(X,-)$. Similarly, we have the spectrum $G^{\mathcal{C}}_{(q)}(X,p)$, defined as the limit of the spectra $G_W(X,p)$, as W runs over finite unions of elements of $X^{\mathcal{C}}_{(p,\leq q)}$. This gives us the simplicial spectrum $G^{\mathcal{C}}_{(q)}(X,-)$, with the natural map

$$G^{\mathcal{C}}_{(q)}(X,-) \to G_{(q)}(X,-) \tag{7.1}$$

If X is a union of equi-dimensional B-schemes, we may label using codimension rather than dimension, giving the subset $X^{(p,q)}_{\mathcal{C}}$ of $X^{(p,q)}$, and the subset $X^{(p,\geq q)}_{\mathcal{C}}$ of $X^{(p,\geq q)}$. In this case, we define the simplicial subgroup $z^q_{\mathcal{C}}(X,-)$ of $z^q(X,-)$, and the simplicial spectrum $G^{(q)}_{\mathcal{C}}(X,-)$ using $X^{(p,q)}_{\mathcal{C}}$ and $X^{(p,\geq q)}_{\mathcal{C}}$ instead of $X^{\mathcal{C}}_{(p,q)}$ and $X^{\mathcal{C}}_{(p,\leq q)}$. We similarly define the spectrum $K^{(q)}_{\mathcal{C}}(X,p)$ as the limit of the spectra $K^W(X,p)$, as W runs over finite unions of elements of $X^{(p,\geq q)}_{\mathcal{C}}$. The $K^{(q)}_{\mathcal{C}}(X,p)$ form a simplicial spectrum $K^{(q)}_{\mathcal{C}}(X,-)$, and we have the natural map of simplicial spectra

$$K^{(q)}_{\mathcal{C}}(X,-) \to K^{(q)}(X,-). \tag{7.2}$$

Conjecture 7.2. Suppose that X is a regular affine B-scheme of finite type. Then the map (7.2) is a weak equivalence for all q.

We can now state our main moving result:

Theorem 7.3. *Suppose that X is smooth and affine over B, and that each $C \in \mathcal{C}$ dominates an irreducible component of B. Then the maps* (7.1) *and* (7.2) *are a weak equivalences for all q. Similarly, the map*

$$z^q_{\mathcal{C}}(X,-) \to z^q(X,-).$$

is a weak equivalence for all q.

Before proceeding to the proof of Theorem 7.3, we describe the consequences for the spectral sequence (2.9).

8. Functoriality

As in [20, Remark 7.5], we may assume that the various pull-back operations on categories of coherent sheaves are strictly functorial, rather than just pseudo-functorial.

8.1. **Pull-back maps.** We note that the spectral sequence (2.9) is the spectral sequence (3.6), together with the identification of the E^1-term as $H_{p+q}(z^{-p}(X,*))$ via the weak equivalence (2.5), and reindexing. Thus, via Proposition 3.5, we have the identification of (2.9) with the spectral sequence (3.5), plus the appropriate reindexing, and the identification of the E^1-term. Finally, we have the natural isomorphisms

$$\mathbb{H}^n(B; f_*\mathcal{K}^{(q/r)}(X,-)) \to \mathbb{H}^n(X, \mathcal{K}^{(q/r)}(X,-)).$$

We have the homotopy category of presheaves of simplicial spectra on X, localized with respect to stalk-wise weak equivalences, $\mathbf{Hot}(X)$. In order to make the spectral sequence (2.9) functorial for maps $f: Y \to X$ of smooth B-schemes, it suffices to define functorial pull-back maps in $\mathbf{Hot}(X)$

$$f^*: \mathcal{K}^{(q/r)}(X,-) \to \mathcal{K}^{(q/r)}(Y,-),$$

compatible with the change of index maps $\mathcal{K}^{(a/b)}(-,-) \to \mathcal{K}^{(a'/b')}(-,-)$ and with the distinguished triangles

$$\mathcal{K}^{(b/c)}(-,-) \to \mathcal{K}^{(a/c)}(-,-) \to \mathcal{K}^{(a/b)}(-,-) \to \Sigma\mathcal{K}^{(b/c)}(-,-).$$

For each B-scheme X, let $\mathcal{C}(X)$ be the set of finite sets of irreducible locally closed subsets of X. We let $\mathcal{C}(X/B)$ be the subset of $\mathcal{C}(X)$ consisting of those $\mathcal{C}$ such that each $C \in \mathcal{C}$ is equi-dimensional over B.

Suppose first of all we have a morphism $f: Y \to X$ of finite type B-schemes, and let W be a closed subset of $X \times \Delta^p$. The map $f \times \mathrm{id}: Y \times \Delta^p \to X \times \Delta^p$ thus induces the map of spectra

$$(f \times \mathrm{id})^*: K^W(X,p) \to K^{f^{-1}(W)}(Y,p), \tag{8.1}$$

and this pull-back is functorial in f.

Let $f: Y \to X$ be a morphism of B-schemes, with both X and Y locally equi-dimensional schemes. Let $\mathcal{C}$ be in $\mathcal{C}(X)$, and let $f^*\mathcal{C}$ be the set of irreducible components of the subsets $f^{-1}(C)$, $C \in \mathcal{C}$. Suppose we have a stratification of X by closed subsets X_j,

$$X = X_0 \supset X_1 \supset \ldots \supset X_N \supset X_{N+1} = \emptyset,$$

such that
(8.2)

1. $X_j \setminus X_{j+1}$ is in $\mathcal{C}$ for each $j \geq 1$.
2. $f: f^{-1}(X_j \setminus X_{j+1}) \to X_j \setminus X_{j+1}$ is equi-dimensional for each $j \geq 0$.

It is easy to see that, for each $W \in X_{\mathcal{C}}^{(p,\geq q)}$, each irreducible component of $(f \times \mathrm{id})^{-1}(W)$ is in $Y_{f^*\mathcal{C}}^{(p,\geq q)}$. Thus, the pull-back maps (8.1) induce the map of simplicial spectra

$$f^*: K_{\mathcal{C}}^{(q)}(X,-) \to K_{f^*\mathcal{C}}^{(q)}(Y,-), \tag{8.3}$$

which are functorial when defined. In addition, if $\mathcal{C}' \supset \mathcal{C}$ also satisfies (8.2), then the diagram

$$
\begin{tikzcd}
K^{(q)}_{\mathcal{C}'}(X,-) \arrow[r, "f"] \arrow[d] & K^{(q)}_{f^*\mathcal{C}'}(Y,-) \arrow[d] \\
K^{(q)}_{\mathcal{C}}(X,-) \arrow[r, "f^*"'] & K^{(q)}_{f^*\mathcal{C}}(Y,-)
\end{tikzcd}
\tag{8.4}
$$

commutes, where the vertical maps are the evident ones.

Let $\mathcal{K}^{(q)}_{\mathcal{C}}(X,-)$ be the presheaf of simplicial spectra on X_{Zar}, $U \mapsto \mathcal{K}^{(q)}_{j_U^*\mathcal{C}}(U,-)$, where $j_U : U \to X$ is the inclusion. Since the maps (8.3) are functorial, they induce the map of presheaves of simplicial spectra on X_{Zar}

$$f^* : \mathcal{K}^{(q)}_{\mathcal{C}}(X,-) \to f_*\mathcal{K}^{(q)}_{f^*\mathcal{C}}(Y,-), \tag{8.5}$$

also functorial when defined. Taking the induced map on the homotopy cofiber defines the functorial pull-back

$$f^* : \mathcal{K}^{(q/r)}_{\mathcal{C}}(X,-) \to f_*\mathcal{K}^{(q/r)}_{f^*\mathcal{C}}(Y,-). \tag{8.6}$$

Since the maps (8.6) are induced from (8.5), the f^* define maps of distinguished triangles

$$
\begin{array}{c}
\mathcal{K}^{(b/c)}_{\mathcal{C}}(X,-) \to \mathcal{K}^{(b/c)}_{\mathcal{C}}(X,-) \to \mathcal{K}^{(b/c)}_{\mathcal{C}}(X,-) \to \Sigma\mathcal{K}^{(b/c)}_{\mathcal{C}}(X,-) \\
\downarrow f^* \\
f_*\mathcal{K}^{(b/c)}_{\mathcal{C}}(Y,-) \to f_*\mathcal{K}^{(b/c)}_{\mathcal{C}}(Y,-) \to f_*\mathcal{K}^{(b/c)}_{\mathcal{C}}(Y,-) \to \Sigma f_*\mathcal{K}^{(b/c)}_{\mathcal{C}}(Y,-)
\end{array}
$$

for $a \le b \le c$.

Suppose now that X is smooth over B, that $\mathcal{C}$ is in $\mathcal{C}(X/B)$ and that the conditions (8.2) hold. Define the map

$$f^* : \mathcal{K}^{(q)}(X,-) \to f_*\mathcal{K}^{(q)}(Y,-) \tag{8.7}$$

in $\mathbf{Hot}(X)$ to be the composition

$$\mathcal{K}^{(q)}(X,-) \xleftarrow{\sim} \mathcal{K}^{(q)}_{\mathcal{C}}(X,-) \xrightarrow{f^*} f_*\mathcal{K}^{(q)}_{f^*\mathcal{C}}(Y,-) \to f_*\mathcal{K}^{(q)}(Y,-).$$

Here the maps $\mathcal{K}^{(q)}_{\mathcal{C}}(X,-) \to \mathcal{K}^{(q)}(X,-)$ and $\mathcal{K}^{(q)}_{f^*\mathcal{C}}(Y,-) \to \mathcal{K}^{(q)}(Y,-)$ are the evident ones, and $\mathcal{K}^{(q)}_{\mathcal{C}}(X,-) \to \mathcal{K}^{(q)}(X,-)$ is an isomorphism in $\mathbf{Hot}(X)$ by Theorem 7.3. For $q \le r$, we define $f^* : \mathcal{K}^{(q/r)}(X,-) \to f_*\mathcal{K}^{(q/r)}(Y,-)$ similarly, using the maps (8.6) instead of (8.5).

Lemma 8.2. *Let X be a smooth finite type B-scheme. Let $f : Y \to X$ be a map of finite type B-schemes, with Y a locally equi-dimensional scheme. Let $\mathcal{C}$ be in $\mathcal{C}(X/B)$ such that the conditions* (8.2) *hold.*

1. *The maps $f^* : \mathcal{K}^{(*/*)}(X,-) \to f_*\mathcal{K}^{(*/*)}(Y,-)$ are compatible with the change-of-index maps $\mathcal{K}^{(q/r)}(-,-) \to \mathcal{K}^{(q'/r')}(-,-)$, and give maps of distinguished triangles*

$$
\begin{array}{c}
\mathcal{K}^{(b/c)}(X,-) \to \mathcal{K}^{(b/c)}(X,-) \to \mathcal{K}^{(b/c)}(X,-) \to \Sigma\mathcal{K}^{(b/c)}(X,-) \\
\downarrow f^* \\
f_*\mathcal{K}^{(b/c)}(Y,-) \to f_*\mathcal{K}^{(b/c)}(Y,-) \to f_*\mathcal{K}^{(b/c)}(Y,-) \to \Sigma f_*\mathcal{K}^{(b/c)}(Y,-)
\end{array}
$$

for $a \le b \le c$.

(2) *The map $f^*: \mathcal{K}^{(q/r)}(X,-) \to f_*\mathcal{K}^{(q/r)}(Y,-)$ is independent of the choice of $\mathcal{C} \in \mathcal{C}(X/B)$.*

(3) *Suppose that Y is smooth over B, and let $g: Z \to Y$ be a map of finite type B-schemes, with Z a locally equi-dimensional scheme. Suppose that $f^*\mathcal{C}$ satisfies the conditions (8.2) for the map g. Then $g^* \circ f^* = (f \circ g)^*$ as maps $\mathcal{K}^{(q/r)}(X,-) \to (f\circ g)_*\mathcal{K}^{(q/r)}(Z,-)$, or $\mathcal{K}^{(q)}(X,-) \to (f\circ g)_*\mathcal{K}^{(q)}(Z,-)$.*

Proof. Since the canonical maps $\mathcal{K}^{(*/*)}_{\mathcal{C}}(X,-) \to \mathcal{K}^{(*/*)}(X,-)$ and $\mathcal{K}^{(*/*)}_{f^*\mathcal{C}}(Y,-) \to \mathcal{K}^{(*/*)}(Y,-)$ are obviously compatible with the change-of-index maps, and define maps of distinguished triangles, (1) follows from the analogous properties of the maps $f^*: \mathcal{K}^{(*/*)}_{\mathcal{C}}(X,-) \to f_*\mathcal{K}^{(*/*)}_{f^*\mathcal{C}}(Y,-)$ described above. For (2), let $\mathcal{C}'$ be another element of $\mathcal{C}(X/B)$ satisfying (8.2). Then $\mathcal{C} \cup \mathcal{C}'$ also satisfies (8.2), so we may assume that $\mathcal{C}' \supset \mathcal{C}$. The commutativity of the diagrams

$$\begin{tikzcd}
\mathcal{K}^{(*/*)}_{\mathcal{C}'}(X,-) \arrow[r] \arrow[dr] & \mathcal{K}^{(*/*)}(X,-) \\
 & \mathcal{K}^{(*/*)}_{\mathcal{C}}(X,-) \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{K}^{(*/*)}_{f^*\mathcal{C}'}(Y,-) \arrow[r] \arrow[dr] & f_*\mathcal{K}^{(*/*)}(Y,-) \\
 & f_*\mathcal{K}^{(*/*)}_{f^*\mathcal{C}}(Y,-) \arrow[u]
\end{tikzcd}$$

together with the commutative of (8.4) prove (2).

For (3), we first note that the hypotheses imply that $\mathcal{C}$ satisfies (8.2) for the map $f \circ g$. The functoriality of the maps (8.6) readily implies (3). $\square$

In case B has dimension zero, the functoriality of (2.9) follows directly from Lemma 8.2 (see Remark 8.6 below). In case some component of B has dimension one, we will factor a morphism $f: Y \to X$ as the composition of the graph $(\mathrm{id}, f): Y \to Y \times_B X$ followed by the projection $Y \times_B X \to X$, and set $f^* = (\mathrm{id}, f)^* \circ p_2^*$. We therefore need some technical results to show that this leads to a well-defined, functorial pull-back.

Let $i: Y \to X$ be a closed embedding, with Y locally equi-dimensional over B, and X smooth over B. Let $\mathcal{C}$ be in $\mathcal{C}(Y/B)$. We let $i\mathcal{C}$ be the set of $i(C)$, $C \in \mathcal{C}_Y$ and $i_*\mathcal{C} := i\mathcal{C} \cup \{i(Y)\}$; by our assumption on Y, $i_*\mathcal{C}$ is in $\mathcal{C}(X/B)$. Thus, the pull-back maps

$$i^*: \mathcal{K}^{(q/r)}(X,-) \to i_*\mathcal{K}^{(q/r)}(Y,-); \quad i^*: \mathcal{K}^{(q)}(X,-) \to i_*\mathcal{K}^{(q)}(Y,-)$$

are defined. Similarly, if $f: Y \to X$ is a map of B-schemes, with X smooth over B, and Y locally equi-dimensional over X, then we may take $\mathcal{C} = \emptyset$, giving the pull-back maps

$$f^*: \mathcal{K}^{(q/r)}(X,-) \to f_*\mathcal{K}^{(q/r)}(Y,-); \quad f^*: \mathcal{K}^{(q)}(X,-) \to f_*\mathcal{K}^{(q)}(Y,-).$$

Lemma 8.3. *Let*

$$\begin{tikzcd}
Z \arrow[r, "i_1"] \arrow[d, "f_1"'] & Y \arrow[d, "f_2"] \\
W \arrow[r, "i_2"'] & X
\end{tikzcd}$$

be a commutative diagram of B-schemes, with i_1, i_2 closed embeddings, and f_1, f_2 are locally equi-dimensional. Suppose that X, Y and W are smooth over B. Then $i_1^* \circ f_2^* = f_1^* \circ i_2^*$ in $\mathbf{Hot}(X)$.

Proof. We give the proof for $\mathcal{K}^{(q)}$; the proof for $\mathcal{K}^{(q/r)}$ is exactly the same.

From our hypotheses, Z is locally equi-dimensional over B. We may assume that X, Y and W are irreducible. Let $\mathcal{C}^Y = \{i_1(Z)\}$ and $\mathcal{C}^X = \{i_2(W)\}$. Since f_2 is locally equi-dimensional, $i_1(Z)$ is locally equi-dimensional over $i_2(W)$. In particular, if A is in $X^{(p,q)}_{\mathcal{C}^X}$, then each irreducible component of $f_2^{-1}(A)$ is in $Y^{(p,q)}_{\mathcal{C}^Y}$. Thus, we have the natural map of presheaves of simplicial spectra $f_2^* : \mathcal{K}^{(q)}_{\mathcal{C}^X}(X, -) \to f_{2*}\mathcal{K}^{(q)}_{\mathcal{C}^Y}(Y, -)$, giving the commutative diagram of presheaves

$$\begin{tikzcd}
\mathcal{K}^{(q)}_{\mathcal{C}^X}(X,-) \arrow[r, "f_2^*"] \arrow[dr, "f_2^*"'] & f_{2*}\mathcal{K}^{(q)}(Y,-) \\
& f_{2*}\mathcal{K}^{(q)}_{\mathcal{C}^Y}(Y,-) \arrow[u, "\iota"']
\end{tikzcd}$$

where ι is the canonical map

Since f_1 and f_2 are locally equi-dimenisonal, we have the commutative diagram of presheaves of simplicial spectra

$$\begin{tikzcd}
\mathcal{K}^{(q)}_{\mathcal{C}^X}(X,-) \arrow[rr, "i_2^*"] \arrow[d, "f_2^*"'] \arrow[dr, "f_2^*"] & & i_{2*}\mathcal{K}^{(q)}(W,-) \arrow[d, "f_1^*"] \\
f_{2*}\mathcal{K}^{(q)}(Y,-) & f_{2*}\mathcal{K}^{(q)}_{\mathcal{C}^Y}(Y,-) \arrow[l, "\iota"] \arrow[r, "i_1^*"'] & g_*\mathcal{K}^{(q)}(Z,-),
\end{tikzcd}$$

where $g = f_2 \circ i_1 = i_2 \circ f_1$. This easily yields the identity $i_1^* \circ f_2^* = f_1^* \circ i_2^*$ in $\mathbf{Hot}(X)$. $\square$

Definition 8.4. Let $f : Y \to X$ be a map of smooth B-schemes. Factor f as the composition

$$Y \xrightarrow{(\mathrm{id},f)} Y \times_B X \xrightarrow{p_2} X,$$

and set

$$f^* := (\mathrm{id}, f)^* \circ p_2^*. \tag{8.8}$$

Lemma 8.5. *Let X, Y and Z be smooth B-schemes.*

1. *Suppose $f : Y \to X$ is an equi-dimensional morphism. Then the two definitions* (8.7) *and* (8.8) *of f^* agree.*
2. *Suppose $f : Y \to X$ is a closed embedding. Then the two definitions* (8.7) *and* (8.8) *of f^* agree.*
3. *Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be B-morphisms. Then $(f \circ g)^* = g^* \circ f^*$.*

Proof. (1) follows from Lemma 8.3 by factoring the composition

$$Y \xrightarrow{(\mathrm{id},f)} Y \times_B X \xrightarrow{p_2} X$$

as $\mathrm{id}_X \circ f$. (2) follows similarly using the factorization $f \circ \mathrm{id}_Y$.

The functoriality (3) in case f and g are both equi-dimensional or both closed embeddings follows from (1), (2) and the functoriality of f^* discussed in Lemma 8.2. In general, we have the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "(\mathrm{id},g)"'] \arrow[rr, bend left, "(\mathrm{id}_Z, f\circ g)"] & Z\times_B Y \arrow[r, "\mathrm{id}_Z\times(\mathrm{id}_Y,f)"'] \arrow[d, "p_2"'] & Z\times_B Y\times_B X \arrow[d, "p_{23}"'] \arrow[dd, bend left, "p_3"] \\ & Y \arrow[r, "(\mathrm{id}_Y,f)"'] & Y\times_B X \arrow[d, "p_2"'] \\ & & X \end{tikzcd}$$

Using this diagram, (3) follows from the functoriality for equi-dimensional maps, closed embeddings, and Lemma 8.3. □

Remark 8.6. Suppose that B has dimension zero. As mentioned above, the pull-back maps discussed in Lemma 8.2 suffice to define pull-backs $f^* : \mathcal{K}^{(q/r)}(X,-) \to f_*\mathcal{K}^{(q/r)}(Y,-)$ for X and Y finite type B-schemes, with X smooth over B, and Y a locally equi-dimensional scheme. Indeed, it is a classical result that, given a map $f : Y \to X$, there is a stratification of X by closed subsets X_j satisfying (8.2). We may then take $\mathcal{C} = \{X_j \setminus X_{j+1} \mid j = 1,\dots N\}$, which is in $\mathcal{C}(X/B)$ since $\dim B = 0$. Similarly, we have the functoriality $(f\circ g)^* = g^*\circ f^*$ for $g : Z \to Y$, in case Y is smooth and Z is a locally equi-dimensional scheme, of finite type over B.

Proposition 8.7. *Let*

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"'] & X \end{tikzcd}$$

be a cartesian square of finite type B-schemes, with X, Y, Z and W smooth over B, and with both f and f' of relative dimension d. Then $g^*f_* = g'_*f'^*$, as maps $f_*\mathcal{K}^{(p/q)}(Y,-) \to g_*\mathcal{K}^{(p-d/q-d)}(Z,-)$ in $\mathbf{Hot}(X)$.

Proof. The case in which g is flat has been verified in Lemma 3.3, so we may assume that g is a closed embedding.

Let $\mathcal{C}_X = \{X, g(Z)\}$ and let $\mathcal{C}_Y = \{Y, g'(W)\}$. The reader will easily verify the following:

(1) $\mathcal{C}_X$ satisfies the conditions (8.2) for the map g, and $\mathcal{C}_Y$ satisfies the conditions (8.2) for the map g'.
(2) Let T be in $Y^{(p,\geq s)}_{\mathcal{C}_Y}$ for some integer s. Then $(f\times\mathrm{id})(T)$ is in $X^{(p,\geq s-d)}_{\mathcal{C}_X}$.

Let $T \supset T'$ be a cosimplicial closed subsets of $Y\times\Delta^*$ such that T^n is in $Y^{(n,\geq p)}_{\mathcal{C}_Y}$ and T'^n is in $Y^{(n,\geq q)}_{\mathcal{C}_Y}$, for all n. Let $f(T) \subset X\times\Delta^*$ be the cosimplicial closed subset $f(T)^n := (f\times\mathrm{id})(T^n)$, and define $f(T')$, $g^{-1}(f(T))$, etc., similarly. We thus have the diagram in $\mathbf{Hot}(X)$

$$\begin{array}{ccc} (f\circ g')_*\mathcal{K}^{g'^{-1}(T)/g'^{-1}(T)}(W,-) & \xleftarrow{g'^*} & f_*\mathcal{K}^{T/T'}(Y,-) \\ \downarrow{\scriptstyle f'_*} & & \downarrow{\scriptstyle f_*} \\ g_*\mathcal{K}^{g^{-1}(T)/g^{-1}(T')}(Z,-) & \xleftarrow[g^*]{} & \mathcal{K}^{f(T)/f(T')}(X,-). \end{array}$$

Here $\mathcal{K}^{T/T'}(Y,-)$ is the cofiber of $\mathcal{K}^{T'}(Y,-) \to \mathcal{K}^{T}(Y,-)$, and the other presheaves of spectra are defined similarly. By Proposition B.17 (one easily verifies the necessary hypotheses) one has $g^*f_* = g'_*f'^*$ in $\mathbf{Hot}(X)$. Taking limits over T and T' completes the proof. □

9. Moving lemmas

This completes the construction of the functorial pull-back maps for the tower of presheaves of simplicial spectra $\mathcal{K}^{(q)}(X,-)$, and for the spectral sequence (2.9). We now turn to the proof of Theorem 7.3. In this section, the base scheme B is affine, $B = \operatorname{Spec} A$, with A a semi-local PID, and X is a finite-type B-scheme, such that each connected component of X is regular.

We will give the proofs for K-theory and G-theory, leaving the essentially notational simplifications necessary for proving the analogous results for the cycle complexes to the reader. We will however indicate the few places where the arguments for K-theory are noticably more difficult, with some indication of the appropriate simplications which suffice for the cycle-complexes.

We begin with some technical results on étale excision in K-theory; the reader only interested in functoriality for the cycle complexes can skip this section. Indeed, the cycle-theoretic analog of the results of this section is just Lemma 9.13(3).

9.1. **Excision in K-theory.** Let $f : Y \to Z$ be a finite map of regular schemes, W a closed subset of Y. Suppose that W is a connected component of $f^{-1}(f(W))$. We define the map

$$f^* : K^{f(W)}(Z) \to K^W(Y)$$

as follows: Let $j : U \to Y$ be an open neighborhood of W in Y such that $W = U \cap f^{-1}(f(W))$. Set f^* equal to the composition

$$K^{f(W)}(Z) \xrightarrow{f^*} K^{f^{-1}(f(W))}(Y) \xrightarrow{j^*} K^W(U) \xrightarrow{(j^*)^{-1}} K^W(Y),$$

the last j^* being the isomorphism $K^W(Y) \to K^W(U)$. This extends without essential modification to a map $f : Y \to Z$ of N-truncated cosimplicial schemes, with a cosimplicial closed subset W of Y such that the map on p-simplices $f_p : Y_p \to Z_p$ is finite, and W_p is a connected component of $f_p^{-1}(f_p(W_p))$ for each p.

Lemma 9.2. *Let $f : Y \to Z$ be a morphism of N-truncated cosimplicial schemes, and let $W, R \subset Y$ be cosimplicial closed subsets, such that $W_p \subset Y_p$ has pure codimension q for each p. Let W'_p be the closure of $f_p^{-1}(f_p(W_p)) \setminus W_p$. Suppose that, for each p,*

(i) *The map on p-simplices $f_p : Y_p \to Z_p$ is a finite surjective map of irreducible smooth B-schemes of finite type.*
(ii) *$f_{p|W_p} : W_p \to f_p(W_p)$ is birational.*
(iii) *R_p contains the ramification locus of $f_p : Y_p \to Z_p$.*

(iv) *For each $g:[p]\to[r]$ in Δ, the diagram*

$$\begin{tikzcd} Y_r \arrow[r, "f_q"] \arrow[d, "Y(g)"'] & Z_r \arrow[d, "Z(g)"] \\ Y_p \arrow[r, "f_p"'] & Z_p \end{tikzcd} \tag{9.1}$$

is cartesian.

Then

(1) *The W'_p form a cosimplicial closed subset W' of Y, with $f^{-1}(f(W)) = W\cup W'$.*

(2) *Let $W_R = W\cap R$. The composition*

$$K^W(Y) \xrightarrow{f^*\circ f_*} K^{W\cup W'}(Y) \to K^{W\setminus(W'\cup W_R)}(Y\setminus(W'\cup W_R)) \tag{9.2}$$

is equal to the restriction map $K^W(Y)\to K^{W\setminus(W'\cup W_R)}(Y\setminus(W'\cup W_R))$ (in the homotopy category).

Proof. The assertion (1) follows easily from the fact that the maps f_p are finite surjective, and that the diagram (9.1) is cartesian. Similarly, it follows that the image $f(R)$ is a cosimplicial closed subset of Z.

For (2), we first show that $f^{-1}(f(W_R))\cap W$ is contained in $(W'\cup W_R)\cap W$. For this, we may assume that $f:Y\to Z$ is a map of schemes. Take $x\in f^{-1}(f(R))\cap W$, and choose $y\in W_R$ with $f(y)=f(x)$. If x is in W_R, we are done, so we may assume that f is unramified in a neighborhood of x. The assertion is local over Z for the étale topology, so we may assume that $Y = Y_0\coprod Z$, with $f_0: Y_0\to Z$ a finite surjective map of regular schemes, $f = f_0\coprod \mathrm{id}$ and $x\in Z$. We write $W = W_0\coprod W_x$, with $x\in W_x\subset Z$, $y\in W_0\subset Y_0$, and similarly write $W' = W'_0\coprod W'_x$. By our assumptions (i) and (ii), and the assumption that W has pure codimension q on Y, it follows that W_x and $f_0(W_0)$ have no irreducible components in common. From this, we see that $W'_x = f_0(W_0)$, whence our claim.

We may factor the composition (9.2) through the restriction map

$$K^W(Y)\to K^{W\setminus f^{-1}(f(W_R))\cap W}(Y\setminus f^{-1}(f(W_R))\cap W).$$

From Quillen's localization theorem [24, §7, Proposition 3.1], we have the excision weak equivalences

$$K^{W\setminus f^{-1}(f(W_R))\cap W}(Y\setminus f^{-1}(f(W_R))\cap W)\sim K^{W\setminus f^{-1}(f(W_R))}(Y\setminus f^{-1}(f(W_R)))$$

$$K^{W\setminus(W'\cup W_R)}(Y\setminus(W'\cup W_R))\sim K^{W\setminus(W'\cup f^{-1}(f(W_R)))}\left(Y\setminus(W'\cup f^{-1}(f(W_R)))\right),$$

so we may remove $f(W_R)$ from Z and $f^{-1}(f(W_R))$ from Y, i.e., we may assume that $W_R=\emptyset$.

Let $W'' = W'\cap W$. Clearly W'' is a cosimplicial closed subset of Y. We claim that

(a) W'' contains no generic point of W.

(b) $f^{-1}(f(W''))\cap(W\setminus W'')=\emptyset$.

(c) The map $f: W\setminus W''\to f(W)\setminus f(W'')$ is an isomorphism.

To see this, we may assume that $f:Y\to Z$ is a map of schemes. We note that the assertions (a)-(c) are are local over Z for the étale topology. Thus we may assume that $Y = Y_0\coprod\coprod_{i=1}^m Z$, with $f_0: Y_0\to Z$ a finite surjective map of regular schemes,

$f = f_0 \coprod \coprod_{i=1}^m \mathrm{id}$, and, since $W_R = \emptyset$, we may assume that $W \cap Y_0 = \emptyset$. W is thus a disjoint union of closed subsets $W_i \subset Z$; since $f : W \to f(W)$ is birational, and W has pure codimension q on Y, the subsets W_i and W_j have no common irreducible components for $i \neq j$. If we write W' as a disjoint union, $W' = W'_0 \coprod \coprod_{i=1}^m W'_i$, then clearly $W'_i = \cup_{j \neq i} W_j$ for $i = 1, \ldots, m$, and similarly $W_i \setminus W''_i = W_i \setminus \cup_{j \neq i} W_j$. Since $f(W) = \cup_i W_i$, the assertions (a)-(c) are now obvious.

Let $W^* = f^{-1}(f(W''))$. Clearly W^* is contained in W'. We may factor the composition (9.2) through the map

$$K^{W \setminus W''}(Y \setminus W^*) \xrightarrow{f^* \circ f_*} K^{(W \cup W') \setminus W^*}(Y \setminus W^*) \to K^{W \setminus W'}(Y \setminus W').$$

Thus, we may remove $f(W'')$ from Z and W^* from Y, and assume that $W' \cap W = \emptyset$. It therefore suffices to show that the maps

$$f^* : K^{f(W)}(Z) \to K^W(Y),\ \ f_* : K^W(Y) \to K^{f(W)}(Z)$$

are inverse isomorphisms.

From (c), the map $f : W \to f(W)$ is an isomorphism. Since Y and Z are regular, we may replace K-theory with G-theory. We have the homotopy equivalences

$$G(W) \xrightarrow{i_{W*}} G_W(Y),\ G(f(W)) \xrightarrow{i_{f(W)*}} G_{f(W)}(Z).$$

Let $\mathcal{F}$ be a coherent sheaf on $f(W_p)$ for some p. Since f_p is étale, the natural map

$$f_p^*(i_{f(W_p)*}(\mathcal{F})) \to i_{W_p*}(f^*_{|W_p}(\mathcal{F}))$$

is an isomorphism on a (Zariski) neighborhood of W_p in Y_p. This natural isomorphism gives a homotopy between the maps $f^* \circ i_{f(W)*}$ and $i_{W*} \circ f^*_{|W}$ of $G(f(W))$ to $G_W(Y)$, i.e., the diagram

$$\begin{tikzcd} G(W) \arrow[r, "i_{W*}"] & G_W(Y) \\ G(f(W)) \arrow[u, "f^*_{|W}"'] \arrow[r, "i_{f(W)*}"'] & G_{f(W)}(Z) \arrow[u, "f^*"] \end{tikzcd}$$

is homotopy commutative. By functoriality of finite push-forward, the diagram

$$\begin{tikzcd} G(W) \arrow[r, "i_{W*}"] \arrow[d, "f_{|W*}"'] & G_W(Y) \arrow[d, "f_*"] \\ G(f(W)) \arrow[r, "i_{f(W)*}"'] & G_{f(W)}(Z) \end{tikzcd}$$

is homotopy commutative as well. Since $f_{|W}$ is an isomorphism, the maps $f_{|W*}$ and $f^*_{|W}$ are inverse isomorphisms, hence f_* and f^* are inverse isomorphisms. □

Lemma 9.3. *Let $f : Y \to Z$ be a morphism of N-truncated cosimplicial schemes, and let $W, W^*, R \subset Y$ be cosimplicial closed subsets with $W^* \supset f^{-1}(f(W^*))$. Suppose that, for each p:*

(1) *The map on p-simplices $f_p : Y_p \to Z_p$ is a finite surjective map of irreducible smooth B-schemes of finite type.*
(2) *$Y_p \setminus R_p \to Z_p$ is étale.*
(3) *$f_{p|W_p} : W_p \setminus W^*_p \to f_p(W_p) \setminus f_p(W^*_p)$ is birational.*
(4) *For each $g : [p] \to [q]$ in Δ, the diagram (9.1) is cartesian.*

Let W'_p be the closure of $f_p^{-1}(f_p(W_p)) \setminus W_p$, and let $W_R = W \cap R$. Let

$$\iota : K^{W\cup W^*}(Y)/K^{W^*}(Y) \to K^{W\cup W^*\cup W'}(Y)/K^{W^*}(Y),$$
$$\iota' : K^{W\cup W^*\cup W'}(Y)/K^{W^*}(Y) \to K^{W\cup W^*\cup W'}(Y)/K^{W^*\cup W'\cup W_R}(Y)$$

be the evident maps. Then the W'_p form a cosimplicial closed subset W' of Y, with $f^{-1}(f(W)) = W \cup W'$, and the composition

$$K^{W\cup W^*}(Y)/K^{W^*}(Y) \xrightarrow{\iota - f^*\circ f_*} K^{W\cup W^*\cup W'}(Y)/K^{W^*}(Y)$$
$$\xrightarrow{\iota'} K^{W\cup W^*\cup W'}(Y)/K^{W^*\cup W'\cup W_R}(Y)$$

is the zero map (in the homotopy category).

Proof. The first assertion follows from Lemma 9.2(1). By Quillen's localization theorem [24, §7, Proposition 3.1], the restriction maps

$$K^{W\cup W^*}(Y)/K^{W^*}(Y) \to K^{W\setminus W^*}(Y \setminus W^*),$$
$$K^{W\cup W^*\cup W'}(Y)/K^{W^*}(Y) \to K^{(W\cup W')\setminus W^*}(Y \setminus W^*)$$
$$K^{W\cup W^*\cup W'}(Y)/K^{W^*\cup W'\cup W_R}(Y) \to K^{W\setminus(W^*\cup W'\cup W_R)}(Y \setminus (W^* \cup W' \cup W_R))$$

are weak equivalences. Thus, we may remove $f(W^*)$ from Z and W^* from Y, i.e., it suffices to prove the result with $W^* = \emptyset$.

We must therefore show that the composition

$$K^W(Y) \xrightarrow{\iota - f^*\circ f_*} K^{W\cup W'}(Y) \xrightarrow{\iota'} K^{W\setminus(W'\cup W_R)}(Y \setminus (W' \cup W_R)) \tag{9.3}$$

is the zero map in the homotopy category. As $\iota' \circ \iota$ is just the restriction map $K^W(Y) \to K^{W\setminus W'}(Y \setminus (W' \cup W_R))$, this follows from Lemma 9.2. □

9.4. **Triangulations and homotopies.** We recall the standard construction of simplicial homotopies of maps of simplicial spaces.

We have the ordered sets $[p] := \{0 < \ldots, < p\}$. We give the product $[p] \times [q]$ the product partial order

$$(a, b) \leq (a', b') \Leftrightarrow a \leq a' \text{ and } b \leq b'.$$

Let $\Delta^{\leq N}$ denote the full subcategory of the category Δ, with objects the ordered sets $[p]$, $0 \leq p \leq N$. For a functor $F : \Delta^{\leq N\text{ op}} \to \mathcal{S}$ (an *N-truncated simplicial space*), we have the geometric realization $|F|_N \in \mathcal{S}$.

Let $F, G : \Delta^{\leq N+1\text{ op}} \to \mathcal{S}$ be functors, giving the geometric realizations $|F|_N$ and $|G|_N$. Suppose we have, for each order-preserving map $h : [p] \to [q] \times [1]$, $0 \leq q \leq N$, $0 \leq p \leq N + 1$, a morphism

$$H(h) : G([q]) \to F([p])$$

such that, for order-preserving $g : [q] \to [r]$, $f : [s] \to [p]$, we have

$$H(h \circ f) = F(f) \circ H(h),\ H((g \times \mathrm{id}) \circ h) = H(h) \circ G(g).$$

Then, restricting to those h with image in $[q] \times 0$ (resp. $[q] \times 1$), and with $p = q$, we have the maps of N-truncated simplicial spaces

$$H_0, H_1 : G \to F,$$

and the map in $\mathcal{S}$

$$H : |G|_N \times I \to |F|_N,$$

with

$$H_{||G|\times 0} = |H_0|,\ \ H_{||G|\times 1} = |H_1|.$$

Indeed, $|G|_N \times I$ is the geometric realization of the $N+1$-truncated simplicial space

$$(G\times I)([p]) = \coprod_{g:[p]\to[q]\times[1]} G([q])/\sim,$$

where $\sim$ is the equivalence relation $(G(g)(x), h) \sim (x, (g\times \mathrm{id})\circ h)$, with the evident map on morphisms. Our assertion follows directly from this.

9.5. **Moving by translation.** We begin by proving the "easy" moving lemma, which requires a transistive action by a connected linear algebraic group. We will only need the case of $\mathbb{A}^n$, with the action of translation.

Let $\mathcal{G}$ be a group scheme over B, and let X be a B-scheme with a (right) $\mathcal{G}$-action $\rho : X \times_B \mathcal{G} \to X$. Let $B' \to B$ be a B-scheme, $\psi : \mathbb{A}^1_{B'} \to \mathcal{G}$ a B-morphism with $\psi(0) = \mathrm{id}_{\mathcal{G}}$. We let $\phi : X_{B'} \times \Delta^1 \to X$ be the composition

$$X_{B'} \times \Delta^1 \xrightarrow{\sim} X \times_B \mathbb{A}^1_{B'} \xrightarrow{\mathrm{id}\times\psi} X \times_B \mathcal{G} \xrightarrow{\rho} X,$$

where the isomorphism $X_{B'} \times \Delta^1 \xrightarrow{\sim} X \times_B \mathbb{A}^1_{B'}$ is induced by the isomorphism $(t_0, t_1) \mapsto t_1$ of Δ^1 with $\mathbb{A}^1$.

We have the standard triangulation of $\Delta^p \times \Delta^q$, given as follows: Let v_i^p be the vertex $t_j = 0$, $j \neq i$ of Δ^p. Let $g = (g_1, g_2) : [n] \to [p] \times [q]$ be an order-preserving map. We have the affine linear map $T(g) : \Delta^n \to \Delta^p \times \Delta^q$ with $T(g)(v_i^{p+q}) = v^p_{g_1(i)} \times v^q_{g_2(i)}$.

Let $\pi : B' \to B$ be a B-scheme, $\psi : \mathbb{A}^1_{B'} \to G_{B'}$ a B'-morphism, and $g : [p] \to [1] \times [q]$ an order-preserving map. We denote the composition

$$\begin{aligned} X_{B'} \times \Delta^p \xrightarrow{\mathrm{id}\times T(g)} X_{B'} \times \Delta^1 \times \Delta^q \xrightarrow{(\phi,\mathrm{id})\times\mathrm{id}} X_{B'} \times \Delta^1 \times \Delta^q \\ \xrightarrow{\pi\times\mathrm{id}} X \times \Delta^1 \times \Delta^q \xrightarrow{p_{13}} X \times \Delta^q \end{aligned}$$

by $T(\phi, g)$. Similarly, we have the composition

$$X_{B'} \times \Delta^p \xrightarrow{\phi(1)\times\mathrm{id}} X_{B'} \times \Delta^p \xrightarrow{\pi\times\mathrm{id}} X \times \Delta^p$$

which we denote by $\phi(1, p)$.

Lemma 9.6. *Let N be a (generalized) integer $0 \le N \le \infty$. Let Y be a B-scheme of finite type, and let $W' \subset W$ be subsets of $\coprod_{p=0}^{N+1}(Y \times_B X)_{(p,q_p)}$, forming cosimplicial closed subsets of $Y \times_B X \times \Delta^{*\le N+1}$. Suppose that, for each $C \in W$, and each order-preserving $g : [p] \to [q] \times [1]$, each irreducible component of $[\mathrm{id} \times T(\phi, g)]^{-1}(C)$ is in W', for all $p, q \le N+1$. Then*

(1) *The maps $\mathrm{id} \times \phi(1, p)$ define the map of N-truncated simplicial spaces*
$$[\mathrm{id} \times \phi(1, -)]^* : G_W(Y \times_B X, -)_N \to G_{W'}(Y \times_B X_{B'}, -)_N.$$

(2) *The maps $T(\phi, g)$ give a homotopy of the compositions*

$$G_W(Y \times_B X, -)_N \xrightarrow{[\mathrm{id}\times\phi(1,-)]^*} G_{W'}(Y \times_B X_{B'}, -)_N \to G_W(Y \times_B X_{B'}, -)_N$$

$$G_{W'}(Y \times_B X, -)_N \to G_W(Y \times_B X, -)_N \xrightarrow{[\mathrm{id}\times\phi(1,-)]^*} G_{W'}(Y \times_B X_{B'}, -)_N$$

with the respective base-extensions

$$\pi^* : G_W(Y \times_B X, -)_N \to G_W(Y \times_B X_{B'}, -)_N,$$
$$\pi^* : G_{W'}(Y \times_B X, -)_N \to G_{W'}(Y \times_B X_{B'}, -)_N.$$

The same holds with the simplicial spaces $G_{W'}(-,-)$, $G_W(-,-)$ replaced with the simplicial abelian groups $z_{W'}(-,-)$, $z_W(-,-)$.

Proof. Let $f : |G_W(Y \times_B X, -)_N| \times I \to |G_{W'}(Y \times_B X_{B'}, -)_N|$ be the map constructed from the maps $\mathrm{id} \times T(\phi, g)$, following the homotopy construction of §9.4. One sees directly that the maps

$$|G_W(Y \times_B X, -)_N| \times 0 \xrightarrow{f} |G_{W'}(Y \times_B X_{B'}, -)_N|$$
$$|G_W(Y \times_B X, -)_N| \times 1 \xrightarrow{f} |G_{W'}(Y \times_B X_{B'}, -)_N|$$

are π^* and $[\mathrm{id}\times\phi(1,p)]^*$, which proves the lemma for the pair $G_{W'}(-,-)$, $G_W(-,-)$. The proof for $z_{W'}(-,-)$, $z_W(-,-)$ is the same. □

We will take $\mathcal{G}$ to be the additive group $\mathbb{A}^n$, and ψ to be the linear map ψ_x, $\psi_x(t) = t \cdot x$, where $x : B' \to \mathbb{A}^n$ is a B-morphism. We take $X = \mathbb{A}^n$, with $\mathcal{G}$ acting on X by translation.

Let $x_1, \ldots, x_n$ be independent variables. We let $A(x_1, \ldots, x_n)$ be the localization of the polynomial ring $A[x_1, \ldots, x_n]$ with respect to the multiplicatively closed set of $f = \sum_I a_I x^I$ such that the ideal in A generated by the coefficients a_I is the unit ideal. The inclusion $A \to A(x_1, \ldots, x_n)$ is a faithfully flat extension of semi-local PID's. We let $B(x) = \mathrm{Spec}\,(A(x_1, \ldots, x_n))$, and let $x : B(x) \to \mathbb{A}^n$ the morphism with $x^* : A[x_1, \ldots, x_n] \to A(x_1, \ldots, x_n)$ the evident inclusion.

Before we state the basic moving lemma for group actions, we introduce one more bit of notation.

Definition 9.7. Let X and Y be finite type B-schemes, with X equi-dimensional over B. Let $\mathcal{C}$ be a finite set of irreducible locally closed subsets of X, and let $e : \mathcal{C} \to \mathbb{N}$ be a function.

(1) We let $Y \times_B X^{\mathcal{C}}_{(p,q)}(e)$ be the subset of $(Y \times_B X)_{(p,q)}$ consisting of those W such that, for each $C \in \mathcal{C}$, each irreducible component of $(Y \times_B C \times \Delta^p) \cap W$ is in $(Y \times_B C)_{(p, \leq q - \mathrm{codim}_X C + e(C))}$. Replacing $(Y \times_B X)_{(p,q)}$ with $(Y \times_B X)_{(p,\leq q)}$ throughout defines the subset $Y \times_B X^{\mathcal{C}}_{(p,\leq q)}(e)$ of $(Y \times_B X)_{(p,\leq q)}$.

(2) Suppose that, in addition to $\mathcal{C}$ and e, we have for each non-generic point b of B a set of locally closed subsets $\mathcal{C}(b)$ of X_b and a function $e(b) : \mathcal{C}_b \to \mathbb{N}$. We set

$$Y \times_B X^{\mathcal{C}(*)}_{(p,q)}(e(*)) := Y \times_B X^{\mathcal{C}}_{(p,q)}(e) \cup \coprod_b Y_b \times_b X^{\mathcal{C}(b)}_{b\ (p,q)}(e(b))$$
$$Y \times_B X^{\mathcal{C}(*)}_{(p,\leq q)}(e(*)) := Y \times_B X^{\mathcal{C}}_{(p,\leq q)}(e) \cup \coprod_b Y_b \times_b X^{\mathcal{C}(b)}_{b\ (p,\leq q)}(e(b)).$$

Of course, $Y \times_B X^{\mathcal{C}}_{(p,q)}(e) = Y \times_B X^{\mathcal{C}(*)}_{(p,q)}(e(*))$ if we take the sets $\mathcal{C}_b$ to be empty, so the notation is somewhat redundant. We often label with codimension rather than dimension, giving for example the subset $X^{(p,q)}_{\mathcal{C}}(e)$ of $X^{(p,q)}$, with $X^{(p,q)}_{\mathcal{C}}(e) = X^{\mathcal{C}}_{(p,d-q)}(e)$ in case $d = \dim X$.

Given a function $e : \mathcal{C} \to \mathbb{N}$, we let $e - 1 : \mathcal{C} \to \mathbb{N}$ be the function $(e-1)(C) = \max(e(C) - 1, 0)$.

We let $G^{\mathcal{C}(*)}_{(q)}(Y \times_B X, -)(e(*))$ be the limit of the spectra $G_W(Y \times_B X, -)$, as W runs over finite unions of elements of $(Y \times_B X)^{\mathcal{C}(*)}_{(p,\leq q)}(e(*))$.

Remark 9.8. Define the simplicial abelian subgroup

$$z^{\mathcal{C}(*)}_q(Y \times_B X, -)(e(*)) \subset z_q(Y \times_B X, -)$$

similarly as for $G^{\mathcal{C}(*)}_{(q)}(Y \times_B X, -)(e(*))$, using $(Y \times_B X)^{\mathcal{C}(*)}_{(p,q)}(e(*))$ instead of $(Y \times_B X)^{\mathcal{C}(*)}_{(p,\leq q)}(e(*))$. Under the same hypotheses as Proposition 9.10, the proof of that proposition shows that

$$z^{\mathcal{C}(*)}_q(Y \times_B X, -)(e(*)) \to z_q(Y \times_B X, -)$$

is a weak equivalence. To make the proof work in this setting, one merely replaces $G_{(q)}$ with z_q and $(p, \leq q)$ with (p, q) throughout. If the reader wishes to work with cycle complexes rather than with the simplicial abelian groups, one need only use the complex associated to a simplicial abelian group throughout, and replace notions of homotopy with the analoguos one for homology.

Lemma 9.9. *Let $\mathcal{C}$ be a finite set of irreducible locally closed subsets of $X := \mathbb{A}^n$, such that each $C \in \mathcal{C}$ dominates B, let Y be a B-scheme of finite type, and let $e : \mathcal{C} \to \mathbb{N}$ be a function. Let $\mathcal{C}(b)$, for each nongeneric point b of B, be a finite set of irreducible locally closed subsets of X_b, and $e(b) : \mathcal{C}(b) \to \mathbb{N}$ a function. Let W be in $(Y \times_B X)^{\mathcal{C}(*)}_{(p,\leq q)}(e(*))$, and let $\psi := \psi_x : B(x) \to \mathbb{A}^n$. Then:*

1. *Each irreducible component of $[\mathrm{id} \times \phi(1, p)]^{-1}(W)$ is in $(B(x) \times_B Y \times_B X)^{\mathcal{C}(*)}_{(p,\leq q)}$,*
2. *For each order-preserving $g : [r] \to [p] \times [1]$, each irreducible component of $[\mathrm{id} \times T(\phi, g)]^{-1}(W)$ is in $(B(x) \times_B Y \times_B X)^{\mathcal{C}(*)}_{(r,\leq q)}(e(*))$.*

The analogous result holds for $W \in Y \times_B X^{\mathcal{C}(*)}_{(p,q)}(e(*))$.

Proof. We give the proof for $W \in Y \times_B X^{\mathcal{C}}_{(p,\leq q)}(e)$; the proof for $W \in Y \times_B X^{\mathcal{C}}_{(p,q)}(e)$ is similar.

In case A is a field, this follows directly from [2, Lemma 2.2]. In general, let η be the generic point of B. Since the inclusion $i : \eta \to B$ is flat, $i^{-1}(W)$ is in $(\eta \times_B Y \times_B X)_{(p,\leq q)}$. From the case of a field, it follows that (1) and (2) are valid with η replacing B, and $\mathcal{C}_\eta$ replacing $\mathcal{C}$, where $\mathcal{C}_\eta$ is the set of generic fibers C_η, $C \in \mathcal{C}$, where $e : \mathcal{C}_\eta \to \mathbb{N}$ is the map $e(C_\eta) = e(C)$.

For a non-generic point b of B, let $\mathcal{C}_b$ be the set of irreducible components of the fibers C_b for $C \in \mathcal{C}$. For $C \in \mathcal{C}_b$, let $e'(C)$ be the minimum of the numbers $e(\tilde{C})$, where $\tilde{C}$ is in $\mathcal{C}$, and C is an irreducible component of $\tilde{C}_b$. We let $\mathcal{C}^*_b = \mathcal{C}_b \cup \mathcal{C}(b)$, and let $e_b : \mathcal{C}^*_b \to \mathbb{N}$ be the function which is e' on $\mathcal{C}_b \setminus \mathcal{C}(b)$, $e(b)$ on $\mathcal{C}(b) \setminus \mathcal{C}_b$ and the minimum of e' and $e(b)$ on $\mathcal{C}_b \cap \mathcal{C}(b)$.

We note that $B(x) \to B$ a bijection on points. Since each $C \in \mathcal{C}$ is equidimensional over B, to complete the proof it suffices to show that, for each closed point $\iota : b \to B$ of B,

(a) Each irreducible component of $(\iota_x \times \mathrm{id})^{-1}([\mathrm{id} \times \phi(1, p)]^{-1}(W))$ is in $(b(x) \times_B Y \times_B X)^{\mathcal{C}^*_b}_{(p,\leq q)}$,

(b) For each order-preserving $g : [r] \to [p] \times [1]$, each irreducible component of $(\iota_x \times \mathrm{id})^{-1}([\mathrm{id} \times T(\phi, g)^{-1}](W))$ is in $(b(x) \times_B Y \times_B X)^{\mathcal{C}^*_b}_{(r,\leq q)}(e_b)$.

Here $\iota_x : b(x) \to B(x)$ is the inclusion induced by ι.

We proceed to verify (a) and (b). Since dimension can only go down with intersection, each irreducible component of $(\iota \times \mathrm{id})^{-1}(W)$ is in $(b \times_B Y \times_B X)^{\mathcal{C}_b^*}_{(p,\leq q)}(e_b)$, so (a) and (b) follows from the case of a field. □

Proposition 9.10. *[Moving by translation] Let $\mathcal{C}$ be a finite set of irreducible locally closed subsets of $X := \mathbb{A}^n$, such that each $C \in \mathcal{C}$ dominates B, and let $e : \mathcal{C} \to \mathbb{N}$ be a function. For each non-generic point b of B, let $\mathcal{C}(b)$ be a finite set of irreducible locally closed subsets of X_b, and let $e(b) : \mathcal{C}(b) \to \mathbb{N}$ be a function. Let Y be a B-scheme of finite type. Then, for each $q \geq 0$, the map*

$$G^{\mathcal{C}(*)}_{(q)}(Y \times_B X, -)(e(*)) \to G_{(q)}(Y \times_B X, -)$$

is a weak equivalences.

Proof. Let $[G_{(q)}/G^{\mathcal{C}(*)}_{(q)}(e(*))](Y \times_B X)$ be the cofiber of the map of simplicial spectra

$$G^{\mathcal{C}(*)}_{(q)}(Y \times_B X, -)(e(*)) \to G_{(q)}(Y \times_B X, -).$$

It suffices to show that $[G_{(q)}/G^{\mathcal{C}(*)}_{(q)}(e(*))](X \times_B Y)$ is weakly equivalent to a point.

We first note that the map

$$[G_{(q)}/G^{\mathcal{C}(*)}_{(q)}(e(*))](Y \times_B X) \to [G_{(q)}/G^{\mathcal{C}(*)}_{(q)}(e(*))](B(x) \times_B X \times_B Y) \tag{9.4}$$

is injective on homotopy groups. Indeed, the scheme $B(x)$ is a filtered inverse limit of open subschemes U of $\mathbb{A}^n_B$, with U faithfully flat and of finite type over B. Thus, since the G-theory spectra transform filtered inverse limits to filtered direct limits, it suffices to show that the map

$$[G_{(q)}/G^{\mathcal{C}(*)}_{(q)}(e(*))](Y \times_B X) \to [G_{(q)}/G^{\mathcal{C}(*)}_{(q)}(e(*))](U \times_B X \times_B Y) \tag{9.5}$$

is injective on homotopy groups for each such U. For such a U, there exist finite étale B-schemes $B_1 \to B$, $B_2 \to B$, of relatively prime degrees over B, and B-morphisms $B_i \to U$ (see [20, Lemma 6.1]). Since the simplicial spectra $G_{(q)}(Y \times_B X, -)$, $G^{\mathcal{C}(*)}_{(q)}(Y \times_B X, -)(e(*))$ are covariantly functorial for finite morphisms, and since that composition of pull-back and push-forward for a finite morphism $B' \to B$ of degree d is multiplication by d, it follows that the map (9.5) is injective.

Thus, we need only show that (9.4) is zero on homotopy groups. This follows directly from Lemma 9.6 and Lemma 9.9. □

9.11. **The projecting cone.** The method of moving by translation takes care of the case $X = \mathbb{A}^n$; for a general smooth affine B-scheme, we need to apply the classical method of the projecting cone.

Let $i : X \to \mathbb{A}^n$ be a closed subscheme of $\mathbb{A}^n$, giving the closure $\bar{X} \subset \mathbb{P}^n$. Let $\mathbb{P}^{n-1}_\infty = \mathbb{P}^n \setminus \mathbb{A}^n$. For each linear subspace $L \subset \mathbb{P}^{n-1}_\infty$, we have the corresponding linear projection

$$\pi_L : \mathbb{A}^n \to \mathbb{A}^m,$$

with m the codimension of L in $\mathbb{P}^{n-1}_\infty$. If $L \cap \bar{X} = \emptyset$, then the restriction of π_L to

$$\pi_{L,X} : X \to \mathbb{A}^m$$

is finite, necessarily dominant if X is equi-dimensional over B with $m = \dim_B X$. If this is the case, the maps

$$\pi_{L,X} \times \mathrm{id} : X \times \Delta^p \to \mathbb{A}^m \times \Delta^p$$

are also finite and dominant for each p.

For the remainder of this section, we assume that X is equi-dimensional over B. We let $\mathcal{U}_X$ be the open subscheme of the Grassmannian of codimension $\dim_B X$ linear subspaces L of $\mathbb{P}^{n-1}_\infty$ with $L \cap \bar{X} = \emptyset$, so $\pi_{L,X}$ is finite and dominant for all $L \in \mathcal{U}_X$; we will always take $L \in \mathcal{U}_X$ unless specific mention to the contrary is made.

Let W be a closed subset of $X \times \Delta^p$. We let $C_L(W) \subset \mathbb{A}^n \times \Delta^p$ be the closed subset $(\pi_L \times \mathrm{id})^{-1}(\pi_{L,X} \times \mathrm{id})(W)$.

Lemma 9.12. *Suppose that each irreducible component of W is in $X^{(p,q)}$. Then each irreducible component of $C_L(W)$ is in $(\mathbb{A}^n)^{(p,q)}_{\{X\}}$.*

Proof. Since $\pi_{L,X} : X \to \mathbb{A}^m$ is finite, it follows that each irreducible component of $(\pi_{L,X} \times \mathrm{id})(W)$ is in $(\mathbb{A}^m)^{(p,q)}$. Since $\pi_L : \mathbb{A}^n \to \mathbb{A}^m$ is equi-dimensional, each irreducible component of $C_L(W) = (\pi_L \times \mathrm{id})^{-1}((\pi_{L,X} \times \mathrm{id})(W))$ is in $(\mathbb{A}^n)^{(p,q)}$. Similarly, each irreducible component of $(\pi_{L,X} \times \mathrm{id})^{-1}((\pi_{L,X} \times \mathrm{id})(W))$ is in $X^{(p,q)}$; since

$$C_L(W) \cap (X \times \Delta^p) = (\pi_{L,X} \times \mathrm{id})^{-1}((\pi_{L,X} \times \mathrm{id})(W)),$$

it follows that $C_L(W)$ is in $(\mathbb{A}^n)^{(p,q)}_{\{X\}}$. □

After replacing $\mathcal{U}_X$ with a smaller open subscheme if necessary, we may assume that $\mathcal{U}_X$ is an open subscheme of some affine space $\mathbb{A}^r$ over B, faithfully flat over B. Let $x := (x_1, \ldots, x_r)$, giving us the semi-local PID $A(x)$, and the B-scheme $B(x) := \mathrm{Spec}\, A(x)$, with the canonical morphism $x : B(x) \to \mathcal{U}_X$. We let L_x be the corresponding linear subspace.

Lemma 9.13. *Let $i : X \to \mathbb{A}^n$ be a closed subscheme of $\mathbb{A}^n$, $\mathcal{C}$ a finite set of irreducible locally closed subsets of X such that each $C \in \mathcal{C}$ dominates B, and $e : \mathcal{C} \to \mathbb{N}$ a function. Suppose that X is smooth over B. Let W be a cosimplicial closed subset of $X \times \Delta^*_N$ such that each W_p is a finite union of elements of $X^{(p,q)}_{\mathcal{C}}(e)$, and let $L = L_x$. Let $f : X \times \Delta^* \to \mathbb{A}^m \times \Delta^*$ denote the map $\pi_{L,X} \times \mathrm{id}$, and let $R_L \subset X$ be the ramification locus of $\pi_{L,X}$. Then*

1. *Each irreducible component of $f_p^{-1}(f_p(W_p))$ different from the irreducible components of $W_{pB(x)}$ is in $X^{(p,q)}_{B(x)\mathcal{C}}(e-1)$.*
2. *Each irreducible component of $W_{pB(x)} \cap (R_L \times \Delta^p)$ is in $X^{(p,q+1)}_{B(x)\mathcal{C}}(e)$.*
3. *Suppose that $B = \mathrm{Spec}\, K$, K a field, and that X is absolutely irreducible over K. Let Z be an effective codimension q cycle on $X \times \Delta^p$ supported in W_p. Let $Z' = \pi^*_{L,X}(\pi_{L,X*}(Z)) - Z$. Then Z' is effective, and each irreducible component of Z' is in $X^{(p,q)}_{\mathcal{C}}(e-1)$.*

Proof. From Lemma 9.12, $C_L(W)$ is in $\mathbb{A}^n_{B(x)\{X\}}$, hence $(i \times \mathrm{id})^{-1}(C_L(W))$ is in $X^{(p,q)}_{B(x)}$.

If A is a field, the result is the classical Chow's moving lemma, as adapted in [19, Chap. II, §3.5]. The result for A a semi-local PID follows from the case of fields by the same argument as used in the proof of Lemma 9.9. □

We call a codimension q closed subset Z of $X \times \Delta^p$ *induced* if $p_2(Z) \subset Z'$ for some codimension p closed subset Z' of Δ^p. Note that, if $Z \subset X \times \Delta^p$ is induced, then $(\pi_{L,X} \times \mathrm{id})(Z)$ is an induced closed subset of $\mathbb{A}^m \times \Delta^p$.

Lemma 9.14. *Suppose that A is a field K. Let W be a proper closed subscheme of $X \times \Delta^p$ of pure codimension q, and let $Z \subset W$ be the union of the induced irreducible components of W. Let $L = L_x$. Then*

$$\pi_{L,X} \times \mathrm{id} : W_{B(x)} \setminus Z_{B(x)} \to (\pi_{L,X} \times \mathrm{id})(W_{B(x)}) \setminus (\pi_{L,X} \times \mathrm{id})(Z_{B(x)})$$

is birational.

Proof. If $A \subset X \times \Delta^p$ is a codimension q induced closed subset, so is each irreducible component of A, hence Z is the unique maximal element of the set of induced closed subsets of W which are the union of irreducible components of W. If $Z' \subset \mathbb{A}^m \times \Delta^p$ is induced, then clearly $(\pi_{L,X} \times \mathrm{id})^{-1}(Z')$ is induced, hence, if W' is an irreducible component of W, then $(\pi_{L,X} \times \mathrm{id})(W')$ is induced if and only if W' is induced. In particular, if W' is an irreducible component of W which is not induced, then $(\pi_{L,X} \times \mathrm{id})(W')$ is not contained in $(\pi_{L,X} \times \mathrm{id})(Z)$. Thus, we may assume that $Z = \emptyset$.

Let $W_0 \subset X$ be the closure of the image of W under the projection $X \times \Delta^p \to X$. We proceed by induction on the maximum d of the dimension of an irreducible component of W_0, starting with $W_0 = \emptyset$. Suppose we know our result for $d - 1$. Let $W' \subset W$ be the union of the irreducible components of W whose projections to X have dimension at most $d - 1$, and let W'' be the union of the remaining components.

Let $W_1 \neq W_2$ be irreducible components of W. Suppose $(\pi_{L,X} \times \mathrm{id})(W_1) \subset (\pi_{L,X} \times \mathrm{id})(W_2)$. Since the projection $\pi_{L,X}$ is finite, and each component of W has the same dimension, it follows that $(\pi_{L,X} \times \mathrm{id})(W_1) = (\pi_{L,X} \times \mathrm{id})(W_2)$.

Suppose a component W_1'' of W'' maps to an irreducible component of $(\pi_{L,X} \times \mathrm{id})(W')$. Then the projection of $(\pi_{L,X} \times \mathrm{id})(W_1'')$ on $\mathbb{A}^m$ is a subset of an irreducible component of

$$p_{\mathbb{A}^m}((\pi_{L,X} \times \mathrm{id})(W')) = (\pi_{L,X} \times \mathrm{id})(p_X(W')),$$

which is impossible, since $p_{\mathbb{A}^m}((\pi_{L,X} \times \mathrm{id})(W_1''))$ has dimension d, while the components of $(\pi_{L,X} \times \mathrm{id})(p_X(W'))$ have dimension $< d$.

Similarly, no component W_1' of W' can map via $\pi_{L,X} \times \mathrm{id}$ surjectively onto an irreducible component of $(\pi_{L,X} \times \mathrm{id})(W'')$. Thus, we may assume that $W = W''$.

We first consider the case $p = 0$. Let C be a pure dimension d proper closed subset of X, let $C_1, \ldots, C_r$ be the irreducible components of C, and let Z be the cycle $\sum_{i=1}^r C_i$. By the classical Chow's moving lemma (or Lemma 9.13(3) for $\mathcal{C} = \{C_1, \ldots, C_r\}$, $Z = \sum_i C_i$), the cycle $\pi_{L,X}^*(\pi_{L,X*}(Z)) - Z$ is effective, and has no component in common with C. In particular, each component of $\pi_{L,X*}(Z)$ has multiplicity one, hence the irreducible cycles $\pi_{L,X*}(C_i)$ each have multiplicity one, and are pair-wise distinct. Since the multiplicity of $\pi_{L,X*}(C_i)$ is the degree of the field extension $K(x)(C_i)/K(x)(\pi_{L,X}(C_i))$, this implies that the map $C \to \pi_{L,X}(C)$ is $K(x)$-birational, as desired.

The case $p = 0$ easily implies the general case, in case the image subset W_0 is a proper subset of X. Thus, we have only to consider the case in which each component of W dominates X. Fix a $\bar{K}$-point z. Suppose that, for all $\bar{K}$-points y in an open subset of X, the fiber $W_y \subset y \times \Delta^p = \Delta^p$ of W over y has an irreducible component Z in common with the fiber W_z. Then clearly $W_{\bar{K}}$ contains $Z \times \Delta^p$, from which it follows that W contains an induced component, namely, the union of the conjugates of $Z \times \Delta^p$ over K. Since we have assumed that this is not the case, it follows that, for all pairs (z, y) in an open subset U of $X \times X$, the fibers W_z and W_y have no components in common. In particular, let η be a geometric generic point of $\mathbb{A}^m$ over K, such that η remains a generic point over $K(x)$. If z and y are distinct points in $\pi_{L,X}^{-1}(\eta)$, then (z, y) is a $\overline{K}$-generic point of $X \times X$, hence W_z and W_y have no common components. Since the fiber of $(\pi_{L,X} \times \mathrm{id})(W)$ over η is the union of the fibers W_y, over $y \in \pi_{L,X}^{-1}(\eta)$, it follows that the map $W_{\pi_{L,X}^{-1}(\eta)} \to (\pi_{L,X} \times \mathrm{id})(W)_\eta$ is $\overline{K(x,\eta)}$-birational (using the reduced scheme structures). Thus $W_{K(x)} \to (\pi_{L,X} \times \mathrm{id})(W)$ is generically one to one. Since, by Lemma 9.13(2), $W_{K(x)}$ is generically étale over $(\pi_{L,X} \times \mathrm{id})(W)$, $W \to (\pi_{L,X} \times \mathrm{id})(W)$ is thus $K(x)$-birational, completing the proof. □

9.15. **Proof of Theorem 7.3.** We can now complete the proof of Theorem 7.3; we use the elegant method of Hanamura [12, §1]. We give the proof for the K-theory spectra; the proof for the simplicial abelian groups $z^q(X, -)$ is similar (but easier) and is left to the reader. Other than the obvious notational changes, the only other changes are to use Lemma 9.13(3) instead of Lemma 9.3 to prove the analog of Lemma 9.16, and to replace the use of Quillen's localization theorem for K-theory with our localization results for the cycle complexes found in Theorem 1.6.

It suffices to consider the case of irreducible X. We fix a closed imbedding $i : X \to \mathbb{A}^n$.

Fix an integer $N \geq 0$. We have the cofiber $K^{(q)}(X, -)_N / K_{\mathcal{C}}^{(q)}(X, -)_N$ of the map of N-truncated simplicial spectra $K_{\mathcal{C}}^{(q)}(X, -)_N \to K^{(q)}(X, -)_N$; it suffices to show that $\pi_m(K^{(q)}(X, -)_N / K_{\mathcal{C}}^{(q)}(X, -)_N) = 0$ for all $m < N$, and all $N \geq 0$.

Let $e \geq 0$ be an integer, which we consider as the constant map $e : \mathcal{C} \to \mathbb{N}$ with value e. Let $K_{\mathcal{C}}^{(q)}(X, p)(e)$ be the limit of the K-theory spectra $K_{\mathcal{C}}^{Z}(X, p)$, with Z a finite union of $W \in X_{\mathcal{C}}^{(p,q)}(e)$. This gives us the simplicial spectrum $K_{\mathcal{C}}^{(q)}(X, -)(e)$ and the N-truncated simplicial spectrum $K_{\mathcal{C}}^{(q)}(X, -)(e)_N$.

Since X is affine, each element of $X^{(p, \geq q)}$ is contained in an element of $X^{(p,q)}$. Thus, the K-theory spectrum $K^{(q)}(X, p)$ may be defined as the direct limit of the spectra $K^W(X, -)$, as W runs over finite unions of elements of $X^{(p,q)}$.

For $p \leq N$, each W in $X^{(p,q)}$ is in $X_{\mathcal{C}}^{(p,q)}(e)$, where e is the constant function with value $\dim_B X + N + 1$. Therefore

$$K^{(q)}(X, -)_N = K_{\mathcal{C}}^{(q)}(X, -)(e)_N.$$

Thus, it suffices to show that

$$\pi_m \left(K_{\mathcal{C}}^{(q)}(X, -)(e)_N / K_{\mathcal{C}}^{(q)}(X, -)(e-1)_N \right) = 0 \tag{9.6}$$

for all $m < N$, all constant e with $1 \leq e \leq \dim_B X + N + 1$, and all $N \geq 0$.

Let $X_{\mathcal{C}\mathrm{fin}}^{(p,q)}(e)$ be the subset of $X_{\mathcal{C}}^{(p,\leq q)}(e)$ consisting of those W which map to a non-generic point of B. Clearly

$$X_{\mathcal{C}\mathrm{fin}}^{(p,q)}(e) = \cup_b X_{b\mathcal{C}_b}^{(p,q-1)}(e), \tag{9.7}$$

where each union is over the non-generic points of B.

Let $K^{(q)}_{\mathcal{C}+}(X,-)(e-1)$ be the limit of the simplicial spectra $K^W(X,-)$, where W runs over finite unions of elements of $X^{(p,q)}_{\mathcal{C}\mathrm{fin}}(e)\cup X^{(p,q)}_{\mathcal{C}}(e-1)$. From (9.7), we see that

$$K^{(q)}_{\mathcal{C}+}(X,-)(e-1)=K^{(q)}_{\mathcal{C}(*)}(X,-)(\tilde{e}(*)),$$

where $\tilde{e}:\mathcal{C}\to\mathbb{N}$ is the constant function $e-1$, $\mathcal{C}(b)=\mathcal{C}_b$, and $\tilde{e}(b)\to\mathbb{N}$ is the constant function e.

Suppose we have shown, for all semi-local regular B, and all smooth affine $X\to B$, that

$$\pi_m\left(K^{(q)}_{\mathcal{C}}(X,-)(e)_N/K^{(q)}_{\mathcal{C}+}(X,-)(e-1)_N\right)=0 \tag{9.8}$$

for all $m<N$, all e with $1\le e\le \dim_B X+N+1$, and all $N\ge 0$. Since $X^{(p,q)}_{\mathcal{C}\mathrm{fin}}(e)$ is empty if $B=\operatorname{Spec} K$, K a field, this suffices to prove our main result in this case. In general, this reduces us to showing that

$$\pi_m\left(K^{(q)}_{\mathcal{C}+}(X,-)(e-1)_N/K^{(q)}_{\mathcal{C}}(X,-)(e-1)_N\right)=0$$

for all $m<N$, all e with $1\le e\le \dim_B X+N+1$, and all $N\ge 0$.

Let W be in $X^{(p,q)}_{\mathcal{C}}(e-1)$, and let b be a non-generic point of B. Since the elements of $\mathcal{C}$ are equi-dimensional over B, and since X_b is affine, each irreducible component of the fiber W_b is a subset of some element of $X^{(p,q-1)}_{b\mathcal{C}_b}(e-1)$. Using Quillen's localization theorem, (9.7), and this fact, we arrive at the weak equivalence

$$\begin{aligned}K^{(q)}_{\mathcal{C}+}&(X,-)(e-1)_N/K^{(q)}_{\mathcal{C}}(X,-)(e-1)_N\\ &\sim\prod_b K^{(q-1)}_{\mathcal{C}_b}(X_b,-)(e)_N/K^{(q-1)}_{\mathcal{C}_b}(X_b,-)(e-1)_N.\end{aligned}$$

Thus, in order to prove the identity (9.6) for all $m<N$, all e with $1\le e\le \dim_B X+N+1$, and all $N\ge 0$, it suffices to prove (9.8) (in the cases described above). We note that (9.8) depends only on the scheme X, not the choice of the base scheme B, so we may replace B with the integral closure of B in the function field of X. Thus, we may assume that the generic fiber of X over B is absolutely irreducible.

Extend e to the map $e: i\mathcal{C}\cup\{X\}\to\mathbb{N}$ by $e(iC)=e(C)$, $e(X)=0$. As in the proof of Proposition 9.10, the map

$$\begin{aligned}K^{(q)}_{\mathcal{C}}&(X,-)(e)_N/K^{(q)}_{\mathcal{C}+}(X,-)(e-1)_N\\ &\xrightarrow{\pi^*} K^{(q)}_{\mathcal{C}}(X_{B(x)},-)(e)_N/K^{(q)}_{\mathcal{C}+}(X_{B(x)},-)(e-1)_N\end{aligned}\tag{9.9}$$

is injective on homotopy groups.

By Lemma 9.12 and Lemma 9.13, we have the well-defined maps of N-truncated simplicial spectra

$$\begin{aligned}K^{(q)}_{\mathcal{C}}(X,-)(e)_N &\xrightarrow{\pi_L^*\circ\pi_{L,X*}} K^{(q)}_{i\mathcal{C}\cup\{X\}}(\mathbb{A}^n_{B(x)},-)(e)_N\\ &\xrightarrow{i^*} K^{(q)}_{\mathcal{C}}(X_{B(x)},-)(e)_N,\end{aligned}\tag{9.10}$$

and similarly with e replaced by $e-1$ and $\mathcal{C}$ replaced with $\mathcal{C}\mathrm{fin}$. The composition is just $\pi^*_{L,X}\circ\pi_{L,X*}$.

As above, we may write $K^{(q)}_{i\mathcal{C}+\cup\{X\}}(\mathbb{A}^n_{B(x)},-)(e-1)$ and $K^{(q)}_{i\mathcal{C}\cup\{X\}}(\mathbb{A}^n_{B(x)},-)(e)$ in the form $K^{(q)}_{\bar{\mathcal{C}}(*)}(\mathbb{A}^n_{B(x)},-)(\bar{e}(*))$, for suitable choices of sets $\bar{\mathcal{C}}$, $\bar{\mathcal{C}}(b)$ tand functions $\bar{e}$, $\bar{e}(b)$. By Proposition 9.10, the cofiber

$$K^{(q)}_{i\mathcal{C}\cup\{X\}}(\mathbb{A}^n_{B(x)},-)(e)/K^{(q)}_{i\mathcal{C}+\cup\{X\}}(\mathbb{A}^n_{B(x)},-)(e-1)$$

is weakly equivalent to a point, hence, using the factorization of $\pi^*_{L,X}\circ\pi_{L,X*}$ given in (9.10), we see that

$$\begin{aligned}(9.11)\quad & K^{(q)}_{\mathcal{C}}(X,-)(e)_N/K^{(q)}_{\mathcal{C}+}(X,-)(e-1)_N\\ &\xrightarrow{\pi^*_{L,X}\circ\pi_{L,X*}} K^{(q)}_{\mathcal{C}}(X_{B(x)},-)(e)_N/K^{(q)}_{\mathcal{C}+}(X_{B(x)},-)(e-1)_N\end{aligned}$$

induces the zero map on homotopy groups π_m, $m<N$.

Lemma 9.16. *The map*

$$\begin{aligned}(9.12)\quad & K^{(q)}_{\mathcal{C}}(X,-)(e)_N/K^{(q)}_{\mathcal{C}+}(X,-)(e-1)_N\\ &\xrightarrow{\pi^*-\pi^*_{L,X}\circ\pi_{L,X*}} K^{(q)}_{\mathcal{C}}(X_{B(x)},-)(e)_N/K^{(q)}_{\mathcal{C}+}(X_{B(x)},-)(e-1)_N\end{aligned}$$

induces the zero map on homotopy groups.

Proof. Let W be a finite union of elements of $X^{(p,q)}(e)$, for $0\le p\le N$, and let $W_{\rm fin}$ be the union of the components of W which map to the non-generic points of B. We assume that W and $W_{\rm fin}$ form a cosimplicial closed subsets of $(X\times\Delta^*)_N$. We write $f_p: X\times\Delta^p\to\mathbb{A}^m\times\Delta^p$ for the map $\pi_{L,X}\times{\rm id}$. Let W'_p be the union of the irreducible components of $f_p^{-1}(f_p(W_p))$ which are not in W_p; by Lemma 9.3, the W'_p form a cosimplicial closed subset W' of $X\times\Delta^*_N$. By Lemma 9.13, each irreducible component of W'_p is in $X^{(p,q)}_{B(x)\mathcal{C}}(e-1)$.

Let $R\subset X$ be the ramification locus of $\pi_{L,X}$, so $R\times\Delta^p$ is the ramification locus of $\pi_{L,X}\times{\rm id}$. We let W_R denote the cosimplicial closed subset $W\cap(R\times\Delta^*_N)$. By Lemma 9.13, each irreducible component of W_{Rp} is in $X^{(p,q+1)}_{B(x)\mathcal{C}}(e)$. Since X is affine, a sufficiently general collection of complete intersections containing W_R forms a cosimplicial closed subset W_1, such that each irreducible component of W_{1p} is in $X^{(p,q)}_{B(x)\mathcal{C}}(e-1)$. Let $W''\supset W_1\cup W'$ be a cosimplicial closed subset of $X\times\Delta^*_N$ such that each component of W'' is in $X^{(p,q)}_{B(x)\mathcal{C}{\rm fin}}(e-1)$.

For each p, let $Z_p\subset W_p$ be the maximal induced closed subset of W_p which is a union of irreducible components of W_p. Write each $Z_p=X\times Z'_p$. Let Z'' be the closure of the Z'_p with respect to the morphisms in $\Delta^{\le N}$; Z'' is then a cosimplicial closed subset of Δ^*_N containing all the Z'_p, and with Z''_p a union of elements of $B^{(p,q)}$ for all p. It is evident that $X\times Z''_p$ is a union of elements of $X^{(p,q)}_{\mathcal{C}}$ for all p.

Let W^* be a union of elements of $X^{(p,q)}_{\mathcal{C}{\rm fin}}(e-1)$ which forms a cosimplicial closed subset of $X\times\Delta^*_N$. We assume that $W^*_p\supset X\times Z''_p\cup W_{{\rm fin}p}$ for all p. Let W^{**} be a cosimplicial closed subset of $X_{B(x)}\times\Delta^*_N$ which is a union of element of $X^{(p,q)}_{\mathcal{C}}(e)$

By Lemma 9.14 and Lemma 9.3, the composition

$$\begin{aligned}
&K^{W\cup W^*}(X,-)_N/K^{W^*}(X,-)_N\\
&\qquad\xrightarrow{\pi^*-f^*\circ f_*} K^{W\cup W^*\cup W'}(X_{B(x)},-)_N/K^{W^*}(X_{B(x)},-)_N\\
&\qquad\qquad\to K^{W\cup W^*\cup W'}(X_{B(x)},-)_N/K^{W^*\cup W'\cup W_R}(X_{B(x)},-)_N
\end{aligned}$$

induces zero on homotopy groups. Thus, the map

$$\begin{aligned}
(9.13)\quad &K^{W\cup W^*}(X,-)_N/K^{W^*}(X,-)_N \xrightarrow{\pi^*-f^*\circ f_*}\\
&\qquad K^{W\cup W^*\cup W''\cup W^{**}}(X_{B(x)},-)_N/K^{W''\cup W^*}(X_{B(x)},-)_N.
\end{aligned}$$

induces zero on homotopy groups.

Taking limits over W, W'', W^* and W^{**} in (9.13) gives the map (9.12), from which it follows that the map (9.12) induces zero on homotopy groups. $\square$

proof of Theorem 7.3. Since both (9.11) and (9.12) induce zero on homotopy groups π_m, $m<N$, the map (9.9) induces zero on homotopy groups π_m, $m<N$. Since we have already seen that the map (9.9) is injective on homotopy groups, it follows that $\pi_m(K^{(q)}_{\mathcal{C}}(X,-)(e)_N/K^{(q)}_{\mathcal{C}+}(X,-)(e-1)_N)=0$ for $m<N$. $\square$

10. Equi-dimensional cycles

The results of §9, suitably modified, easily carry over to the setting of equi-dimension cycles. We state the main results, and describe the necessary modifications in the proofs.

10.1. **The simplicial spectrum $G_{(q)}(X/B,-)$.** Let $X\to B$ be a finite type B-scheme. From §2.7, we have the simplicial spectrum $G_{(q)}(X/B,-)$, and, if each connected component of X is equi-dimensional, the simplicial spectrum $G^{(q)}(X/B,-)$.

Let $\mathcal{C}$ be a finite set of irreducible locally closed subsets of X, such that each $C\in\mathcal{C}$ has pure codimension on X. We let $(X/B)^{\mathcal{C}}_{(p,\le q)}$ be the subset of $(X/B)_{(p,\le q)}$ consisting of those W such that, for each $b\in B$, each irreducible component of the fiber W_b is in $(X_b/b)^{\mathcal{C}_b}_{(p,\le q)}$. We define the subset $(X/B)^{\mathcal{C}}_{(p,q)}$ of $(X/B)_{(p,q)}$ similarly. Also for a B-scheme Y, we let $(Y\times_B X/B)^{\mathcal{C}}_{(p,q)}(e)$ be the subset of $(Y\times_B X/B)_{(p,q)}$ of those W such that each irreducible component of W_b is in $(Y\times_B X_b/b)^{\mathcal{C}_b}_{(p,q)}(e)$, for each $b\in B$.

Taking the limit of the $G_W(Y\times_B X,p)$ as W runs over finite unions of elements of $(Y\times_B X/B)^{\mathcal{C}}_{(p,\le q)}$ gives us the spectrum $G^{\mathcal{C}}_{(q)}(Y\times_B X/B,p)$ and the simplicial spectrum $G^{\mathcal{C}}_{(q)}(Y\times_B X/B,-)$. We similarly have the K-theory spectrum $K^{\mathcal{C}}_{(q)}(Y\times_B X/B,-)$ and the simplicial abelian group $z^{\mathcal{C}}_q(Y\times_B X/B,-)$.

We have the following analog of Theorem 7.3:

Theorem 10.2. *Let $X\to B$ be a smooth affine B-scheme of finite type, and let $\mathcal{C}$ be a finite set of irreducible locally closed subsets of X. Then the natural maps*

$$G^{\mathcal{C}}_{(q)}(X/B,-)\to G_{(q)}(X/B,-),\ K^{\mathcal{C}}_{(q)}(X/B,-)\to K_{(q)}(X/B,-),$$
$$z^{\mathcal{C}}_q(X/B,-)\to z_q(X/B,-)$$

are weak equivalences.

Note that we no longer have the condition that all $C\in\mathcal{C}$ dominate B.

proof of Theorem 10.2. The proof is similar to that of Theorem 7.3; we proceed to describe the necessary modifications.

We may assume that B is irreducible.

Since the definition of the sets $(Y \times_B X/B)^{\mathcal{C}}_{(p,q)}(e)$ is fiber-wise, the equi-dimensional analogs of Lemma 9.9, Lemma 9.12, and Lemma 9.13, with $\mathcal{C}$ an arbitrary finite collection of irreducible locally closed subsets of X, follow directly from these results in case $B = \operatorname{Spec} K$, K a field.

Choose for each non-generic point b of B, a finite set of irreducible locally closed subsets $\mathcal{C}(b)$ of the fiber X_b, and let $e(b) : \mathcal{C}(b) \to \mathbb{N}$ be a function. Let $(Y \times_B X/B)^{\mathcal{C}(*)}_{(p,q)}(e(*))$ be the union of $(Y \times_B X/B)^{\mathcal{C}}_{(p,q)}(e)$ with the sets $(Y \times_B X_b/b)^{\mathcal{C}(b)}_{(p,q)}(e(b))$, b a non-generic point of B, and let $G^{\mathcal{C}(*)}_{(q)}(Y \times_B X/B, -)(e(*))$ be the limit of the $G_W(Y \times_B X/B, -)$, as W runs over finite unions of elements of $(Y \times_B X/B)^{\mathcal{C}(*)}_{(p,q)}(e(*))$. Making the evident replacements, the proof of the equi-dimensional analog of Proposition 9.10 follows from the equi-dimensional analogs of Lemma 9.9 and Lemma 9.12, using the same argument as Proposition 9.10.

As in the proof of Theorem 7.3, it suffices to show that

$$\pi_m\left(K^{(q)}_{\mathcal{C}}(X/B, -)(e)_N / K^{(q)}_{\mathcal{C}}(X/B, -)(e-1)_N\right) = 0 \tag{10.1}$$

for all $m < N$, where $e : \mathcal{C} \to \mathbb{N}$ is the constant map with value e.

For each non-generic point b of B, let $\mathcal{C}(b) = \mathcal{C}_b$, and let $e(b) : \mathcal{C}(b) \to \mathbb{N}$ be the function $e(b)(C') = e + 1$.

Since X is affine, each $C' \in (X_b/b)^{(p,q)}_{\mathcal{C}(b)}(e(b) - 1)$ is an irreducible component of C_b, for some $C \in (X/B)^{(p,q)}_{\mathcal{C}(b)}(e)$, so we have the natural maps of simplicial spectra

$$K^{(q)}_{\mathcal{C}}(X/B, -)(e-1) \to K^{(q)}_{\mathcal{C}(*)}(X/B, -)(e(*) - 1) \to K^{(q)}_{\mathcal{C}}(X/B, -)(e).$$

By Quillen's localization theorem, the cofiber

$$K^{(q)}_{\mathcal{C}(*)}(X/B, -)(e(*) - 1)_N / K^{(q)}_{\mathcal{C}}(X/B, -)(e-1)_N$$

is weakly equivalent to the product over the non-generic points b of B of the cofibers

$$K^{(q)}_{\mathcal{C}(b)}(X_b, -)(e(b) - 1)_N / K^{(q)}_{\mathcal{C}(b)}(X_b, -)(e(b) - 2)_N.$$

This in turn has $\pi_m = 0$ for $m < N$ by (9.6) for $B = b$, so we need only show that

$$\pi_m(K^{(q)}_{\mathcal{C}}(X/B, -)(e)_N / K^{(q)}_{\mathcal{C}(*)}(X/B, -)(e(*) - 1)_N) = 0$$

for all $m < N$. This follows from the argument of Lemma 9.16 and the discussion after Lemma 9.16, completing the proof of Theorem 10.2. □

10.3. **Equi-dimensional functoriality.** The construction of pull-back maps for $G_{(q)}(X/B, -)$, for X affine and smooth over B, is now quite easy. Let $f : Y \to X$ be a morphism, with X affine and smooth over B, and Y of finite type over B. Suppose that Y is irreducible. As is well-known, there is a finite stratification of X by irreducible locally closed subsets C such that $f^{-1}(C) \to C$ is equi-dimensional. Let $\mathcal{C}_f$ be the set of the irreducible components of these C. If Y is not irreducible, let $\mathcal{C}_f$ be the union of the $\mathcal{C}_{f_i}$, where $f_i : Y_i \to X$ is the restriction of f to the irreducible component Y_i of Y. Then, for each $W \in (X/B)^{(p,q)}_{\mathcal{C}_f}$, each irreducible component of $(f \times \mathrm{id})^{-1}(W)$ is in $Y^{(p,q)}$. Thus, the maps

$$(f \times \mathrm{id})^* : K^{(q)}_{\mathcal{C}_f}(X/B, p) \to K^{(q)}(Y, p)$$

are well-defined, and give the map of simplicial spectra

$$f^* : K^{(q)}_{\mathcal{C}_f}(X/B, -) \to K^{(q)}(Y, -).$$

Using Theorem 10.2, we have the well-defined map

$$f^* : K^{(q)}(X/B, -) \to K^{(q)}(Y, -)$$

in the homotopy category.

Suppose that Y is equi-dimensional over B, and let Y_{fin} be the disjoint union of the closed fibers of Y. Let $Y' := Y \coprod Y_{\mathrm{fin}}$, and let $f' : Y' \to X$ be the induced map. If W is in $X^{(p,q)}_{\mathcal{C}_{f'}}$, then each irreducible component W' of $(f \times \mathrm{id})^{-1}(W)$ is in $Y^{(p,q)}$, and each irreducible component of $W' \cap (Y_b \times \Delta^p)$ is in $Y_b^{(p,q)}$, for each closed point b of B. Thus W' is in $(Y/B)^{(p,q)}$. Let $\mathcal{C}_{f/B} = \mathcal{C}_{f'}$, giving the map of simplicial spectra

$$f^* : K^{(q)}_{\mathcal{C}_{f/B}}(X/B, -) \to K^{(q)}(Y/B, -),$$

and the well-defined map

$$f^* : K^{(q)}(X/B, -) \to K^{(q)}(Y/B, -)$$

in the homotopy category.

Suppose that Y is affine and smooth over B, and that $g : Z \to Y$ is a B-morphism of finite type. Let Y' be the disjoint union of Y with the elements of $\mathcal{C}_g$, the closed fibers of Y and the closed fibers of each $C \in \mathcal{C}_g$, and let $f' : Y' \to X$ be the map induced by f. Let $\mathcal{C}_{f/B,g}$ denote the set $\mathcal{C}_{f'}$. Then, for each $W \in (X/B)^{(p,q)}_{\mathcal{C}_{f/B,g}}$, each irreducible component of $(f \times \mathrm{id})^{-1}(W)$ is in $(Y/B)^{(p,q)}_{\mathcal{C}_g}$. This gives us the identity

$$(g \circ f)^* = f^* \circ g^*$$

in the homotopy category, with

$$f^* : K^{(q)}(X/B, -) \to K^{(q)}(Y/B, -),\ g^* : K^{(q)}(Y/B, -) \to K^{(q)}(Z, -)$$

the maps defined above. If Z is smooth and affine over B, we make a similar construction, replacing $\mathcal{C}_g$ with $\mathcal{C}_{g/B}$, giving the set $\mathcal{C}_{f/B.g/B}$. This gives the functoriality for the maps

$$f^* : K^{(q)}(X/B, -) \to K^{(q)}(Y/B, -),\ g^* : K^{(q)}(Y/B, -) \to K^{(q)}(Z/B, -).$$

This makes the assignment $X \mapsto K^{(q)}(X/B, -)$ a contravariant functor from the category of smooth affine B-schemes to the homotopy category of simplicial spectra.

Remark 10.4. We have concentrated on the affine case, since, for the case of B a union of closed points, this suffices to give a general functoriality, using the Mayer-Vietoris property for the spectra $G_{(q)}(X, -)$. The analog of Theorem 7.3 and Theorem 10.2 for X smooth and *projective* over B is proved exactly the same way, replacing $\mathbb{A}^n$ with $\mathbb{P}^n$, and using the transitive action of the group of elementary matrices on $\mathbb{P}^n$ instead of the translation action of $\mathbb{A}^n$ on $\mathbb{A}^n$ to prove the analog of Proposition 9.10.

If Conjecture 2.8 were true, the Mayer-Vietoris property for the $G_{(q)}(X,-)$ would imply the same for the spectra $G_{(q)}(X/B,-)$ which would then lead to a full functoriality for X smooth over B, or even smooth over its image.

11. Products

Bloch introduced external products on the complexes $z^q(X,*)$ in [2, §5]; we use a variant of his technique to define external products for the complexes $z^q(X,*)$ and the simplicial spectra $G^{(q)}(X,-)$. For smooth X, pull-back by the diagonal gives us products for the $z^q(X,*)$ and the $G^{(q)}(X,-)$, which in turn give a product structure to the spectral sequence (2.9).

Actually, in the case of a base scheme of mixed characteristic, Conjecture 2.8 comes into play here, since the external products can only be defined for a product of a cycle with an *equi-dimensional* cycle. Even with this restriction, some use can be made of the product structure, even in mixed characteristic.

11.1. **The bi-simplicial spectra $G_{(q)}(X,-,-)$.** Let X be a finite type B-scheme. For each p and p', let $X_{(p,p',q)}$ be the set of irreducible closed subsets W of $X \times \Delta^p \times \Delta^{p'}$ such that, for each pair of faces $F \subset \Delta^p$, $F' \subset \Delta^{p'}$, each component W' of $W \cap (X \times F \times F')$ satisfies

$$\dim(W') = \dim_B F + \dim_B F' + q.$$

As noted in Remark B.14, we may extend the notations of §B.2 to bi-cosimplicial schemes. In particular, for $U \subset X \times \Delta^p \times \Delta^{p'}$, we let $\mathcal{M}_U(\partial)$ be the exact category of coherent sheaves on U with vanishing higher Tor's with respect to the subschemes $U \cap (X \times F \times F')$, $F \subset \Delta^p$ and $F' \subset \Delta^{p'}$ faces. For $U = X \times \Delta^p \times \Delta^{p'}$, we write $\mathcal{M}_X(p,p')$ for $\mathcal{M}_U(\partial)$. and let $G(X,p,p')$ be the K-theory spectrum $K(\mathcal{M}_X(p,p'))$. For $W \subset X \times \Delta^p \times \Delta^{p'}$, we define the spectrum with support $G_W(X,p,p')$ as the homotopy fiber of the restriction map

$$K(\mathcal{M}_X(p,p')) \to K(\mathcal{M}_U(\partial)),$$

where U is the complement of W.

Taking the limit of the $G_W(X,p,p')$, where W runs over finite unions of elements of $X_{(p,p',q)}$ defines the spectrum $G_{(q)}(X,p,p')$. These clearly form a bisimplicial spectrum $G_{(q)}(X,-,-)$.

11.2. **A moving lemma.** Recall the maps

$$T(g) : \Delta^n \to \Delta^p \times \Delta^q$$

defined in §9.4. We let $G^T(X,-,-)$ be the bi-simplicial spectrum defined as $G(X,-,-)$, where we use the categories of coherent sheaves on $X \times \Delta^p \times \Delta^q$ which are Tor-independent with respect to all the subschemes $X \times T(g)(F)$, F a face of Δ^n, in addition to the subschemes $X \times F \times F'$. We define the spectrum with support $G^T_W(X,p,p')$ similarly.

We let $X^T_{(p,p',q)}$ be the subset of $X_{(p,p',q)}$ consisting of those W which intersect $X \times T(g)(F)$ properly, for all faces F of Δ^n, and let $G^T_{(q)}(X,p,p')$ be the limit of the $G^T_W(X,p,p')$, as W runs over all finite unions of elements of $X^T_{(p,p',q)}$. This gives us the bi-simiplicial spectrum $G^T_{(q)}(X,-,-)$, and the natural map

$$\iota : G^T_{(q)}(X,-,-) \to G_{(q)}(X,-,-).$$

Lemma 11.3. *For each p, the map*

$$\iota : G^T_{(q)}(X, p, -) \to G_{(q)}(X, p, -).$$

is a weak equivalence.

Proof. Let W be a closed subset of $X \times \Delta^p \times \Delta^{p'}$. By Quillen's resolution theorem [24, §4, Corollary 1], the natural map

$$G^T_W(X, p, p') \to G_W(X, p, p')$$

is a weak equivalence (see the proof of [20, Lemma 7.6] for details). Thus, if we let $G^T_{(q)!}(X, -, -)$ be the spectrum gotten by taking the limit of the $G^T_W(X, p, p')$, as W runs over finite unions of elements of $X_{(p,p',q)}$, it suffices to show that the natural map

$$G^T_{(q)}(X, p, -) \to G^T_{(q)!}(X, p, -)$$

is a weak equivalence.

Fix an injective map $g : [n] \to [p] \times [p']$, let F be a face of Δ^p, and let F' be the projection of $T(g)(F)$ to $\Delta^{p'}$. Clearly F' is a face of $\Delta^{p'}$.

We identify Δ^p with $\mathbb{A}^p$ by using the barycentric coordinates $(t_1, \ldots, t_p)$, giving us the action of $\mathcal{G} := \mathbb{A}^p$ on Δ^p by translation. We let $\mathcal{G}$ act on $\Delta^p \times \Delta^{p'}$ via the action on Δ^p. Then $\mathcal{G} \cdot T(g)(F) = \Delta^p \times F'$, from which it follows that, if W is in $X_{(p,p',q)}$, and x is the generic point of $\mathcal{G}$, then $x \cdot T(g)(F)$ is in $(B(x) \times_B X)^T_{(p,p',q)}$.

The proof now follows by applying Lemma 9.6 and Lemma 9.9, as in the proof of Proposition 9.10. $\square$

For each $p \geq 0$, we have the natural map of simplicial spectra

$$\pi_p^* : G_{(q)}(X, -) \to G_{(q)}(X, p, -), \tag{11.1}$$

gotten by composing the identity $G_{(q)}(X, -) = G_{(q)}(X, 0, -)$ with the canonical degeneracy map $G_{(q)}(X, 0, -) \to G_{(q)}(X, p, -)$.

Lemma 11.4. *For each $p \geq 0$, the map* (11.1) *is a weak equivalence.*

Proof. Let $\mathcal{C}_p$ be the set of faces of Δ^p. The simplicial spectrum $G_{(q)}(X, p, -)$ is the same as the simplicial spectrum $G^{\mathcal{C}_p}_{(q)}(X \times \Delta^p, -)$, and the map (11.1) is just the pull-back via the projection $\pi : X \times \Delta^p \to Z$. We have the commutative diagram

$$\begin{tikzcd}
G^{\mathcal{C}_p}_{(q)}(X \times \Delta^p, -) \arrow[r, "\iota"] & G_{(q)}(X \times \Delta^p, -) \\
G_{(q)}(X, -) \arrow[u, "\pi_p^*"] \arrow[ur, "\pi^*"'] &
\end{tikzcd}$$

By Theorem 5.1, $\pi^* : G_{(q)}(X, -) \to G_{(q)}(X \times \Delta^p, -)$ is a weak equivalence, and by Proposition 9.10, the map $\iota : G^{\mathcal{C}_p}_{(q)}(X \times \Delta^p, -) \to G_{(q)}(X \times \Delta^p, -)$ is a weak equivalence. $\square$

For a bi-simplicial space (or spectrum) $T(-, -)$, let $T(- = -)$ denote the associated diagonal simplicial space (or spectrum) $p \mapsto T(p, p)$.

We consider $G_{(q)}(X, -)$ as a bi-simplicial spectrum, with (p, p')-simplices the spectrum $G_{(q)}(Z, p')$. The maps $\pi_p^* : G_{(q)}(X, -) \to G_{(q)}(X, p, -)$ gives the map of bi-simplicial spectra

$$\pi^* : G_{(q)}(X, -) \to G_{(q)}(X, -, -).$$

Similarly, we have the map of simplicial spectra

$$\pi^* : G_{(q)}(X, -) \to G_{(q)}(X, - = -).$$

For each p, let $\delta_p : \Delta^p \to \Delta^p \times \Delta^p$ be the diagonal embedding. Since $\delta_p = T(g)$ for $g : [p] \to [p] \times [p]$ the diagonal map, we have the natural pull-back map

$$\delta^* : G^T_{(q)}(X, - = -) \to G_{(q)}(X, -).$$

Proposition 11.5. *Let X be a B-scheme of finite type. Then for all $q \leq r$*

(1) *The maps*

$$\iota : G^T_{(q/r)}(X, -, -) \to G_{(q/r)}(X, -, -),$$
$$\iota : G^T_{(q/r)}(X, - = -) \to G_{(q/r)}(X, - = -)$$

are weak equivalences.

(2) *The map $\pi^* : G_{(q/r)}(X, -) \to G_{(q/r)}(X, -, -)$ is a weak equivalence.*
(3) *The map $\pi^* : G_{(q/r)}(X, -) \to G_{(q/r)}(X, - = -)$ is a weak equivalence.*
(4) *The map $\delta^* : G^T_{(q/r)}(X, - = -) \to G_{(q/r)}(X, -)$ is a weak equivalence.*

Proof. Let $T(-, -)$ be a bi-simplicial space. It is a standard result (see e.g. [21]) that the diagonal maps $[p] \to [p] \times [p]$ induce a homeomorphism $|T(- = -)| \to |T(-, -)|$, so, to prove (1), it suffices to consider the map of bi-simplicial spectra. The result in this case follows from Lemma 11.3 and the standard E^1 spectral sequence for a bi-simplicial space $T(-, -)$:

$$E^1_{p,q} = \pi_q(T(p, -)) \Longrightarrow \pi_{p+q}(T(-, -)).$$

The second assertion is proved similarly, using Lemma 11.4.

The assertion (3) follows from (2) and the homeomorphism $|T(- = -)| \cong |T(-, -)|$ mentioned above. For (4), the map $\pi^* : G_{(q/r)}(X, -) \to G_{(q/r)}(X, - = -)$ factors through ι by a similarly defined map $\pi^*_T : G_{(q/r)}(X, -) \to G^T_{(q/r)}(X, - = -)$. Since $\delta^* \circ \pi^*_T = \mathrm{id}$, (4) follows from (1) and (3). □

Remark 11.6. Suppose we have indices $q > r > s$, giving the distinguished triangle

$$G_{(r/s)}(X, -) \to G_{(q/s)}(X, -) \to G_{(q/r)}(X, -) \to \Sigma G_{(r/s)}(X, -).$$

It follows from Proposition 11.5 that

$$G_{(r/s)}(X, -, -) \to G_{(q/s)}(X, -, -) \to G_{(q/r)}(X, -, -) \to \Sigma G_{(r/s)}(X, -, -)$$
$$G^T_{(r/s)}(X, -, -) \to G^T_{(q/s)}(X, -, -) \to G^T_{(q/r)}(X, -, -) \to \Sigma G^T_{(r/s)}(X, -, -)$$
$$G_{(r/s)}(X, - = -) \to G_{(q/s)}(X, - = -) \to G_{(q/r)}(X, - = -) \to \Sigma G_{(r/s)}(X, - = -)$$

and

$$G^T_{(r/s)}(X, - = -) \to G^T_{(q/s)}(X, - = -) \to G^T_{(q/r)}(X, - = -) \to \Sigma G^T_{(r/s)}(X, - = -)$$

are distinguished triangles, and the maps of Proposition 11.5 give maps of distinguished triangles.

11.7. **External products for $G_{(q)}(X,-)$.** Let X and Y be B-schemes of finite type, with Y flat over B.

For W a closed subset of $X\times\Delta^p$, and W' a closed subset of $Y\times\Delta^{p'}$, we let $W\times^* W'$ denote the image of $W\times W'$ under the permutation isomorphism

$$(X\times\Delta^p)\times(Y\times\Delta^{p'})\to X\times Y\times\Delta^p\times\Delta^{p'}.$$

We note that, for each $C\in X_{(p,r)}$, $C'\in (Y/B)_{(p',s)}$, each irreducible component of $C\times^* C'$ is in $(X\times_B Y)_{(p,p',r+s)}$; this is an immediate consequence of the formula

$$\dim Z''=\dim Z+\dim_B Z'$$

if $Z\to B$, $Z'\to B$ are irreducible finite type B-schemes with Z' equi-dimensional over B, and Z'' is an irreducible component of $Z\times_B Z'$.

The products (C.9) thus give us the natural map of bi-simplicial spectra (in the homotopy category)

$$\boxtimes^{r,s}_{X,Y}: G_{(r)}(X,-)\wedge G_{(s)}(Y/B,-)\to G_{(r+s)}(X\times_B Y,-,-). \tag{11.2}$$

Taking the map on the associated diagonal simplicial spectra gives the map (in the homotopy category)

$$\boxtimes^{r,s}_{X,Y}: G_{(r)}(X,-)\wedge_\delta G_{(s)}(Y/B,-)\to G_{(r+s)}(X\times_B Y,-=-). \tag{11.3}$$

Here $T(-)\wedge_\delta S(-)$ is the simplicial space $p\mapsto T(p)\wedge S(p)$.

We now apply Proposition 11.5. We have the diagram of weak equivalences

$$G_{(r+s)}(X\times_B Y,-=-)\xleftarrow{\iota} G^T_{(r+s)}(X\times_B Y,-=-)\xrightarrow{\delta^*} G_{(r+s)}(X\times_B Y,-).$$

Composing the product (11.3) with $\delta^*\circ\iota^{-1}$ gives the natural external product map (in the homotopy category of simplicial spectra)

$$\cup^{r,s}_{X,Y/B}: G_{(r)}(X,-)\wedge_\delta G_{(s)}(Y/B,-)\to G_{(r+s)}(X\times_B Y,-). \tag{11.4}$$

Since each irreducible component of $C\times^* C'$ is in $(X\times_B Y)_{(p,p',r+s-t)}$ if C is in $X_{(p,r)}$ and C' is in $(Y/B)_{(p',s-t)}$, or if C is in $X_{(p,r-t)}$ and C' is in $(Y/B)_{(p',s)}$, the same construction gives the natural external product map (in the homotopy category of simplicial spectra)

(11.5)
$$\cup^{r/r-t,s/s-t}_{X,Y/B}: G_{(r/r-t)}(X,-)\wedge_\delta G_{(s/s-t)}(Y/B,-)\to G_{(r+s/r+s-t)}(X\times_B Y,-),$$

compatible with (11.4) via the evident maps.

Suppose that X and Y are flat over B, and admit ample families of line bundles. For $C\in (X/B)_{(p,r)}$ and $C'\in (Y/B)_{(p',s)}$, each irreducible component of $C\times^* C'$ is in $(X\times_B Y)_{(p,p',r+s)}$. We therefore have natural products (in the homotopy category of simplicial spectra)

$$\cup^{r,s}_{X/B,Y/B}: G_{(r)}(X/B,-)\wedge_\delta G_{(s)}(Y/B,-)\to G_{(r+s)}(X\times_B Y/B,-). \tag{11.6}$$

and
(11.7)
$$G_{(r/r-t)}(X/B,-)\wedge_\delta G_{(s/s-t)}(Y/B,-)\xrightarrow{\cup^{r/r-t,s/s-t}_{X/B,Y/B}} G_{(r+s/r+s-t)}(X\times_B Y/B,-),$$

compatible with the products (11.4) and (11.5).

Proposition 11.8. *Let X and Y be finite type B-schemes, with Y flat over B, and admitting an ample family of line bundles. Then the product* (11.4) *induces a product*

$$\cup_{X,Y/B} : \pi_a(G_{(r)}(X,-)) \otimes \pi_b(G_{(s)}(Y/B,-)) \to \pi_{a+b}(G_{(r+s)}(X \times_B Y,-)).$$

Similarly, the product (11.5) *induces a product*

$$\pi_a(G_{(r/r-t)}(X,-)) \otimes \pi_b(G_{(s/s-t)}(Y/B,-)) \xrightarrow{\cup_{X,Y/B}} \pi_{a+b}(G_{(r+s/r+s-t)}(X \times_B Y,-)).$$

In case X is flat over B, and admits an ample family of line bundles, the products (11.6) *and* (11.7) *induce products*

$$\cup_{X/B,Y/B} : \pi_a(G_{(r)}(X/B,-)) \otimes \pi_b(G_{(s)}(Y/B,-)) \to \pi_{a+b}(G_{(r+s)}(X \times_B Y/B,-)).$$

and

$$\pi_a(G_{(r/r-t)}(X/B,-)) \otimes \pi_b(G_{(s/s-t)}(Y/B,-)) \xrightarrow{\cup_{X/B,Y/B}} \pi_{a+b}(G_{(r+s/r+s-t)}(X \times_B Y/B,-)).$$

These products are graded-commutative and associative (in all cases where the triple product is defined). Finally, the products (11.5) *define a pairing of towers of spectra (cf. Appendix D)*

$$\cup_{X,Y/B} : |G_{(*)}(X,-)| \wedge |G_{(*)}(Y/B,-)| \to G_{(*)}(X \times_B Y,-)|.$$

Proof. We give the proof for the products on $G_{(r)}$; the proof for the products on $G_{(r/r-t)}$ is exactly the same.

Let $T(-)$ and $S(-)$ be pointed simplicial spaces. We have the natural homeomorphism

$$|T(-) \wedge_\delta S(-)| \cong |T(-)| \wedge |S(-)|.$$

Thus, the product (11.4) induces a map (in the homotopy category of spectra)

$$\cup^{r,s}_{X,Y/B} : |G_{(r)}(X,-)| \wedge |G_{(s)}(Y/B,-)| \to |G_{(r+s)}(X \times_B Y,-)|.$$

Taking the associated product on the homotopy groups gives the desired product.

Let Z be a B-scheme of finite type, and let

$$\tau_Z : G_{(q)}(Z,-=-) \to G_{(q)}(Z,-=-);\ \tau^T_Z : G^T_{(q)}(Z,-=-) \to G^T_{(q)}(Z,-=-)$$

be the maps induced by the exchange of factors $Z \times \Delta^p \times \Delta^p \to Z \times \Delta^p \times \Delta^p$. Since $\iota : G^T_{(q)}(Z,-=-) \to G_{(q)}(Z,-=-)$ intertwines τ_Z and τ^T_Z, and $\tau^T \circ \delta = \delta$, we have

$$\delta^* \circ \iota^{-1} \circ \tau_Z = \delta^* \circ \iota^{-1}. \tag{11.8}$$

Suppose that X is also flat over B, and let $t_{Y,X} : Y \times_B X \to X \times_B Y$ be the exchange of factors. It follows from (11.8) and the commutativity of the products (C.9) that the diagram

$$\begin{tikzcd}
{|G_{(r)}(X/B,-)| \wedge |G_{(s)}(Y/B,-)|} \arrow[r, "\tau"] \arrow[d, "\cup^{r,s}_{X/B,Y/B}"'] & {|G_{(s)}(Y/B,-)| \wedge |G_{(r)}(X/B,-)|} \arrow[d, "\cup^{s,r}_{Y/B,X/B}"] \\
{|G_{(r+s)}(X \times_B Y,-)|} \arrow[r, "t^*_{Y,X}"'] & {|G_{(r+s)}(Y \times_B X,-)|}
\end{tikzcd}$$

is homotopy commutative, which gives the graded-commutativity of the product $\cup_{X,Y}$.

The associativity of the products

$$\pi_a(G_{(r)}(X,-))\otimes\pi_b(G_{(s)}(Y/B,-))\otimes\pi_c(G_{(t)}(Z/B,-)) \\ \to\pi_{a+b+c}(G_{(r+s+t)}(X\times_B Y\times_B Z,-)),$$

$$\pi_a(G_{(r)}(X/B,-))\otimes\pi_b(G_{(s)}(Y/B,-))\otimes\pi_c(G_{(t)}(Z,-)) \\ \to\pi_{a+b+c}(G_{(r+s+t)}(X\times_B Y\times_B Z,-)),$$

and

$$\pi_a(G_{(r)}(X/B,-))\otimes\pi_b(G_{(s)}(Y/B,-))\otimes\pi_c(G_{(t)}(Z/B,-)) \\ \to\pi_{a+b+c}(G_{(r+s+t)}(X\times_B Y\times_B Z/B,-))$$

follows similarly from the homotopy associativity of the products (C.9).

Finally, we verify that the products the products (11.5) define a pairing of towers of spectra. First of all, the products respect the change of indices maps $G_{a/b}(X,-)\to G_{a'/b'}(X,-)$ since the products in the K-theory spectrum $K(Z)$ are natural in the scheme Z. The compatibility of the products (11.5) with the distinguished triangles formed from the layers in the tower follows from Remark 11.6 and Lemma C.4. □

Remark 11.9. One has a similar construction of cup products

$$G_{(r)}(X,-)\wedge_\delta K_{(s)}(Y/B,-)\to G_{(r+s)}(X\times_B Y,-),$$

etc., using the natural tensor products $\mathcal{M}_X\otimes\mathcal{P}_Y\to\mathcal{M}_{X\times_B Y}$. Similarly, we map replace G-theory with K-theory throughout, giving cup products

$$K_{(r)}(X,-)\wedge_\delta K_{(s)}(Y/B,-)\to K_{(r+s)}(X\times_B Y,-),$$

etc. These products are compatible with the G-theory products, with respect to the natural transformation $K\to G$; if X, Y and $X\times_B Y$ are all regular, $K\to G$ induces weak equivalences on all the relevant spectra, so we may use either G-theory or K-theory, as is convenient.

11.10. **Cup products.** Suppose that X is smooth over B. Let $\delta_X: X\to X\times_B X$ be the diagonal embedding. We have the cup products

$$\cup_X^{r,s}: G_{(r)}(X,-)\wedge_\delta G_{(s)}(X/B,-)\to G_{(r+s)}(X,-) \tag{11.9}$$

$$\cup_X^{r/r-t,s/s-t}: G_{(r/r-t)}(X,-)\wedge_\delta G_{(s/s-t)}(X/B,-)\to G_{(r+s/r+s-t)}(X,-) \tag{11.10}$$

defined by composing the products (11.4) and (11.5) with

$$\delta_X^*: G_{(r+s)}(X\times_B X,-)\to G_{(r+s)}(X,-),$$

using the functoriality proved in §8.

More generally, let $p: Y\to B$ be a B-scheme of finite type, such that each connected component Y' of Y is smooth over $p(Y')$. Since X is smooth over B, each connected component Z of $Y\times_B X$ is smooth over $p(p_1(Z))\subset B$, so the functoriality discussed in §8 applies to arbitrary B-morphisms $h: T\to Y\times_B X$.

Let $f : Y \to X$ be a B-morphism, giving us the closed embedding $\delta_f : Y \to Y \times_B X$. We define the products

$$\cup_f^{r,s} : G_{(r)}(Y,-) \wedge_\delta G_{(s)}(X/B,-) \to G_{(r+s)}(Y,-) \tag{11.11}$$

$$\cup_f^{r/r-t,s/s-t} : G_{(r/r-t)}(Y,-) \wedge_\delta G_{(s/s-t)}(X/B,-) \to G_{(r+s/r+s-t)}(Y,-) \tag{11.12}$$

by setting

$$\cup_f^{r,s} := \delta_f^* \circ \cup_{Y,X/B}^{r,s}, \quad \cup_f^{r/r-t,s/s-t} := \delta_f^* \circ \cup_{Y,X/B}^{r/r-t,s/s-t},$$

respectively.

If X is affine, we have the cup products

$$\cup_{X/B}^{r,s} : G_{(r)}(X/B,-) \wedge_\delta G_{(s)}(X/B,-) \to G_{(r+s)}(X/B,-) \tag{11.13}$$

(11.14)
$$\cup_{X/B}^{r/r-t,s/s-t} : G_{(r/r-t)}(X/B,-) \wedge_\delta G_{(s/s-t)}(X/B,-) \to G_{(r+s/r+s-t)}(X/B,-),$$

where we use the equi-dimensional analog of δ_X^* discussed in §10.3.

Proposition 11.11. *(1) Let $X \to B$ be a smooth affine B-scheme of finite type. The product* (11.13) *gives $\oplus\pi_a(G_{(b)}(X/B,-))$ the structure of a bi-graded ring, graded-commutative in a, and natural in X. Similarly, the product* (11.14) *gives $\oplus\pi_a(G_{(b/b-t)}(X/B,-))$ the natural structure of a bi-graded ring, graded-commutative in a, and natural in X.*

(2) Let $X \to B$ be a smooth affine B-scheme of finite type, $p : Y \to B$ a B-scheme of finite type, and $f : Y \to X$ a morphism. Suppose that each connected component Y' of Y is smooth over its image $p(Y')$ in B. Then the product (11.11) *gives $\oplus\pi_a(G_{(b)}(Y,-))$ the structure of a bi-graded $\oplus\pi_a(G_{(b)}(X/B,-))$-module, natural in the triple (X,Y,f). Similarly, the product* (11.12) *make $\oplus\pi_a(G_{(b/b-t)}(Y,-))$ a bi-graded $\oplus\pi_a(G_{(b/b-t)}(X/B,-))$-module, natural in (X,Y,f).*

(3) Suppose that each point of B is closed, and that X is a smooth B-scheme of finite type (not necessarily affine). Then

$$G_{(s)}(X/B,-) = G_{(s)}(X,-),\ G_{(s/s-t)}(X/B,-) = G_{(s/s-t)}(X,-),$$

and the products (11.9) *and* (11.10) *make $\oplus\pi_a(G_{(b)}(X,-))$ and $\oplus\pi_a(G_{(b/b-t)}(X,-))$ bi-graded rings, graded-commutative in a, and natural in X.*

(4) Under the hyptheses of (1), the products (11.13) *define a multiplicative structure on the tower $G_{(*)}(X/B,-)$. Under the hypotheses of (2), the products* (11.11) *give a pairing of towers $G_{(*)}(X/B,-) \wedge G_{(*)}(Y,-) \to G_{(*)}(Y,-)$ and under the hypotheses of (3), the products* (11.10) *define a multiplicative structure on the tower $G_{(*)}(X,-)$.*

Proof. The result follows from Proposition 11.8, together with the functoriality of pull-back, as discussed in §8 and §10.3. □

Suppose X is an equi-dimensional B-scheme of finite type. The equi-dimensional G-theory spectra $G_{(q)}(X/B-)$ form the tower

(11.15)
$$\ldots G_{(p-1)}(X/B,-) \to G_{(p)}(X/B,-) \to \ldots \to G_{(\dim_B X)}(X/B,-) \sim G(X),$$

giving the spectral sequence

$$E^1_{p,q} = \pi_{p+q}(G_{(p/p-1)}(X/B,-)) \Longrightarrow G_{p+q}(X). \tag{11.16}$$

We have the obvious natural map of the tower (11.15) to (2.1), giving the map of the spectral sequence (2.3) to the spectral sequence (11.16). In case each point of B is closed, the spectra $G_{(q)}(X/B,-)$ and $G_{(q)}(X,-)$ are weakly equivalent, so the spectral sequences (2.3) and (11.16) agree.

We collect our results in the following:

Theorem 11.12. *(1) Let X be a smooth affine B-scheme of finite type. Then the cup products* (11.10) *and* (11.14) *give a natural associative multiplicative structure to the spectral sequence* (11.16).

(2) Let X be a smooth affine B-scheme of finite type, let $p : Y \to B$ be a finite type B-scheme, and let $f : Y \to X$ be a B-morphism. Suppose each connected component Y' of Y is smooth over $p(Y')$. Then the cup products (11.11) *and* (11.12) *make the spectral sequence* (2.3) *(for Y) a natural module over the spectral sequence* (11.16) *(for X).*

(3) Suppose each point of B is closed, and let X be a smooth B-scheme of finite type (not necessarily affine). Then the spectral sequences (2.3) *and* (11.16) *agree, and the cup products* (11.9) *and* (11.10) *give a natural associative multiplicative structure to the spectral sequence* (2.3).

Proof. This all follows from Proposition 11.11 and Lemma D.6. □

Remarks 11.13. (1) If B is a general one-dimensional regular scheme (not necessarily semi-local), $p : X \to B$ a smooth B-scheme, the cup products (11.9)-(11.12) extend to define cup products on the sheaves $p_*G_{(q)}(X,-)$ described in §2 (one needs to assume that X is affine over B for the cup products on $p_*G_{(q)}(X/B,-)$ to be defined). The results of Theorem 11.12 extend to gives product structures for the spectral sequence (2.4), and the sheafified version of (11.16).

(2) The E_2 spectral sequence (2.8) and the (sheafified) E_2 reindexed analog of (11.16) have products as well; one merely reindexes to pass from the E^1-sequences to the E_2 sequences. Similarly, we get products for the E_2 spectral sequence (2.9) and the E_2 reindexed analog of sheafified version of (11.16). As in (1), we need to assume that X is affine over B in order to have products on the equi-dimensional spectral sequence.

11.14. **Products for cycle complexes.** The results of the proceeding section carry over directly to give external products for the simplicial abelian groups $z_q(X/B,-)$, cup products for $z^q(X/B,-)$ and a multiplication of $z^q(X/B,-)$ on $z^q(X,-)$. We give an explicit descroption of this product.

Let $z_{r+s}(X \times_B Y,-,-)$, $z^T_{r+s}(X \times_B Y,-,-)$ be the bisimplicial abelian groups with

$$z_{r+s}(X \times_B Y, p, p') = \mathbb{Z}[(X \times_B Y)_{(p,p',r+s)}],$$
$$z^T_{r+s}(X \times_B Y, p, p') = \mathbb{Z}[(X \times_B Y)^T_{(p,p',r+s)}].$$

The external product

$$\cup^{r,s}_{X.Y/B} : z_r(X,-) \wedge_\delta z_s(Y/B,-) \to z_{r+s}(X \times_B Y,-)$$

is given by the zig-zag diagram

$$z_r(X,-)\wedge_\delta z_s(Y/B,-)\xrightarrow{\boxtimes^{r,s}_{X,Y/B}} z_{r+s}(X\times_B Y,-=-)\xleftarrow{\iota} z^T_{r+s}(X\times_B Y,-=-)$$
$$\xrightarrow{\delta^*} z_{r+s}(X\times_B Y,-).$$

Here $z_{r+s}(X\times_B Y,-=-)$ and $z^T_{r+s}(X\times_B Y,-=-)$ are the associated diagonal simplicial abelian groups, ι is the inclusion, and δ^* is given by the pull-backs via the diagonal maps

$$\mathrm{id}\times\delta : X\times Y\times\Delta^p\to X\times Y\times\Delta^p\times\Delta^p.$$

The proof of Proposition 11.5 shows that the map ι is a weak equivalence.

Since the product in K_0 with supports is compatible with cycle intersection products via the cycle class map (this follows from Serre's intersection multiplicity formula), the maps

$$\mathrm{cl}_q : G_{(q/q-1)}(X,-)\to z_q(X,-);\quad \mathrm{cl}_q : G_{(q/q-1)}(Y/B,-)\to z_q(Y/B,-)$$

respect the products.

11.15. **Products for cycle complexes.** Using the Dold-Kan correspondence, we have products for the complexes $z_q(X,*)$, $z_q(Y/B,*)$ associated to the simplicial abelian groups $z_q(X,-)$, $z_q(Y/B,-)$; these two products thus give the same result on the homotopy/homology under the Dold-Kan isomorphism. We proceed to make the product on the complexes explicit.

Let $z_{r+s}(X\times_B,*=*)$ be the complex associated to the simplicial abelian group $z_{r+s}(X\times_B,-=-)$. Explicitly, the external product

$$\boxtimes^{r,s}_{X,Y/B} : z_r(X,*)\otimes z_s(Y/B,*)\to z_{r+s}(X\times_B,*=*)$$

is given by the formula

$$\boxtimes^{r,s}_{X,Y/B}(W\otimes W') := \sum_{g=(g_1,g_2)} \mathrm{sgn}(g)[\mathrm{id}\times T(g_1)\times T(g_2)]^*(W\times' W').$$

To explain the formula: The sum is over all injective order-preserving maps $g=(g_1,g_2) : [r+s]\to[r]\times[s]$, and $T(g_1):\Delta^{r+s}\to\Delta^r$, $T(g_2):\Delta^{r+s}\to\Delta^s$ are the maps defined in §9.4. To define the sign $\mathrm{sgn}(g)$, we will define a permutation $\sigma(g)$ of $\{1,\ldots,r+s\}$, and set $\mathrm{sgn}(g)=\mathrm{sgn}(\sigma(g))$. To define σ, we first identify $\{1,\ldots,r\}\coprod\{1,\ldots,s\}$ with $\{1,\ldots,r+s\}$ by sending $i\in\{1,\ldots,r\}$ to i and $j\in\{1,\ldots,s\}$ to $r+j$. Define a second bijection of $\{1,\ldots,r\}\coprod\{1,\ldots,s\}$ with $\{1,\ldots,r+s\}$ by sending $i\in\{1,\ldots,r+s\}$ to $g_1(i)\in\{1,\ldots,r\}$ if $g_1(i-1)<g_1(i)$, and to $g_2(i)\in\{1,\ldots,s\}$ if $g_2(i-1)<g_2(i)$. The composition of these two bijections gives the permutation $\sigma(g)$.

Via the Dold-Kan correspondence, the product

$$\cup^{r,s}_{X.Y/B} : z_r(X,*)\otimes z_s(Y/B,*)\to z_{r+s}(X\times_B Y,*) \tag{11.17}$$

is then given by the zig-zag diagram

$$z_r(X,*)\otimes z_s(Y/B,*)\xrightarrow{\boxtimes^{r,s}_{X,Y/B}} z_{r+s}(X\times_B Y,*=*)\xleftarrow{\iota} z^T_{r+s}(X\times_B Y,*=*)$$
$$\xrightarrow{\delta^*} z_{r+s}(X\times_B Y,*).$$

In particular, suppose we have $z = \sum_i n_i W_i \in z_r(X,p)$, $z' = \sum_j m_j W'_j \in z_s(Y/B,p')$ such that each $W_i \times' W'_j$ is in $(X \times_B Y)^T_{(p,p',r+s)}$. Then $\boxtimes^{r,s}_{X,Y/B}(z \otimes z')$ lands in the subcomplex $z^T_{r+s}(X \times_B Y, * = *)$, giving the formula:

$$z \cup_{X,Y/B} z' = \sum_g \mathrm{sgn}(g)(\mathrm{id} \times T(g))^*(p^*_{13}z \cdot p^*_{24}z') \tag{11.18}$$

with $T(g) : \Delta^{r+s} \to \Delta^r \times \Delta^s$ the map defined in §9.4.

The formula (11.18) shows that the cycle products (11.17) agree with the products defined in [2, §5] and [9, §8].

12. Lambda operations

We give a construction of natural λ-operations on the K-theory version of the tower (2.1), which endows the spectral sequence (2.3) with λ-operations in case X is regular. The idea of the construction is essentially to build up from case of an affine scheme $X = \mathrm{Spec}\, A$ to the general case using the Mayer-Vietoris property of K-theory, the case of $\mathrm{Spec}\, A$ being handled by identifying the identity component of $K(A)$ with $\mathrm{BGL}^+(A)$ and using Quillen's method to construct the lambda operations via representation theory; our approach follows the technique for the construction of a natural special λ-algebra structure for the relative K-groups with support $K^W_*(Y, D_1, \ldots, D_n)$ given in [18, Corollary 5.6].

We will first give the argument for the following model result:

Theorem 12.1. *Let $X : \Delta \to \mathbf{Sch}_B$ be a cosimplicial scheme such that each $X([p])$ is of finite type over B and has a B-ample family of line bundles, and let W be a cosimplicial closed subset of X. Then the graded group $\oplus_p K^W_p(X)$ has the structure of a special $K_0(B)$-λ-algebra, natural in W and X.*

Afterwards, we will indicate the modifications necessary to remove the condition that B be affine, and give an extension to cover the hyperhomotopy groups of the associated presheaf $f_*\mathcal{K}^W(X)$ on B, where $f : X \to B$ is the structure morphism. We begin with some preliminary material.

12.2. Let $\mathcal{S}^*$ be the category of pointed simplicial sets. As in §6.3, let $\square^n_0$ be the category of non-empty subsets of $\{1, \ldots, n\}$, with morphisms the inclusions, and let $0 < 1 > *$ be the category associated to the partially ordered set $0 < 1 > *$, i.e., there is a unique morphism $0 \to 1$, a unique morphism $* \to 1$, and no other non-identity morphisms.

12.3. Let $T : \mathbf{Ord}^{\mathrm{op}} \to \mathcal{S}$ be a simplicial space. We may restrict T to the subcategory $\mathbf{Ord}^{\mathrm{op}}_{\mathrm{inj}}$ of $\mathbf{Ord}$ having the same objects, but with the morphisms $[p] \to [q]$ being the injective order-preserving maps, giving the functor $T^{\mathrm{inj}} : \mathbf{Ord}^{\mathrm{op}}_{\mathrm{inj}} \to \mathcal{S}$. Similarly, we have the full subcategory $\mathbf{Ord}^{\le N}_{\mathrm{inj}}$ of $\mathbf{Ord}_{\mathrm{inj}}$ with objects $[p]$, $0 \le p \le N$, and the restriction $T^{\mathrm{inj}}_N : \mathbf{Ord}^{\le N\mathrm{op}}_{\mathrm{inj}} \to \mathcal{S}$ of T^{inj}. Define the geometric realization $|T^{\mathrm{inj}}|$ of T^{inj} as one defines the geometric realizations $|T|$, i.e., the quotient of $\coprod_p T_p \times \mathbf{Ord}_p$ mod the relations $(t, g(x)) \sim (g(t), x)$ for x in $\mathbf{Ord}_q$, t in T_p and $g : [q] \to [p]$ a map in $\mathbf{Ord}_{\mathrm{inj}}$. Restricting to $p, q \le N$ gives the geometric realization $|T^{\mathrm{inj}}_N|$ of T^{inj}_N.

We have the natural maps

$$|T^{\mathrm{inj}}| \to |T|; \quad |T^{\mathrm{inj}}_N| \to |T_N|$$

realizing $|T|$ as a quotient of $|T^{\mathrm{inj}}|$ and $|T_N^{\mathrm{inj}}|$ as a quotient of $|T_N|$.

Lemma 12.4. *The map $|T^{\mathrm{inj}}| \to |T|$ is a weak equivalence.*

Proof. This follows from [28, Proposition A.1], noting that a simplicial space (i.e., a bisimplicial set) is always *good* in the sense of [28]. □

By Lemma 12.4, we have the formula

$$\pi_n(|T|) = \varinjlim_N \pi_n(|T_N^{\mathrm{inj}}|). \tag{12.1}$$

The advantage of replacing **Ord** with $\mathbf{Ord}^{\leq N_{\mathrm{inj}}}$ is that $\mathbf{Ord}^{\leq N_{\mathrm{inj}}}$ is a *finite* category, i.e., the nerve of the category $\mathbf{Ord}^{\leq N_{\mathrm{inj}}}$ has only finitely many non-degenerate simplices.

12.5. **Affine covers.** Let $f : X \to B$ be a cosimplicial scheme of finite type over a noetherian scheme B, and let W be a cosimplicial closed subset of X. Let $j : U := X \setminus W \to X$ be the inclusion.

We note that, since each injective map $g : [q] \to [p]$ in **Ord** is split, the induced map $g : X([q]) \to X([p])$ is a closed embedding. Thus, if $\mathcal{U}$ is an affine open cover of $X([N])$ and $g : [q] \to [N]$ is injective, then the pull-back open cover $g^{-1}(\mathcal{U})$ is an affine open cover of $X([q])$. Let $\mathcal{U}([q])$ be the canonical refinement of the covers $g^{-1}(\mathcal{U})$, as g runs over all injective maps $[q] \to [N]$ in **Ord**, i.e., if $\mathcal{U} = \{U_1, \ldots, U_n\}$, and $g_1, \ldots, g_M$ are the injective maps $[q] \to [N]$, then the elements of $\mathcal{U}([q])$ are all open subsets of the form $\cap_{j=1}^M g_j^{-1}(U_{j_i})$. In particular, $\mathcal{U}([q])$ is an affine open cover of $X([q])$ for each $q \leq N$, and is functorial in $[q]$ over $\mathbf{Ord}^{\leq N_{\mathrm{inj}}}$.

Similarly, taking an affine refinement $\mathcal{V}$ of the induced open cover $j_N^{-1}(\mathcal{U})$ of $U([N])$, and performing the same construction, we have the affine open cover $\mathcal{V}([q])$ of $U([q])$, functorial in $[q]$, and with a natural refinement $\rho_q : \mathcal{V}([q]) \to j_q^{-1}(\mathcal{U}([q]))$. Using the degeneracy maps in **Ord**, we may replace $\mathcal{U}$ with a finer affine cover, and assume that the set of elements in $g^{-1}(\mathcal{U})$, for $g : [p] \to [N]$ injective, is independent of the choice of g. Similarly for $\mathcal{V}$. Thus (allowing some of the open sets to be repeated or empty if necessary) $\mathcal{U}([N])$ and $\mathcal{U}([p])$ have the same number of elements for each $p \leq N$, and similarly for $\mathcal{V}([N])$ and $\mathcal{V}([p])$.

Suppose that $\mathcal{U}([N])$ has n elements $U_1, \ldots, U_n$, and $\mathcal{V}([N])$ has m elements $V_1, \ldots, V_m$. By repeating elements of $\mathcal{U}([N])$ and reordering, we may assume that $n = m$ and that the refinement ρ_N includes V_i into $j_N^{-1}(U_i)$. Similarly, choose for each p an ordering of the $n = m$ elements $U_1^p, \ldots, U_n^p$ of $\mathcal{U}([p])$, and use an ordering on the n elements $V_1^p, \ldots, V_n^p$ of $\mathcal{V}([p])$ so that ρ_p is the inclusion of V_i^p into $j_p^{-1}(U_i^p)$.

Since the category $\mathbf{Ord}^{\leq N}_{\mathrm{inj}}$ is finite, one easily shows that we may further refine $\mathcal{U}$ and $\mathcal{V}$ so that the assignments

$$p \mapsto \mathcal{U}([p]); \quad p \mapsto \mathcal{V}([p])$$

extends to a functor from $\Delta^{\leq N}_{\mathrm{inj}}$ to open covers, i.e., for each injective $g : [p] \to [q]$, we have a choice of a permutation σ_g of $\{1, \ldots, n\}$ such that $U_i^p \subset X(g)^{-1}(U^q_{\sigma_g(i)})$ and that $\sigma_{gh} = \sigma_g \sigma_h$, and similarly for $\mathcal{V}$ (with the same σ_g).

12.6. **Functorial covers.** Define the category $\mathbf{Ord}_{\rm inj}^{\le N}\rtimes(\square_0^n)^{\rm op}$ to have objects the set of pairs $([q], I)$, with $q \le N$ and with $I \subset \{1, \ldots, n\}$, $I \neq \emptyset$, where a morphism $([q], I) \to ([p], J)$ is a pair consisting of a morphism $g : [q] \to [p]$ in $\mathbf{Ord}_{\rm inj}^{\le N}$, and an inclusion of sets $J \subset \sigma_g(I)$. Composition is induced by the composition in $\mathbf{Ord}_{\rm inj}^{\le N}$, and by combining a pair of inclusions $J \subset \sigma_g(I)$, $K \subset \sigma_h(J)$ into the inclusion $K \subset \sigma_{hg}(I)$, using the identity $\sigma_{hg} = \sigma_h\sigma_g$.

For $I \subset \{1, \ldots, n\}$, $I \neq \emptyset$, let

$$\mathcal{U}([q])_I = \cap_{i\in I} U_i^q.$$

We have the functor

$$\mathcal{U} : \mathbf{Ord}_{\rm inj}^{\le N}\rtimes(\square_0^n)^{\rm op} \to \mathbf{Sch}_B$$

sending $([q], I)$ to $\mathcal{U}([q])_I$. If $J \subset I \subset \{1, \ldots, n\}$ are non-empty, we send $\mathcal{U}([q])_I$ to $\mathcal{U}([q])_J$ by the inclusion. If $g : [q] \to [p]$ is injective, send $\mathcal{U}([q])_I$ to $\mathcal{U}([p])_{\sigma_g(I)}$ by the map $X(g)$. The inclusions $\mathcal{U}([q])_I \to X([q])$ define the natural transformation

$$\epsilon_{\mathcal{U}} : \mathcal{U} \to X \circ p_1.$$

Similarly, we have the functor

$$\mathcal{V} : \mathbf{Ord}_{\rm inj}^{\le N}\rtimes(\square_0^n)^{\rm op} \to \mathbf{Sch}_B$$

and the natural transformation

$$\epsilon_{\mathcal{V}} : \mathcal{V} \to (X \setminus W) \circ p_1.$$

12.7. **K-theory and the plus construction.** Since the λ-operations are *not* stable operations, we use the K-theory space $\Omega BQ\mathcal{P}_X$ of a scheme X rather than the full K-theory spectrum. We will denote the K-theory space of X by $K(X)$; since we will be using this construction only in this section on λ-operations, there should be no confusion with the use of the same notation for the K-theory spectrum of X in the rest of the paper.

For each q, we have the functor

$$K(\mathcal{U}([q])) : \square_0^n \to \mathcal{S}^*,$$

with $K(\mathcal{U}([q]))(I)$ the K-theory space of the open subscheme $\mathcal{U}([q])_I$. We have the similar construction with $\mathcal{V}$ replacing $\mathcal{U}$. This gives the functors

$$K(\mathcal{U}) : \mathbf{Ord}_{\rm inj}^{\le N}\rtimes\square_0^n \to \mathbf{Sp},\ K(\mathcal{V}) : \mathbf{Ord}_{\rm inj}^{\le N}\rtimes\square_0^n \to \mathbf{Sp},$$

and the natural transformations

$$\epsilon_{\mathcal{U}} : K(X, -)_N \circ p_1 \to K(\mathcal{U}),\ \epsilon_{\mathcal{U}} : K(X \setminus W, -)_N \circ p_1 \to K(\mathcal{V}).$$

For a B-scheme Y, let $K_0(Y/B)$ denote the image of $K_0(B)$ in $K_0(Y)$. If $\mathcal{W}$ is a finite open cover of Y, we let $K_0(Y, \mathcal{W}/B)$ denote the subgroup of $K_0(Y)$ consisting of the classes of vector bundles E such that the restriction of E to each $W \in \mathcal{W}$ is in $K_0(W/B)$.

Let $\mathcal{A}([q])_I$ be the ring of functions on $\mathcal{U}([q])_I$. We have the plus-construction $\mathrm{BGL}^+(\mathcal{A}([q])_I)$. There is a natural weak equivalence of $\mathrm{BGL}^+(\mathcal{A}([q])_I)$ with the connected component of 0 in $K(\mathcal{U}([q]))_I$ (see [10]).

Let

$$\mathrm{BGL}^+(\mathcal{U}([q])) \times K_0(\mathcal{U}([q])/B) : \square_0^n \to \mathcal{S}^*$$

be the functor $I \mapsto \mathrm{BGL}^+(\mathcal{A}([q])_I) \times K_0(\mathcal{U}([q])_I/B)$, giving the functor

$$\mathrm{BGL}^+(\mathcal{U}) \times K_0(\mathcal{U}/B) : \mathbf{Ord}_{\rm inj}^{\le N\,\rm op}\rtimes\square_0^n \to \mathcal{S}^*.$$

We similarly have the functors

$$\mathrm{BGL}^+(\mathcal{V}([q])) \times K_0(\mathcal{V}([q])/B) : \square_0^n \to \mathcal{S}^*$$

and

$$\mathrm{BGL}^+(\mathcal{V}) \times K_0(\mathcal{V}/B) : \mathbf{Ord}_{\mathrm{inj}}^{\leq N \mathrm{op}} \rtimes \square_0^n \to \mathcal{S}^*.$$

The maps ρ_p define the natural transformation of functors

$$\rho^* : K(\mathcal{U}) \to K(\mathcal{V}); \quad \rho^* : \mathrm{BGL}^+(\mathcal{U}) \times K_0(\mathcal{U}/B) \to \mathrm{BGL}^+(\mathcal{V}) \times K_0(\mathcal{V}/B).$$

Define the functors

$$K(\mathcal{U}, \mathcal{V}) : \mathbf{Ord}_{\mathrm{inj}}^{\leq N \mathrm{op}} \rtimes \square_0^n \times (0 < 1 > *) \to \mathcal{S}^*,$$

$$\mathrm{BGL}^+(\mathcal{U}, \mathcal{V}) \times K_0(\mathcal{U}, \mathcal{V}/B) : \mathbf{Ord}_{\mathrm{inj}}^{\leq N \mathrm{op}} \rtimes \square_0^n \times (0 < 1 > *) \to \mathcal{S}^* \tag{12.2}$$

by setting

$$K(\mathcal{U}, \mathcal{V})(0) = K(\mathcal{U}),\ K(\mathcal{U}, \mathcal{V})(1) = K(\mathcal{V}),\ K(\mathcal{U}, \mathcal{V})(*) = *,\ K(\mathcal{U}, \mathcal{V})(0 < 1) = \rho^*,$$

and $K(\mathcal{U}, \mathcal{V})(* < 1)$ the inclusion of the base-point; we define $\mathrm{BGL}^+(\mathcal{U}, \mathcal{V}) \times K_0(\mathcal{U}, \mathcal{V}/B)$ analogously.

It follows from [18, Theorem 5.3] that the homotopy limit of $K(\mathcal{U}([q]))$ over $\square_0^n$ is, via $\epsilon_{\mathcal{U}([q])}$, weakly equivalent to $K(X([q]))$, and the homotopy limit of $\mathrm{BGL}^+(\mathcal{U}([q])) \times K_0(\mathcal{U}([q])/B)$ over $\square_0^n$ is weakly equivalent to the disjoint union of the connected components of $K(X([q]))$ corresponding to the subgroup

$$K_0(X([q]), \mathcal{U}([q])/B) \subset K_0(X([q])).$$

We have the analogous facts for the functors $K(\mathcal{V}([q]))$ and $\mathrm{BGL}^+(\mathcal{V}([q])) \times K_0(\mathcal{V}([q])/B)$, with $X([q])$ replaced by $X([q]) \setminus W([q])$.

We have thus proved

Proposition 12.8. *The homotopy limit of $K(\mathcal{U}, \mathcal{V})([q])$ over $\square_0^n \times (0 < 1 > *)$ is naturally weakly equivalent to the space $K^{W([q])}(X([q]))$. The homotopy limit of $(\mathrm{BGL}^+(\mathcal{U}, \mathcal{V}) \times K_0(\mathcal{U}, \mathcal{V}/B))([q])$ is weakly equivalent to the union of connected components of $K^{W([q])}(X([q]))$ corresponding to the inverse image of $K_0(X([q]), \mathcal{U}([q])/B)$ by the natural homomorphism*

$$K_0^{W([q])}(X([q])) \to K_0(X([q])).$$

12.9. **Homotopy fibers.** The category $\mathbf{Ord}_{\mathrm{inj}}^{\leq N \mathrm{op}} \rtimes \square_0^n$ is cofibered over $\mathbf{Ord}_{\mathrm{inj}}^{\leq N \mathrm{op}}$, where, given a morphism $g : [p] \to [q]$ in $\mathbf{Ord}_{\mathrm{inj}}^{\leq N \mathrm{op}}$, and an $I \in \square_0^n$, we take $g_*([p], I) = ([q], \sigma_g(I))$, with the cobasechange morphism $(g, \mathrm{id}) : ([p], I) \to ([q], \sigma_g(I))$. In fact, $\mathbf{Ord}_{\mathrm{inj}}^{\leq N \mathrm{op}} \rtimes \square_0^n$ is cofibered over $\mathbf{Ord}_{\mathrm{inj}}^{\leq N \mathrm{op}}$ is *strictly* cofibered over $\mathbf{Ord}_{\mathrm{inj}}^{\leq N \mathrm{op}}$, since we have the identity $h_* \circ g_* = (hg)_*$, rather than merely a natural isomorphism $h_* \circ g_* \cong (hg)_*$.

Extending by taking the product with $0 < 1 > *$ makes $I := \mathbf{Ord}_{\mathrm{inj}}^{\leq N \mathrm{op}} \rtimes \square_0^n \times 0 < 1 > *$ strictly cofibered over $\mathbf{Ord}_{\mathrm{inj}}^{\leq N \mathrm{op}}$. Now, let $F : I \to \mathcal{S}$ be a functor, and let $F([q]) : \square_0^n \times 0 < 1 > *$ denote the restriction of F to $[q] \rtimes \square_0^n \times 0 < 1 > *$. Define $\mathrm{Fib}F([q])$ to be the homotopy limit of $F([q])$ over $\square_0^n \times 0 < 1 > *$. The cobasechange morphisms over $g : [q] \to [p]$ define the morphisms $\mathrm{Fib}F(g) : \mathrm{Fib}F([q]) \to \mathrm{Fib}F([p])$. Since I is strictly cofibered over $\mathbf{Ord}_{\mathrm{inj}}^{\leq N \mathrm{op}}$, we have the functoriality $\mathrm{Fib}F(g) \circ \mathrm{Fib}F(h) = \mathrm{Fib}F(gh)$, giving us the functor

$$\mathrm{Fib}F : \mathbf{Ord}_{\mathrm{inj}}^{\leq N \mathrm{op}} \to \mathcal{S}.$$

12.10. **The proof of Theorem 12.1.** We can now assemble everything to give the proof of Theorem 12.1.

By the identity (12.1), it suffices to show that $\pi_*(|K^W(X,-)^{\mathrm{inj}}_N|)$ has the structure of a special $K_0(B)$-λ-algebra, natural in W, X and N.

Let I be a small category. A functor $C : I \to \mathcal{S}$ is called *finite* if $C(i)$ has only finitely many non-degenerate simplices for each $i \in I$. The category of functors $I \to \mathcal{S}$ has the structure of a simplicial model category (see [18, Appendix A]), so we may speak of cofibrant, fibrant, or bifibrant functors. We let $\mathrm{BGL}^+(\mathcal{U},\mathcal{V}) \times K_0(\mathcal{U},\mathcal{V}/B)^*$ be a bifibrant model of $\mathrm{BGL}^+(\mathcal{U},\mathcal{V}) \times K_0(\mathcal{U},\mathcal{V}/B)$.

Let $I = \mathbf{Ord}^{\le N\mathrm{op}}_{\mathrm{inj}} \rtimes \square^n_0 \times (0 < 1 > *)$, and let $\mathcal{S}^I$ denote the category of functors $I \to \mathcal{S}$. For $X, Y \in \mathcal{S}^I$, we let $[X,Y]$ be the set of homotopy classes of maps $X \to Y$. We let $\mathcal{S}^I_{\mathrm{fin}}$ be the full subcategory of finite $X : I \to \mathcal{S}$. By [18, Theorem 4.5], the functor

$$[-, \mathrm{BGL}^+(\mathcal{U},\mathcal{V}) \times K_0(\mathcal{U},\mathcal{V}/B)^*] : \mathcal{S}^I_{\mathrm{fin}} \to \mathbf{Sets}$$

has the natural structure of a functor to special $K_0(B)$-λ-algebras. In addition, by [18, Lemma 4.1] the bifibrant object $\mathrm{BGL}^+(\mathcal{U},\mathcal{V}) \times K_0(\mathcal{U},\mathcal{V}/B)^*$ is a filtered direct limit of subfunctors $X_\alpha : I \to \mathcal{S}^*$, with $X_\alpha \in \mathcal{S}^I_{\mathrm{fin}}$.

We have the space $\mathrm{Fib}X_\alpha([q])$, defined as the homotopy limit of the functor

$$X_\alpha([q]) : \square^n_0 \times (0 < 1 > *) \to \mathcal{S}^*,$$

and the functor $\mathrm{Fib}X_\alpha : \mathbf{Ord}^{\le N}_{\mathrm{inj}} \to \mathcal{S}$. Let $\Gamma(X_\alpha(\mathcal{U},\mathcal{V}))$ be the geometric realization of $\mathrm{Fib}X_\alpha$.

Let $\Gamma(\mathrm{BGL}^+(\mathcal{U},\mathcal{V}) \times K_0(\mathcal{U},\mathcal{V}/B)^*)$ be the similar construction applied to the functor $\mathrm{BGL}^+(\mathcal{U},\mathcal{V}) \times K_0(\mathcal{U},\mathcal{V}/B)^*$. If we apply the $K_0(B)$-λ-algebra structure on the functor $[-, \mathrm{BGL}^+(\mathcal{U},\mathcal{V}) \times K_0(\mathcal{U},\mathcal{V}/B)^*]$ to the inclusion maps

$$i_\alpha : X_\alpha \to \mathrm{BGL}^+(\mathcal{U},\mathcal{V}) \times K_0(\mathcal{U},\mathcal{V}/B)^*,$$

we have maps $\lambda^q(i_\alpha) : X_\alpha \to \mathrm{BGL}^+(\mathcal{U},\mathcal{V}) \times K_0(\mathcal{U},\mathcal{V}/B)^*$, satisfying the special λ-ring identities, and compatible, up to homotopy, with respect to the maps in the directed system $\{X_\alpha\}$. Applying the functor $\Gamma(-)$ gives an up to homotopy compatible family of maps

$$\Gamma(\lambda^q(i_\alpha)) : \Gamma(X_\alpha) \to \Gamma(\mathrm{BGL}^+(\mathcal{U},\mathcal{V}) \times K_0(\mathcal{U},\mathcal{V}/B)^*).$$

Since $\mathrm{BGL}^+(\mathcal{U},\mathcal{V}) \times K_0(\mathcal{U},\mathcal{V}/B)^*$ is the direct limit of the X_α, this gives us maps

$$\pi_n(\Gamma(\mathrm{BGL}^+(\mathcal{U},\mathcal{V}) \times K_0(\mathcal{U},\mathcal{V}/B)^*)) \xrightarrow{\pi_n(\Gamma(\lambda^q(i_\alpha)))} \pi_n(\Gamma(\mathrm{BGL}^+(\mathcal{U},\mathcal{V}) \times K_0(\mathcal{U},\mathcal{V}/B)^*)),$$

making $\pi_*(\Gamma(\mathrm{BGL}^+(\mathcal{U},\mathcal{V}) \times K_0(\mathcal{U},\mathcal{V}/B)^*))$ a special $K_0(B)$-λ-algebra.

Using the naturality of the $K_0(B)$-λ-algebra structure on $[-, \mathrm{BGL}^+(\mathcal{U},\mathcal{V}) \times K_0(\mathcal{U},\mathcal{V}/B)^*]$, we may pass to the limit over the open covers $(\mathcal{U},\mathcal{V})$; using the weak equivalence of $(\mathrm{BGL}^+(\mathcal{U},\mathcal{V}) \times K_0(\mathcal{U},\mathcal{V}/B))([q])$ with the various connected components of $K^{W([q])}(X([q]))$ described in Proposition 12.8, this gives $\pi_*(|K^W(X)^{\mathrm{inj}}_N|)$ the desired special $K_0(B)$-λ-algebra structure. One shows in a similar fashion that the $K_0(B)$-λ-algebra structure on $\pi_*(|K^W(X)^{\mathrm{inj}}_N|)$ for $* > 0$ is independent of the choice of open covers $\mathcal{U}$ and $\mathcal{V}$. For details on this point, we refer the reader to the proof of [18, Corollary 5.6].

The naturality of the $K_0(B)$-λ-algebra structure on $[-, \mathrm{BGL}^+(\mathcal{U}, \mathcal{V}) \times K_0(\mathcal{U}, \mathcal{V}/B)^*]$ gives us the desired naturality in W, X and N, completing the proof of the theorem.

Remarks 12.11. (1) Let $W' \subset W$ be cosimplicial closed subsets of a cosimplicial B-scheme X; we suppose that each $X([q])$ is of finite type over B and admits an ample family of line bundles. We have the homotopy fiber sequence

$$K^{W'}(X) \to K^W(X) \to K^{W\setminus W'}(X \setminus W'),$$

inducing the long exact sequence of homotopy groups

$$\ldots \to K_p^{W'}(X) \to K_p^W(X) \to K_p^{W\setminus W'}(X \setminus W') \to K_{p-1}^{W'}(X) \to \ldots \tag{12.3}$$

Then the λ-operations given by Theorem 12.1 are compatible with the maps in this sequence.

Indeed, we have the natural homotopy equivalence of $K^{W'}(X)$ with the iterated homotopy fiber $K^{W'}(X)*$ over the diagram

$$\begin{array}{ccc} K(X) & \longrightarrow & K(X \setminus W) \\ \downarrow & & \downarrow \\ K(X \setminus W') & \longrightarrow & K((X \setminus W) \setminus W') = K(X \setminus W). \end{array}$$

The naturality of the lambda-ring structure on $[-, \mathrm{BGL}^+]$ with respect to change of categories implies that the λ-operations for $K_*^{W'}(X)$ and $K_*^{W'}(X)*$ agree, and that the λ-operations for $K_*^{W'}(X)*$ are compatible with the long exact sequence of homotopy groups resulting from the above diagram. Since this is the same as the sequence (12.3), our claim is verified.

(2) Suppose that B is a noetherian scheme (not necessarily affine). By replacing B with an affine open cover $\mathcal{V} := \{V_1, \ldots, V_m\}$, replacing the affine cover $\mathcal{U}$ with affine covers $\mathcal{U}_i$ of $X \times_B V_i$, $i = 1, \ldots, m$, replacing the parameter category $\square_0^n$ with $\square_0^n \times \square_0^m$, and making the evident modifications to the construction given above, we may remove the condition that B is affine from Theorem 12.1.

(3) Let $f : X \to B$ be the structure morphism. Let $\mathcal{V} = \{V_1, \ldots, V_m\}$ be an open cover of B. Suppose we have, for each $I \subset \{1, \ldots, m\}$, a cosimplicial closed subset W_I of $f^{-1}(V_I)$, such that $W_I \cap f^{-1}(V_J) \subset W_J$ for each $I \subset J$. Sending $I \subset \{1, \ldots, m\}$ to $|K^{W_I}(f^{-1}(V_I))|$ gives the functor

$$|K^{W_*}(f^{-1}(V_*))| : \square_0^m \to \mathcal{S}^*.$$

Adding in the $\square_0^m$-variable to the construction of λ-operations on $\pi_*(K^W(X))$ given above extends the special $K_0(B)$-λ-algebra structure on $\pi_*(K^W(X))$ to a special $K_0(B)$-λ-algebra structure on $\pi_*(\mathrm{holim}_{\square_0^m} |K^{W_*}(f^{-1}(V_*))|)$.

Take the base B to have Krull dimension ≤ 1, and let $V_i = \mathrm{Spec}\,(\mathcal{O}_{B,b_i})$ for points b_1, b_2 of B. Then the fiber product $V_1 \times_B V_2$ is a local scheme, from which it follows that the homotopy group $\pi_n(B; f_*K^{(q)}(X, -))$ is the direct limit of the homotopy groups $\pi_n(\mathrm{holim}_{\square_0^m} K^{(q)}(f^{-1}(V_*), -)_N)$, where we take $\mathcal{V} = \{V_1, \ldots, V_m\}$ a finite Zariski open cover and $N \geq n + m + 1$. Thus, we have constructed a natural special $K_0(B)$-λ-algebra structure on $\pi_n(B, f_*\mathcal{K}^{(q)}(X, -))$. The same construction gives a natural special $K_0(B)$-λ-algebra structure on the homotopy groups

$\pi_n(B; f_*\mathcal{K}^{(q/q+r)}(X,-)$, and, by (1), the λ-operations are natural with respect to the maps in the long exact homotopy sequence associated to the cofiber sequence

$$f_*\mathcal{K}^{(q+r)}(X,-) \to f_*\mathcal{K}^{(q)}(X,-) \to f_*\mathcal{K}^{(q/q+r)}(X,-).$$

Theorem 12.12. *Let $f : X \to B$ be a regular B-scheme of finite type, B a regular scheme of Krull dimension ≤ 1. Then the spectral sequence* (2.9) *admits Adams operations ψ_k with the following properties:*

1. *The ψ_k are natural in the category of smooth B-schemes.*
2. *The ψ_k are compatible with the Adams operations ψ_k on $K_*(X)$ given by* [18, Corollary 5.2].
3. *On the $E_2^{p,2q}$-term $H^p(X, \mathbb{Z}(-q))$, ψ_k acts by multiplication by k^q.*

The analogous statements hold for the associated mod n spectral sequence.

Proof. We give the proof for the integral sequence; the proof for the mod n sequence is essentially the same.

It follows from Theorem 12.1 and Remark 12.11(3) that we have the structure of a $K_0(B)$-λ-algebra on $\pi_*(B; f_*\mathcal{K}^{(q)}(X,-))$, natural in q, and, for X smooth over B, natural in X. Also by Remark 12.11(3), we have the structure of a special $K_0(B)$-λ-algebra on $\pi_*(B; f_*\mathcal{K}^{(q/q+1)}(X,-))$, and the resulting λ-operations are compatible with the long exact sequence

$$\begin{aligned} \ldots \to \pi_n(B; f_*\mathcal{K}^{(q+1)}(X,-)) &\to \pi_n(B; f_*\mathcal{K}^{(q)}(X,-)) \\ &\to \pi_n(B; f_*\mathcal{K}^{(q/q+1)}(X,-)) \to \pi_{n-1}(B; f_*\mathcal{K}^{(q+1)}(X,-)) \to \ldots \end{aligned}$$

Since the Adams operations in a special λ-ring are group homomorphisms, this proves (1) and (2). For (3), it follows from [20, Corollary 7.6] that the cycle class map

$$\mathrm{cl}^q : f_*\mathcal{K}^{(q/q+1)}(X,-) \to f_*\mathcal{Z}^q(X,-)$$

is a weak equivalence. The map cl^q factors through the sheaf of simplicial abelian groups

$$p \mapsto f_*\pi_0(\mathcal{K}^{(q/q+1)}(X,p)),$$

so it suffices to take B semi-local, and to show that the Adams operation ψ_k acts on the image of $K_0^{(q/q+1)}(X,p)$ in $z^q(X,p)$, and acts by multiplication by k^q. For this, it suffices to show that ψ_k acts on the image of $K_0^{(q/q+1)}(X,p)$ in $z^q(X \times \Delta^p)$, and acts there by multiplication by k^q.

Let Z be a regular B-scheme of finite type, let $K^{(q)}(Z)$ be the limit of $K^W(Z)$, as W runs over all codimensions q closed subsets, and let $K^{(q/q+1)}(Z)$ denote the cofiber of

$$K^{(q+1)}(Z) \to K^{(q)}(Z).$$

It is well-known that the cycle class map gives an isomorphism

$$\mathrm{cl}^q : K^{(q/q+1)}(Z) \to z^q(Z).$$

By the Adams-Riemann-Roch theorem [8, Theorem 6.3], the ψ_k act on $K^{(q/q+1)}(Z)$ by multiplication by k^q. Taking $Z = X \times \Delta^p$, we have the natural map

$$K_0^{(q/q+1)}(X,p) \to K_0^{(q/q+1)}(X \times \Delta^p),$$

compatible with the ψ_k, and the commutative diagram

$$\begin{tikzcd} K_0^{(q/q+1)}(X,p) \arrow[r] \arrow[d, "\mathrm{cl}^q"'] & K_0^{(q/q+1)}(X\times\Delta^p) \arrow[d, "\mathrm{cl}^q"] \\ z^q(X,p) \arrow[r] & z^q(X\times\Delta^p). \end{tikzcd}$$

This completes the proof of (3), and the theorem. □

13. Comparison of filtrations

Let X be a regular scheme, essentially of finite type over a one-dimensional regular base. The spectral sequence (2.9) gives rise to the filtration $F^*_{\mathrm{HC}}K_n(X)$ of $K_n(X)$, with $F^q_{\mathrm{HC}}K_n(X)$ the image of $\pi_n(K^{(q)}(X,-))$ in $K_n(X)$. In analogy with the situation for K_0, we call this filtration the *homotopy coniveau filtration*. We have the *gamma* filtration $F^*_\gamma K_n(X)$, gotten from the structure of $K_n(X)$ as a $K_0(X)$-λ-algebra. In this section, we give a comparison of $F^*_{\mathrm{HC}}K_n(X)$ with $F^*_\gamma K_n(X)$.

13.1. **The Friedlander-Suslin theorem.** Let $F:\mathbf{Ord}^{\mathrm{op}}\to\mathcal{S}$ be a pointed simplicial space. For fixed n, we have the n-cube $\mathbf{Ord}_{\mathrm{inj}}/[n]$ of injective maps $[-]\to[n]$ in $\mathbf{Ord}$. Let $F(\Delta^n,\partial^n)$ be the iterated homotopy fiber over $(\mathbf{Ord}_{\mathrm{inj}}/[n])^{\mathrm{op}}$ of the functor

$$F_n:(\mathbf{Ord}_{\mathrm{inj}}/[n])^{\mathrm{op}}\to\mathcal{S},$$
$$[p]\xrightarrow{f}[n]\mapsto F([p]).$$

In [6], Friedlander and Suslin construct a natural map

(13.1) $$\Phi_n:F(\Delta^n,\partial^n)\to\Omega^n|F|.$$

If F is N-connected, then Φ_n induces an isomorphism on π_m for $m<N$ and a surjection for $m=N$.

Let sk_nF denote the n-skeleton of F, and $|\mathrm{sk}_nF|$ the geometric realization. The map Φ_n factors through the canonical map $p_n:\Omega^n|\mathrm{sk}_nF|\to\Omega^n|F|$ via a natural map $\phi_n:F(\Delta^n,\partial)\to\Omega^n|\mathrm{sk}_nF|$. In particular, if we have a map $\lambda:\mathrm{sk}_{n+r}F'\to\mathrm{sk}_{n+r}F$, then the diagram

$$\begin{tikzcd} \pi_i(F'(\Delta^n,\partial^n)) \arrow[r, "\Phi_n"] \arrow[d, "\lambda_*"'] & \pi_{n+i}(|F'|) \arrow[r, equal] & \pi_{n+i}(\mathrm{sk}_{n+r}|F'|) \arrow[d, "\lambda_*"] \\ \pi_i(F(\Delta^n,\partial^n)) \arrow[r, "\Phi_n"'] & \pi_{n+i}(|F|) \arrow[r, equal] & \pi_{n+i}(\mathrm{sk}_{n+r}|F|) \end{tikzcd}$$

commutes for all $i<r$.

Examples 13.2. (1) Let $K(X,p)[r]$ denote the rth delooping in the spectrum $K(X,p)$, i.e., $K(X,p)[r]$ is $BQ^r\mathcal{P}_{X\times\Delta^p}$, with $Q^r\mathcal{P}_{X\times\Delta^p}$ the r-fold Q-construction on the exact category $\mathcal{P}_{X\times\Delta^p}$ of locally free coherent sheaves on $X\times\Delta^p$. Taking the appropriate limit of homotopy fibers gives the rth delooping $K^{(q)}(X,p)[r]$ for the spectrum $K^{(q)}(X,p)$. Taking $F(p)=K^{(q)}(X,p)[r]$ gives the space $K^{(q)}(X\times\Delta^n,X\times\partial^n)[r]$. Since $K^{(q)}(X,-)$ is an Ω-spectrum, we have the natural weak equivalence $\Omega^N(K^{(q)}(X\times\Delta^n,X\times\partial^n)[M])$ with $\Omega^{N-M}K^{(q)}(X\times\Delta^n,X\times\partial^n)[0]$ for all $N\geq M$, giving us the spectrum $K^{(q)}(X\times\Delta^n, X\times\partial^n)$. The Friedlander-Suslin construction thus gives the natural map of spectra

$$\Phi_n : K^{(q)}(X\times\Delta^n, X\times\partial^n)\to\Omega^n|K^{(q)}(X,-)|.$$

The method of construction of λ-operations in §12 gives λ-operations for $K_p^{(q)}(X\times\Delta^n, X\times\partial^n)$ (see [18] for a detailed construction). By the comments above, the λ-operations on $K_p^{(q)}(X\times\Delta^n, X\times\partial^n)$ and on $\pi_{n+p}(K^{(q)}(X,-))$ are compatible via the map Φ_n.

(2) Let $\mathcal{K}^{(q)}(X\times\Delta^n, X\times\partial^n)$ be the presheaf of spectra on X_{Zar} given by $U\mapsto K^{(q)}(U\times\Delta^n, U\times\partial^n)$ Since Φ_n is natural in the simplicial space F, the map of (1) gives the map of presheaves of spectra

$$\Phi_n : \mathcal{K}^{(q)}(X\times\Delta^n, X\times\partial^n)\to\Omega^n|\mathcal{K}^{(q)}(X,-)|.$$

As in (1), the map on hypercohomology induced by Φ_n is compatible with the respective λ-operations. In addition, we have the commutative diagram

$$\begin{CD} K_m^{(q)}(X\times\Delta^n, X\times\partial^n) @>{\tilde\Phi_n}>> \pi_{m+n}(|K^{(q)}(X,-)|)\\ @VVV @VVV\\ \mathbb{H}^{-m}(X,\mathcal{K}^{(q)}(X\times\Delta^n, X\times\partial^n)) @>>{\tilde\Phi_n}> \mathbb{H}^{-m-n}(X,\mathcal{K}^{(q)}(X-)), \end{CD}\tag{13.2}$$

where the vertical arrows are the canonical maps.

(3) Let $F(-;-):\mathbf{Ord}^{\rm op}\times\mathbf{Ord}^{\rm op}\to\mathcal{S}$ be a bisimpicial space. We may construct the simplicial spaces $p\mapsto F(\Delta^n,\partial^n;p)$, and $p\mapsto\Omega^n|F(-;p)|$, which we denote by $F(\Delta^n,\partial^n;-)$ and $\Omega^n|F(-)|$, respectively. We may then apply the Friedlander-Suslin construction degreewise, giving the map of simplicial spaces

$$\Phi_n : F(\Delta^n,\partial^n;-)\to\Omega^n|F|(-).$$

Applying this to the bisimplicial spectrum $K^{(q)}(X,-,-)$ gives the map of simplicial spectra

$$\Phi_n : K^{(q)}(X\times\Delta^n, X\times\partial^n;-)\to\Omega^n|K^{(q)}(X,-;-)|$$

and the map of presheaves on X_{Zar}.

$$\Phi_n : \mathcal{K}^{(q)}(X\times\Delta^n, X\times\partial^n;-)\to\Omega^n|\mathcal{K}^{(q)}(X,-;-)|.$$

As above, the homotopy (hypercohomology) of these objects have λ-operations, and the maps Φ_n respect the λ-operations.

As mentioned above, the map $\Phi_n : F(\Delta^n,\partial^n)\to\Omega^n|F|$ is an isomorphism on homotopy groups of sufficiently large degree. We need a slight improvement of the bounds described above in the special case of the simplicial spectrum $K^{(q)}(X,-)$.

Lemma 13.3. *For each $n\ge 0$, the map*

$$\tilde\Phi_n : K_0^{(q)}(X\times\Delta^n, X\times\partial^n)\to\pi_n(|K^{(q)}(X,-)|)\tag{13.3}$$

induced by the map (13.1) *is a surjective map of $K_0(X)$-λ-algebras.*

Proof. We have already seen that $\tilde\Phi_n$ is compatible with the respective λ-operations, so it suffices to prove the surjectivity.

We let Λ^n denote the collection of faces $t_i=0$, $i=0,\ldots,n-1$ of Δ^n, giving us the relative K-theory spectra $K^{(q)}(\Delta^n,\Lambda^n)$ with deloopings $K^{(q)}(\Delta^n,\Lambda^n)[r]$.

Restriction to the face $t_{n+1} = 0$ gives us the homotopy fiber sequence

$$K^{(q)}(X \times \Delta^{n+1}, X \times \partial^{n+1})[r] \to K^{(q)}(X \times \Delta^{n+1}, X \times \Lambda^{n+1})[r] \\ \to K^{(q)}(X \times \Delta^{n}, X \times \partial^{n})[r]$$

Using the simplicial degeneracies, it is easy to show that

$$\pi_i(K^{(q)}(X \times \Delta^{n+1}, X \times \Lambda^{n+1})[r]) = 0$$

for $i < r$, giving the surjection

$$\theta : \pi_r(K^{(q)}(X \times \Delta^{n}, X \times \partial^{n})[r]) \to \pi_{r-1}(K^{(q)}(X \times \Delta^{n+1}, X \times \partial^{n+1})[r]).$$

The map θ is induced by a map of spaces

$$\Theta : \Omega K^{(q)}(X \times \Delta^{n}, X \times \partial^{n})[r] \to K^{(q)}(X \times \Delta^{n+1}, X \times \partial^{n+1})[r].$$

The map Θ is in turn induced by the inclusion of categories

$$\Delta_{\mathrm{inj}}/[n] \to \Delta_{\mathrm{inj}}/[n+1]$$

given by composition with the injective map

$$\delta_0^n : [n] \to [n+1], \\ \delta_0^n(i) = i.$$

The naturality of the maps Φ_n thus implies that we have the commutative diagram

$$\begin{tikzcd}
\pi_0(K^{(q)}(X \times \Delta^n, X \times \partial^n)) \arrow[d, equal] & \\
\pi_r(K^{(q)}(X \times \Delta^n, X \times \partial^n)[r]) \arrow[r, "\theta"] \arrow[d, "\Phi_n"'] & \pi_{r-1}(K^{(q)}(X \times \Delta^{n+1}, X \times \partial^{n+1})[r]) \arrow[d, "\Phi_{n+1}"] \\
\pi_{n+r}(|K^{(q)}(X,-)[r]|) \arrow[r, equal] \arrow[d, equal] & \pi_{n+r}(|K^{(q)}(X,-)[r]|) \\
\pi_n(|K^{(q)}(X,-)|) &
\end{tikzcd}$$

Since the space $K^{(q)}(X,p)[r]$ is $r-1$ connected, the map Φ_{n+1} is surjective, hence the map (13.3) is surjective. □

13.4. **Relative** K_0**.** We now study the relative K_0 group $K_0(X \times \Delta^n, X \times \partial^n)$ and the group with support $K_0^{(q)}(X \times \Delta^n, X \times \partial^n)$. We briefly recall some of the constructions used in [17] to which we refer the reader for further details.

If we have a scheme Y and a closed subscheme D, we may glue two copies of Y along D, forming the *double* $Y \coprod_D Y$. More generally, if we have n closed subschemes $D_1, \ldots, D_n$, we may iterate this procedure, forming the n-fold double $\coprod_n Y/D_*$, which is naturally a quotient of 2^n copies of Y. Indexing these copies by the set $\{0,1\}^n$, the gluing data is given by identifying $Y_{i_1,\ldots,i_j=0,\ldots i_n}$ with $Y_{i_1,\ldots,i_j=1,\ldots i_n}$ along D_j. We have the closed subschemes $\mathcal{D}_j$, being the union of the components $Y_{i_1,\ldots,i_n}$ with $i_j = 1$. We identify Y with the copy $Y_{0,\ldots,0}$, giving the identity

$$Y \cap \mathcal{D}_j = D_j.$$

The n double covers $\coprod_n Y/D_* \to \coprod_{n-1} Y/D_{*\neq i}$ give a splitting π to the natural map

$$K_n(\coprod_n Y/D_*; \mathcal{D}_1, \ldots, \mathcal{D}_n) \xrightarrow{\iota} K_n(\coprod_n Y/D_*).$$

The collection of closed subschemes $D_1, \ldots, D_n$ is called *split* if there are maps

$$p_i : Y \to D_i$$

splitting the inclusions $D_i \to Y$ such that $p_{j|D_i} \circ p_i = p_{i|D_j} \circ p_j$ for each $i \neq j$.

We recall the *homotopy K-theory* KH_* of Weibel [44], which we may apply in the relative situation. There is a canonical transformation of functors $K_m \to KH_m$ for all $m \geq 0$. If T is K_m-regular for all $m \leq p$, then $K_m(T) \to KH_m(T)$ is an isomorphism for all $0 \leq m \leq p$.

We recall the following result:

Theorem 13.5 (Theorem 1.6 and Theorem 1.10, [17])**.** *Suppose that Y is smooth over a regular noetherian ring R, and $D_1, \ldots, D_n$ form a relative normal crossing divisor on Y. Suppose further that there are elements $f_1, \ldots f_k$ of R, generating the unit ideal such that the restriction of $D_1, \ldots, D_n$ to $Y \setminus \{f_i = 0\}$ is split for each i. Then the natural maps*

$$K_0(\coprod_n Y/D_*; \mathcal{D}_1, \ldots, \mathcal{D}_n) \to K_0(Y; D_1, \ldots, D_n)$$

$$K_0(\coprod_n Y/D_*; \mathcal{D}_1, \ldots, \mathcal{D}_n) \to KH_0(\coprod_n Y/D_*; \mathcal{D}_1, \ldots, \mathcal{D}_n)$$

$$K_0(\coprod_n Y/D_*) \to KH_0(\coprod_n Y/D_*)$$

(the first map induced by the restriction to $Y_{0,\ldots,0}$) are isomorphisms.

As our primary example, take $Y = X \times \Delta^n$, $D_* = X \times \partial^n$. Since ∂^n is locally split on the affine scheme Δ^n, we may apply Theorem 13.5 in this case, or more generally to any subset of the D_*.

Lemma 13.6. *Suppose that X is smooth over a regular noetherian ring R. For $Y = X \times \Delta^n$, $D_* = X \times \partial^n$, the canonical maps*

$$K_0(X) \xrightarrow{p^*} K_0(\coprod_{n+1} Y/D_*), \quad K_0(\coprod_{n+1} Y/D_*; \mathcal{D}_0, \ldots, \mathcal{D}_n) \xrightarrow{\iota} K_0(\coprod_{n+1} Y/D_*)$$

induce an isomorphism

$$K_0(\coprod_{n+1} Y/D_*) \cong K_0(X) \oplus K_0(\coprod_{n+1} Y/D_*; \mathcal{D}_0, \ldots, \mathcal{D}_n).$$

If we make $K_0(X) \oplus K_0(\coprod_{n+1} Y/D_*; \mathcal{D}_0, \ldots, \mathcal{D}_n)$ a ring by

$$(x, y)(x', y') = (xx, p^*x'y + p^*xy'),$$

the above isomorphism is an isomorphism of rings.

Proof. By the excision property of KH-theory, restriction to $Y_{0,\ldots,0,i_r=1,0,\ldots,0}$ gives the isomorphism

$$KH_m(\mathcal{D}_r; \mathcal{D}_r \cap \mathcal{D}_0, \ldots, \mathcal{D}_r \cap \mathcal{D}_{r-1}) \cong KH_m(Y; D_0, \ldots, D_{r-1}) \tag{13.4}$$

for all m and all $0 < r \leq n$. The homotopy property for KH-theory gives

$$KH_m(Y; D_0, \ldots, D_{r-1}) = 0$$

for $1 \leq r \leq n$. Thus, the long exact sequences

$$\ldots \to KH_m(\coprod_{n+1} Y/D_*; \mathcal{D}_0, \ldots, \mathcal{D}_r) \to KH_m(\coprod_{n+1} Y/D_*; \mathcal{D}_0, \ldots, \mathcal{D}_{r-1}) \\ \to KH_m(\mathcal{D}_r; \mathcal{D}_r \cap \mathcal{D}_0, \ldots, \mathcal{D}_r \cap \mathcal{D}_{r-1}) \to \ldots$$

give us

$$KH_m(\coprod_{n+1} Y/D_*; \mathcal{D}_0, \ldots, \mathcal{D}_n) \cong KH_m(\coprod_{n+1} Y/D_*; \mathcal{D}_0). \tag{13.5}$$

Now, $\coprod_{n+1} Y/D_*$ is the double of $\coprod_n Y/D_{*\geq 1}$ along the closed subscheme D given as the union of the subschemes $D_0 \subset Y_{i_1,\ldots,i_n}$:

$$\coprod_{n+1} Y/D_* = (\coprod_n Y/D_{*\geq 1}) \coprod_D (\coprod_n Y/D_{*\geq 1}).$$

With respect to this description, $\mathcal{D}_0$ is the component $(\coprod_n Y/D_{*\geq 1})_1$. Thus, we have the split exact sequence

$$0 \to KH_m(\coprod_{n+1} Y/D_*; \mathcal{D}_0) \to KH_m(\coprod_{n+1} Y/D_*) \to KH_m(\coprod_n Y/D_{*\geq 1}) \to 0.$$

Using (13.4) and (13.5) gives the isomorphism

$$KH_m(\coprod_{n+1} Y/D_*) \cong KH_m(\coprod_{n+1} Y/D_*; \mathcal{D}_0, \ldots, \mathcal{D}_n) \oplus KH_m(\coprod_n Y/D_{*\geq 1}).$$

Repeating the argument with $\coprod_{n-r+1} Y/D_{*\geq r}$, $r \geq 1$, replacing $\coprod_{n+1} Y/D_*$ gives the isomorphisms

$$\begin{aligned} KH_m(\coprod_n Y/D_{*\geq 1}) &\cong KH_m(\coprod_{n-1} Y/D_{*\geq 2}) \\ &\vdots \\ &\cong KH_m(Y). \end{aligned}$$

This together with Theorem 13.5 completes the proof of the first assertion.

For the assertion on the product structure, it suffices to show that $yy' = 0$ for $y, y' \in KH_0(\coprod_{n+1} Y/D_*; \mathcal{D}_0, \ldots, \mathcal{D}_n)$. We have the isomorphisms

$$KH_0(\coprod_{n+1} Y/D_*; \mathcal{D}_0, \ldots, \mathcal{D}_n) \cong K_0(X \times \Delta^n, X \times \partial^n) \cong K_n(X),$$

compatible with the various products (induced by the structure as λ-rings). Since the λ-ring product on $K_n(X)$ is zero [13], all products in $KH_0(\coprod_{n+1} Y/D_*; \mathcal{D}_*)$ are zero. □

The $K_0(X)$ λ-algebra structure on $KH_0(\coprod_{n+1} Y/D_*)$ and $KH_0(\coprod_{n+1} Y/D_*; \mathcal{D}_*)$ defines the respective γ-filtrations.

Proposition 13.7. *Suppose that X is smooth over a regular ring R, and let $Y = X \times \Delta^n$, $D_* = X \times \partial^n$. Then*

$$F^q_\gamma K_0(\coprod_{n+1} Y/D_*) = F^q_\gamma K_0(\coprod_{n+1} Y/D_*; \mathcal{D}_0, \ldots, \mathcal{D}_n) \oplus F^q_\gamma K_0(X).$$

Proof. Take $x \in F^1_\gamma K_0(\coprod_{n+1} Y/D_*)$, and write $x = p^*x_0 + x_1$, with $x_0 \in K_0(X)$ and $x_1 \in K_0(\coprod_{n+1} Y/D_*; \mathcal{D}_0, \ldots, \mathcal{D}_n)$, via the splitting of Lemma 13.6. Then x_0 is in $F^1K_0(X)$, and we have

$$\begin{aligned}\gamma^k(x) &= p^*\gamma^k(x_0) + \sum_{i=1}^k p^*\gamma^{k-i}(x_0)\gamma^i(x_1)\\ &\in F^k_\gamma K_0(X) \oplus F^k_\gamma K_0(\coprod_n Y/D_*; \mathcal{D}_0, \ldots, \mathcal{D}_n).\end{aligned}$$

This, together with the product structure on K_0 given by Lemma 13.6, gives

$$F^q_\gamma K_0(\coprod_{n+1} Y/D_*) \subset F^q_\gamma K_0(\coprod_{n+1} Y/D_*; \mathcal{D}_0, \ldots, \mathcal{D}_n) \oplus F^q_\gamma K_0(X).$$

Since the other containment is obvious, the proposition is proved. □

13.8. **A patching lemma.** Before passing to the proof of our main theorem comparing the γ-fitration and the spectral sequence filtration, we need a simple technical lemma

Lemma 13.9. *Let A_0 be a commutative domain, with quotient field F. Let R and A be localizations of A_0. Let $G = \prod_{i=1}^N \mathrm{SL}_{n_i}$. Let $g = (g_1, \ldots, g_N)$ be in $G(A)$, $g' = (g'_1, \ldots, g'_N)$ be in $G(R)$. Suppose that each g_i is in the subgroup of elementary matrices $E_{n_i}(A) \subset \mathrm{SL}_{n_i}$ and similarly, each g'_i is in $E_{n_i}(R)$. Then there is a morphism*

$$h : \mathbb{A}^2_F \to G_F$$

such that

1. *$h_{A^1\times 0}$ is the constant map with value id_G.*
2. *$h_{0\times\mathbb{A}^1}$ is the extension to F of a morphism $h_A : \mathbb{A}^1_A \to G_A$*
3. *$h_{1\times\mathbb{A}^1}$ is the extension to F of a morphism $h_R : \mathbb{A}^1_R \to G_R$*
4. *$h(0,1) = g$, $h(1,1) = g'$.*

In addition, let $f_1, \ldots, f_r : Y \to H = \prod_i \mathrm{Gr}_A(k_i, n_i)$ be morphisms of an A_0-scheme Y to a product of Grassmannians, and let $W_1, \ldots, W_r$ be closed subschemes of H. Suppose that $g \cdot W_j$ is f_j-flat for each j (after extending scalars to Q). Suppose further that R is semi-local and that each residue field of R is infinite. Then one can choose $g' \in G(R)$ satisfying all the above conditions and such that, after extending scalars to R, $g' \cdot W_j$ is f_j-flat for each j and W_j is F_j-flat for each j, where $F_j : Y \times \mathbb{A}^1 \to H$ is the morphism

$$F_j(y,t) = h(t,1)^{-1} \cdot f_j(y).$$

Proof. Let e^{is}_{ij} be the $n_i \times n_i$ elementary matrix with ij-entry s. To construct h, we may write

$$g_i = e^{i,s^i_1}_{i_1j_1} \ldots e^{i,s^i_n}_{i_nj_n}; \quad g'_i = e^{i,s'^i_1}_{i_1j_1} \ldots e^{i,s'^i_n}_{i_nj_n}.$$

To save the notation, we take the row and column indices the same for each i, using the convention that $e^{is}_{ij} = \mathrm{id}$ if $i > n_i$ or $j > n_i$. Let $s^i_j : \mathbb{A}^2_F \to \mathbb{A}^1_F$ be the map

$$s^i_j(x,y) = y((1-x)s^i_j + xs'^i_j),$$

and define $h_i : \mathbb{A}^2_F \to \mathrm{SL}_{n_i}$ be the morphism

$$h_i(x,y) = e^{i,s^i_1(x,y)}_{i_1j_1} \ldots e^{i,s^i_n(x,y)}_{i_nj_n}.$$

Letting $h : \mathbb{A}^2_F \to G$ be the product of the h_i clearly satisfies (1)-(4).

For the last assertion, we may take the products $\tilde{g} := e^{i,\tilde{s}^i_1}_{i_1 j_1} \dots e^{i,\tilde{s}^i_n}_{i_n j_n}$ so that $\tilde{g}$ is a geometric generic point of SL_{n_i} over F, if the $\tilde{s}^i_j$ are independent geometric generic points of $\mathbb{A}^1$ over F: just choose a way of row-reducing the generic $n_i \times n_i$ matrix with determinant one to the identity matrix, and add extra terms if necessary so that one can also write g in the desired form. It is then clear that, with this choice of elements $\tilde{s}^j_i$, the maps f_j and F_j have the desired flatness properties. Since the choice of elements $\{s^j_i\}$ which fail to satisfy the flatness conditions clearly form a finite union of locally closed subsets of $\mathbb{A}^{nN}_R$, and since the finitely many residue fields of R are infinite, we may specialize the $\tilde{s}^j_i$ to elements $\mathfrak{s}^j_i \in R$ for which the desired flatness conditions are met. □

We will need a few more simple constructions in the proof of our main theorem. Let X be a regular scheme. Recall the sheaf of simplicial specta $\mathcal{K}^{(q)}(X \times \Delta^n; X \times \partial^n, -)$ described in Example 13.2(3). Forgetting the support gives the natural map

$$\mathcal{K}^{(q)}(X \times \Delta^n; X \times \partial^n, -) \to \mathcal{K}(X \times \Delta^n; X \times \partial^n, -);$$

since all the faces of $X \times \Delta^n$ are regular, the homotopy property for K-theory implies that the natural map $\mathcal{K}(X \times \Delta^n; X \times \partial^n) \to \mathcal{K}(X \times \Delta^n; X \times \partial^n; -)$ induces a weak equivalence of presheaves. Thus, we have the natural map on hypercohomology

$$\text{(13.6)} \qquad \mathbb{H}^N(X, \mathcal{K}^{(q)}(X \times \Delta^n; X \times \partial^n, -)) \to \mathbb{H}^N(X; \mathcal{K}(X \times \Delta^n; X \times \partial^n).$$

for each N.

We define the simplicial spectrum $K^{(q)}(\coprod_{n+1} Y/D_*, -)$ by taking $K^{(q)}(\coprod_{n+1} Y/D_*, p)$ to be the limit of the spectra $K^W((\coprod_{n+1} Y/D_*) \times \Delta^p)$, where W is a closed subset of $(\coprod_{n+1} Y/D_*) \times \Delta^p$ such that the intersection of W with each copy $X \times \Delta^n \times \Delta^p$ inside $(\coprod_{n+1} Y/D_*) \times \Delta^p$ is in $X^{(n,p,\geq q)}$. For each open subsecheme U of X, we make the same construction, replacing X throughout with U. This gives us the presheaf of simplicial spectra $\mathcal{K}^{(q)}(\coprod_{n+1} Y/D_*, -)$ on X_{Zar}.

We may apply the constructions of section 13.4 to $\mathcal{K}^{(q)}(\coprod_{n+1} Y/D_*, -)$, as follows: Using the structure of the $n+1$-fold double of $\coprod_{n+1} Y/D_*$, we have the splitting $\mathcal{K}^{(q)}(\coprod_{n+1} Y/D_*, -) \to \mathcal{K}^{(q)}(\coprod_{n+1} Y/D_*; \mathcal{D}_0, \dots, \mathcal{D}_n, -)$ to the natural map $\mathcal{K}^{(q)}(\coprod_{n+1} Y/D_*; \mathcal{D}_0, \dots, \mathcal{D}_n, -) \to \mathcal{K}^{(q)}(\coprod_{n+1} Y/D_*, -)$. Combining this with the restriction map $\mathcal{K}^{(q)}(\coprod_{n+1} Y/D_*; \mathcal{D}_0, \dots, \mathcal{D}_n, -) \to \mathcal{K}^{(q)}(Y, D_*, -)$ gives the natural map

$$\text{(13.7)} \qquad \Psi : \mathbb{H}^N(X, \mathcal{K}^{(q)}(\coprod_{n+1} Y/D_*, -)) \to \mathbb{H}^N(X, \mathcal{K}^{(q)}(Y, D_*, -)).$$

Theorem 13.10. *Suppose that $f : X \to B$ is smooth over a regular noetherian one-dimensional scheme B. Then the image of $\mathbb{H}^0(X, \mathcal{K}^{(q)}(X \times \Delta^n; X \times \partial^n, -))$ in $\mathbb{H}^0(X; \mathcal{K}(X \times \Delta^n; X \times \partial^n))$ contains the image of $F^q_\gamma K_0(X \times \Delta^n; X \times \partial^n)$.*

Proof. . Write $Y = X \times \Delta^n$, $D_* = X \times \partial^n$, and let z be an element of $F^q_\gamma K_0(Y; D_*)$. By Proposition 13.7, we may lift z to an element $\tilde{z}$ of $F^q_\gamma K_0(\coprod_{n+1} Y/D_*)$.

Since X is regular, X admits an ample family of line bundles. Since $\coprod_{n+1} Y/D_*$ is an affine X-scheme, $\coprod_{n+1} Y/D_*$ also admits an ample family of line bundles. From this it is easy to see that each vector bundle E on $\coprod_{n+1} Y/D_*$ is isomorphic to a pull-back bundle $f^*\mathcal{E}$, where $f : Y/D_* \to H$ is a B-morphism, $\mathcal{E}$ is a vector bundle on H, and H is a B-scheme of the form $\prod_{j=1}^N \mathrm{Gr}_B(k_j, n_j)$ for some integers k_j, n_j and N.

In fact (see for example [7]),

$$K_0(\coprod_{n+1} Y/D_*) = \lim_{\substack{\to \\ f:\coprod_{n+1} Y/D_*\to H}} K_0(H),$$

where the limit is over maps of $\coprod_{n+1} Y/D_*$ to B-schemes of the form $\prod_{j=1}^N \mathrm{Gr}_B(k_j, n_j)$. Thus

$$F^q_\gamma K_0(\coprod_{n+1} Y/D_*) = \lim_{\substack{\to \\ f:\coprod_{n+1} Y/D_*\to H}} F^q_\gamma K_0(H),$$

where the limit is over maps of $\coprod_{n+1} Y/D_*$ to B-schemes H of the form $\prod_{j=1}^N \mathrm{Gr}_B(k_j, n_j)$. We may therefore assume that $\tilde z = f^*\eta$ for $f : \coprod_{n+1} Y/D_* \to H$ a morphism of this form, and $\eta \in F^q_\gamma K_0(H)$.

Suppose at first that $B = \operatorname{Spec} R$, where R is a semi-local ring with infinite residue fields. In [17, proof of Theorem 2.3], we have shown that, if T is a closed subscheme of codimension $\geq q$ of such an H, there is a closed subscheme $T' \subset H$ such that $[\mathcal{O}_T] = [\mathcal{O}_{T'}]$ in $K_0(H)$, and such that the projection of $f^*[\mathcal{O}_{T'}]$ to

$$KH_0(\coprod_{n+1} Y/D_*; \mathcal{D}_1, \ldots, \mathcal{D}_n) \cong K_0(Y; D_*)$$

lands in the image of $K_0^{(q)}(Y; D_*)$. To see this, let $G = \prod_{j=1}^N \mathrm{SL}_{n_j}/R$, which has H as a homogeneous space. One takes $T' = g\cdot T$, where g is a suitably general element $g \in G(R)$, and uses an elementary extension of Kleiman's transversality result [16] to show that the projection of $f^*[\mathcal{O}_{T'}]$ is in fact in the image of $K_0^{(q)}(Y; D_*)$; one need only take g so that $g\cdot T$ is flat with respect to all subschemes of the form $X \times \Delta^r$ inside $\coprod_{n+1} Y/D_*$. Since $G(R)$ acts trivially on $K_0(H)$, it follows that $[\mathcal{O}_T] = [\mathcal{O}_{T'}]$ in $K_0(H)$. Since $F^q_\gamma K_0(H)$ is contained in the topological filtration $F^q_{\mathrm{top}} K_0(H)$ by [8, Theorem 3.9 and Proposition 5.5], and since $F^q_{\mathrm{top}} K_0(H)$ is generated by the classes $[\mathcal{O}_T]$ with $\mathrm{codim} T \geq q$, the theorem follows in this case, using the usual tricks with norms in case R has some finite residue fields.

In general, using norm tricks, we have assume that any given finite set of closed points of B have infinite residue fields. We may also assume that B is irreducible. As above, we may assume there is a closed codimension q subscheme T of H such that $\eta = [\mathcal{O}_T]$ in $K_0(H)$. Take a closed point x in B, and let R be the local ring of x. We may assume that $k(x)$ is infinite. Let $g \in G(R)$ be such that $g \cdot T_R$ is flat with respect to all the faces $X_R \times \Delta^r$ inside $\coprod_{n+1} Y_R/D_{R*}$.

Next, we may spread out the above construction to an affine open neighborhood $U = \operatorname{Spec} A$ of x in B, so that g is in $G(A)$, and $g \cdot T_A$ is flat with respect to all the faces $X_A \times \Delta^r$ inside $\coprod_{n+1} Y_R/D_{A*}$. Write $g = (g_1, \ldots, g_N)$, with $g_i \in \mathrm{SL}_{n_i}(A)$. Localizing A if necessary, we may assume that each g_i is a product of elementary matrices:

$$g_i = \prod_a e_{i_a j_a}^{\lambda^i_a}; \quad i = 1, \ldots, N,$$

with each $\lambda^i_a \in A$.

Now let R' be the semi-local ring of the finitely many points $x_1, \ldots, x_M$ of $B \setminus \operatorname{Spec} A$. We may assume that each residue field of R' is infinite. Let F be the quotient field of R'. By Lemma 13.9, there is a $g' \in G(R')$ and a morphism

$h : \mathbb{A}^2_F \to G_F$ such that g'_i is in $E_{n_i}(R')$ for each i, and the conditions (1)-(4) of Lemma 13.9 are satisfied. In addition, we may choose g' and h so that

(1) $g' \cdot T_{R'}$ is flat with respect to all the faces $X_{R'} \times \Delta^r$ inside $\coprod_{n+1} Y_{R'}/D_{R'*}$.
(2) T_F is flat with respect to all the faces $\mathbb{A}^1 \times X_{R'} \times \Delta^r$ inside $\mathbb{A}^1 \times \coprod_{n+1} Y_F/D_{F*}$, via the map

$$\phi : \mathbb{A}^1 \times \coprod_{n+1} Y_F/D_{F*} \to H$$
$$(t, y) \mapsto h(t, 1)^{-1} \cdot f(y).$$

This gives us the following data: We have the subscheme $f_A^{-1}(g \cdot T)$ of $\coprod_{n+1} Y_A/D_{A*}$, the subscheme $f_{R'}^{-1}(g' \cdot T)$ of $\coprod_{n+1} Y_{R'}/D_{R'*}$, and the subscheme $\phi^{-1}(T)$ of $\mathbb{A}^1 \times \coprod_{n+1} Y_F/D_{F*}$. These are all subschemes of finite projective dimension, with respective KH_0-classes

$$[\mathcal{O}_{f_A^{-1}(g\cdot T)} = f_A^*[\mathcal{O}_{g\cdot T}]$$
$$[\mathcal{O}_{f_{R'}^{-1}(g'\cdot T)} = f_{R'}^*[\mathcal{O}_{g'\cdot T}]$$
$$[\mathcal{O}_{\phi^{-1}(T)}] = \phi^*[\mathcal{O}_{\mathbb{A}^1\times T}].$$

In addition, letting $i_0, i_1 : \coprod_{n+1} Y_F/D_{F*} \to \mathbb{A}^1 \times \coprod_{n+1} Y_F/D_{F*}$ be the 0-section and 1-section, respectively, we have

$$i_0^{-1}\phi^{-1}(T) = f^{-1}(g \cdot T)_F; \quad i_1^{-1}\phi^{-1}(T) = f^{-1}(g' \cdot T)_F.$$

In particular, if we take a finite resolution $\mathcal{E}^* \to \mathcal{O}_T$ by vector bundles on H, we have the resolutions $f_A^*(\tau_{g^{-1}}^*\mathcal{E}_A^*) \to \mathcal{O}_{f^{-1}(g\cdot T)}$, $f_{R'}^*(\tau_{g'^{-1}}^*\mathcal{E}_{R'}^*) \to \mathcal{O}_{f^{-1}(g'\cdot T)}$ and $\phi^*\mathcal{E}_F \to \phi^*(\mathcal{O}_T)$, where τ_g denotes translation by g on H.

The data

$$(f_A^*(\tau_{g^{-1}}^*\mathcal{E}_A^*), f_{R'}^*(\tau_{g'^{-1}}^*\mathcal{E}_{R'}^*), \phi^*\mathcal{E}_F^*)$$

gives an element of $\mathbb{H}^0(X, \mathcal{K}^{(q)}(\coprod_{n+1} Y/D_*, -))$; applying the map (13.7) gives an element of $\mathbb{H}^0(X, \mathcal{K}^{(q)}(Y, D_*, -))$. Mapping to $\mathbb{H}^0(X, \mathcal{K}(Y, D_*))$ by (13.6) gives the element $\bar{T}$ of $\mathbb{H}^0(X, \mathcal{K}(Y, D_*))$. To complete the proof, it suffices to see that $\bar{T}$ is the image of $f^*\eta$ in $\mathbb{H}^0(X, \mathcal{K}(Y, D_*))$.

To see this, define maps

$$\Phi_F : \mathbb{A}^1 \times \mathbb{A}^1 \times \coprod_{n+1} Y_F/D_{F*} \to H_F$$
$$F_A : \mathbb{A}^1 \times \coprod_{n+1} Y_A/D_{A*} \to H_A$$
$$F_{R'} : \mathbb{A}^1 \times \coprod_{n+1} Y_{R'}/D_{R'*} \to H_{R'}$$

by

$$\Phi_F(t_1, t_2, y) = h(t_2, t_1)^{-1} \cdot f(y)$$
$$F_A(t, y) = h(0, t)^{-1} \cdot f(y)$$
$$F_{R'}(t, y) = h(1, t)^{-1} \cdot f(y)$$

Replacing X with $\mathbb{A}^1 \times X$ gives us the sheaf of simplicial spectra on X, $\mathcal{K}(\mathbb{A}^1 \times \coprod_{n+1} Y/D, -)$. The data

$$(F_A^*\mathcal{E}, F_{R'}^*\mathcal{E}, \Phi_F^*\mathcal{E})$$

defines an element of $\mathbb{H}^0(X, \mathcal{K}(\mathbb{A}^1 \times \coprod_{n+1} Y/D, -))$. Passing to $\mathbb{H}^0(X, \mathcal{K}(\mathbb{A}^1 \times Y, \mathbb{A}^1 \times D_*, -))$ and then to $\mathbb{H}^0(X, \mathcal{K}(\mathbb{A}^1 \times Y, \mathbb{A}^1 \times D_*))$ as above, we have the element $\bar{T}'$ of $\mathbb{H}^0(X, \mathcal{K}(\mathbb{A}^1 \times Y, \mathbb{A}^1 \times D_*))$. From the properties of h, it is clear that the restriction of $\bar{T}'$ to $1 \times Y$ is $\bar{T}$, and the restriction to $0 \times Y$ is the image of $f^*\eta$ in $\mathbb{H}^0(X, \mathcal{K}(Y, D_*))$.

Using the homotopy property of K-theory, we see that the pull-back

$$p^* : \mathbb{H}^0(X, \mathcal{K}(Y, D_*)) \to \mathbb{H}^0(X, \mathcal{K}^{(q)}(\mathbb{A}^1 \times Y, \mathbb{A}^1 \times D_*))$$

is an isomorphism. Thus $\bar{T}$ equals the image of $f^*\eta$ in $\mathbb{H}^0(X, \mathcal{K}(Y, D_*))$, completing the proof. □

Corollary 13.11. *Let X be a scheme which is a localization of a smooth scheme of finite type over a regular one-dimensional scheme B. Then*

$$F^q_{\mathrm{HC}} K_n(X) \supset F^q_\gamma K_n(X)$$

for all n and q.

Proof. By Proposition 11.5, the natural map $\mathcal{K}^{(q)}(X, -) \to \mathcal{K}^{(q)}(X, -, -)$ is a weak equivalence. Combining this with the Friedlander-Suslin map, we have the natural map

$$\tilde{\Phi}^{(q)}_n : \mathbb{H}^0(X, \mathcal{K}^{(q)}(X \times \Delta^n, X \times \partial^n; -)) \to \mathbb{H}^{-n}(X, \mathcal{K}^{(q)}(X, -)).$$

Taking $q = 0$, the homotopy and Mayer-Vietoris properties of K-theory yields the isomorphism $K_n(X) \cong \mathbb{H}^{-n}(X, \mathcal{K}(X, -))$. Thus, we have the natural map

$$\tilde{\Phi}_n : \mathbb{H}^0(X, \mathcal{K}(X \times \Delta^n, X \times \partial^n)) \to K_n(X).$$

This gives us the commutative diagram

$$\begin{tikzcd}
\mathbb{H}^0(X, \mathcal{K}^{(q)}(X \times \Delta^n, X \times \partial^n; -)) \arrow[r, "\bar{\Phi}^{(q)}_n"] \arrow[d, "\alpha"'] & F^q_{\mathrm{HC}} K_n(X) \arrow[d, "\cap"] \\
\mathbb{H}^0(X, \mathcal{K}(X \times \Delta^n, X \times \partial^n)) \arrow[r, "\tilde{\Phi}_n"] & K_n(X) \\
F^q_\gamma K_0(X \times \Delta^n, X \times \partial^n) \arrow[r, "\tilde{\Phi}_n"] \arrow[u] & F^q_\gamma K_n(X) \arrow[u, "\cup"']
\end{tikzcd}$$

By Lemma 13.3, the bottom horizontal arrow is surjective. By Theorem 13.10, the image of α contains $F^q_\gamma K_0(X \times \Delta^n, X \times \partial^n)$, which proves the corollary. □

14. Computations

In this section, B is a regular noetherian one-dimensional scheme. If $X \to B$ is a B-scheme of finite type, we say $\dim X \leq d$ if each irreducible component X' of X has $\dim X \leq d$.

Lemma 14.1. *Let $f : X \to B$ be a finite type B-scheme. Then $\mathrm{CH}_q(X, p) = 0$ for all $p < 0$.*

Proof. In case B is semi-local, we have $\mathrm{CH}_q(X, p) = H_p(z_q(X, *))$, which gives the result immediately, since $z_q(X, p) = 0$ if $p < 0$.

Let Z be a finite set of closed points of B. We have the localization sequence

$$\ldots \to \mathrm{CH}_q(f^{-1}(Z),p) \to \mathrm{CH}_q(X,p) \to \mathrm{CH}(f^{-1}(B\setminus Z),p) \\ \to \mathrm{CH}_q(f^{-1}(Z),p-1) \to \ldots$$

Since

$$\mathrm{CH}_q(f^{-1}(Z),p) = \oplus_{z\in Z}\mathrm{CH}_q(f^{-1}(z),p),$$

the map

$$\mathrm{CH}_q(X,p) \to \mathrm{CH}(f^{-1}(B\setminus Z),p)$$

is an isomorphism for $p<0$. Let η be the generic point of B. Taking the limit over Z, we have the isomorphism

$$\mathrm{CH}_q(X,p) \to \mathrm{CH}_q(X_\eta,p) = 0$$

for $p<0$, completing the proof. □

Lemma 14.2. *Let $f: X\to B$ be a finite type B-scheme with $\dim X \le d$. Then*

(1) $\mathrm{CH}_q(X,p)=0$ *for all* $q>d$.
(2) $\mathrm{CH}_d(X,p)=0$ *for all* $p\neq 0$.
(3) $\mathrm{CH}_d(X,0)\cong z_d(X)$, *the isomorphism induced by the inclusion of* $z_d(X) = z_d(X,0)$ *into* $z_d(X,*)$.
(4) $\mathrm{CH}_{d-1}(X,p)=0$ *for all* $p\neq 0,1$.

Moreover, if X is regular and $\dim X = d$, then $\mathrm{CH}_{d-1}(X,0)\cong \mathrm{Pic}(X)$, and $\mathrm{CH}_{d-1}(X,1)\cong \Gamma(X,\mathcal{O}_X^*)$. The first isomorphism is induced by the inclusion $z_{d-1}(X) = z_{d-1}(X,0)$ into $z_{d-1}(X,*)$, and the second by the map sending $u\in \Gamma(X,\mathcal{O}_X^*)$ to the graph of the rational map $(\frac{1}{1-u},\frac{u}{u-1}): X\to \Delta^1$.

Proof. Let $X_1,\ldots,X_r$ be the irreducible components of X having dimension d. We have

$$z_q(X,p)\subset z_{p+q}(X\times\Delta^p) = 0 \text{ for } q>d$$
$$z_d(X,p) = z_{p+d}(X\times\Delta^p)\cong \mathbb{Z}^r,$$

with generators the cycles $X_i\times\Delta^p$. The differentials in $z_d(X,*)$ alternate between the identity and the zero map, which proves the assertions (1)-(3) in case B is semi-local.

Let η be the generic point of B. Taking the direct limit of the localization sequences

$$\mathrm{CH}_q(f^{-1}(Z),p)\to \mathrm{CH}_q(X,p)\to \mathrm{CH}_q(f^{-1}(B\setminus Z),p)\to$$

for Z a finite union of closed points of B reduces (1)-(4) to the case of B a single point; this completes the proof of (1)-(3). Using localization on X and the vanishing (1) we similarly reduce (4) to the case of $X=\mathrm{Spec}\,F$, F a field, in which case we have

$$\mathrm{CH}_{-1}(F,p) = \mathrm{CH}^1(F,p) = \begin{cases} F^* & p=1\\ 0 & p\neq 1\end{cases}$$

by [2, Theorem 6.1]. The isomorphism $F^*\to \mathrm{CH}^1(F,1)$ is induced by sending $u\neq 1\in F^*$ to the point $(\frac{1}{1-u},\frac{u}{u-1})\in\Delta^1_F$.

If X is regular and irreducible of dimension d, and $Z \subset X$ is a proper closed subset with complement U, we have the localization sequence

$$0 = \mathrm{CH}_{d-1}(Z,1) \to \mathrm{CH}_{d-1}(X,1) \to \mathrm{CH}_{d-1}(U,1) \to \mathrm{CH}_{d-1}(Z,0)$$
$$\to \mathrm{CH}_{d-1}(X,0) \to \mathrm{CH}_{d-1}(U,0) \to 0$$

Taking the limit over all Z (we can take Z to be pure codimension one) gives the exact sequence

$$0 \to \mathrm{CH}_{d-1}(X,1) \to \mathrm{CH}_{d-1}(k(X),1) = k(X)^* \xrightarrow{\delta} z_{d-1}(X)$$
$$\to \mathrm{CH}_{d-1}(X,0) \to 0$$

As in the case of smooth quasi-projective varities over a field, one can check directly that $\delta(f) = \pm\mathrm{div}(f)$, giving the isomorphisms

$$\mathrm{CH}_{d-1}(X,1) \cong \Gamma(X, O_X^*); \quad \mathrm{CH}_{d-1}(X,0) \cong \mathrm{Pic}(X).$$

□

Lemma 14.3. *Let $f : X \to B$ be a finite type B-scheme. Then $\mathrm{CH}_q(X,p) = 0$ for $p + q < 0$.*

Proof. Using localization as above, we reduce to the case $B = \mathrm{Spec}\, k$, k a field. Since $z_q(X,p)$ is a subgroup of $z_{p+q}(X \times \Delta^p)$, the result in this case is obvious. □

Proposition 14.4. *(1) Let $f : X \to B$ be a finite type B-scheme with $\dim X \le d$. Then the terms $E^1_{p,q}$ in the spectral sequence (2.4) are zero in the following cases:*

(i) $p + q < 0$,
(ii) $p > d$,
(iii) $p = d$ *and* $q \neq -d$,
(iv) $p = d - 1$ *and* $q \neq 1 - d, 2 - d$,
(v) $2p + q < 0$.

Also, $E^1_{d,-d} = z_d(X)$. The terms $E^2_{p,q}$ in the spectral sequence (2.8) are zero in the following cases:

(i) q *odd.*
(ii) $p + q < 0$,
(iii) $q < -2d$,
(iv) $q = -2d$ *and* $p \neq 2d$,
(v) $q = -2d + 2$ *and* $p \neq 2d - 2, 2d - 1$,
(vi) $2p + q < 0$,

and $E^2_{2d,-2d} = z_d(X)$.

(2) Suppose that X is regular, $\dim X = d$. The terms $E_2^{p,q}$ in the spectral sequence (2.9) are zero in the following cases:

(i) q *odd.*
(ii) $p + q > 0$,
(iii) $q > 0$,
(iv) $q = 0$ *and* $p \neq 0$,
(v) $q = -2$ *and* $p \neq 1, 2$,
(vi) $2p + q > 2 \dim X$.

In addition, we have

$$E_2^{0,0} = z^0(X),\ E_2^{2,-2} = \mathrm{Pic}(X),\ E_2^{1,-2} = \Gamma(X, \mathcal{O}_X^*).$$

Proof. Since

$$E^1_{p,q} = \mathrm{CH}_p(X, p+q),\ E^2_{p,q} = \mathrm{CH}_{-q/2}(X, p+q),\ E_2^{p,q} = \mathrm{CH}^{-q/2}(X, -p-q),$$

this follows from Lemma 14.1, Lemma 14.2 and Lemma 14.3. $\square$

Theorem 14.5. *Let $f : X \to B$ be a regular finite type B-scheme with* $\dim X = d$. *Then*

(1) $F^{d+n+1}_{\mathrm{HC}} K_n(X) = F^{d+n+1}_\gamma K_n(X) = 0$.

(2) For $n \geq 2$, $F^2_{\mathrm{HC}} K_n(X) = F^2_\gamma K_n(X) = K_n(X)$. In addition, $F^1_{\mathrm{HC}} K_1(X) = F^1_\gamma K_1(X) = K_1(X)$.

Proof. We first prove (1). Since $F^{d+n+1}_\gamma K_n(X) \subset F^{d+n+1}_{\mathrm{HC}} K_n(X)$ by Corollary 13.11, it suffices to verify the vanishing of $F^{d+n+1}_{\mathrm{HC}} K_n(X)$. This follows from the vanishing of $E_2^{p,-2q}$ for $p - q > \dim X$ (Proposition 14.4(2)(vi)), and the truncated version

$$E_2^{p,q} = \begin{cases} H^p(X, \mathbb{Z}(-q/2)) & -2q \geq r \\ 0 & \text{otherwise} \end{cases} \Longrightarrow \pi_{-p-q}(K^{(r)}(X, -))$$

of the spectral sequence (2.9). This proves (1).

For (2), the identity $F^2_\gamma K_n(X) = K_n(X)$ for $n \geq 2$ is a theorem of Soulé [29, §5.2, Théorèm 4(iv)]; the identity $F^1_\gamma K_1(X) = K_1(X)$ follows from the definition of $F^1_\gamma K_*(X)$ as the kernel of the augmention $K_*(X) \to K_0(X) \to H^0(X, \mathbb{Z})$. The corresponding identities for F^*_{HC} follow from the inclusion $F^*_\gamma K_n(X) \subset F^*_{\mathrm{HC}} K_n(X)$ of Corollary 13.11. $\square$

Let $0 \leq a \leq q \leq b$, $k \geq 2$ be integers, and let

$$n_k^q(a,b) = \prod_{\substack{a \leq i \leq b \\ i \neq q}} k^q - k^i,\ n_k^{\geq q}(a,b) = \prod_{q \leq r \leq b} n_k^r(a,b),$$

$$n^{\geq q}(a,b) = \gcd_{k \geq 2} n_k^{\geq q}(a,b),\ n(a,b) = n^{\geq b}(a,b).$$

It is easy to check that a prime l divides $n(a,b)$ if and only if $l \leq b - a + 1$, so inverting $n(a,b)$ is the same as inverting $(b-a+1)!$.

Lemma 14.6. *Let $0 \leq a \leq b$ be integers, and let M be a $\mathbb{Z}$-module with a collection of commuting endomorphisms ψ_k, $k = 2, 3, \ldots$. Let*

$$M^{(q)} = \{m \in M \mid \psi_k(m) = k^q m,\ k = 2, 3, \ldots\},\ a \leq q \leq b.$$

(1) Suppose that M has a decreasing filtration F^*M, $* \geq 0$ such that

(i) *$\psi_k(F^qM) \subset F^qM$ for all k and q.*
(ii) *$\psi_k = \times k^q$ on $\mathrm{gr}^q M$.*
(iii) *$F^{b+1}M = 0$.*
(iv) *$F^aM = F^0M$.*

Then $M \otimes \mathbb{Z}[\frac{1}{(b-a+1)!}] = \oplus_{q=a}^b (M \otimes \mathbb{Z}[\frac{1}{(b-a+1)!}])^{(q)}$ and $F^rM \otimes \mathbb{Z}[\frac{1}{(b-a+1)!}] = \oplus_{q \geq r} (M \otimes \mathbb{Z}[\frac{1}{(b-a+1)!}])^{(q)}$ for $r = a, \ldots, b$.

(2) Suppose that M has two decreasing filtrations $F_1^*M \subset F_2^*M$, both satisfying (i)-(iv). Then

$$n^{\geq q}(a,b) F_2^q M \subset F_1^q M$$

for all q.

Proof. We first prove (1); we write n_k^q for $n_k^q(a,b)$, etc, and we may assume that M is already a $\mathbb{Z}[\frac{1}{(b-a+1)!}]$-module. Replacing M with F^rM reduces us to proving the statement for M. Choose an integer $k \geq 2$, and consider the operator

$$\Pi_q^k := \frac{1}{n_k^q} \prod_{\substack{a \leq i \leq b \\ i \neq q}} \psi_k - k^i \mathrm{id}$$

on $M \otimes \mathbb{Z}[\frac{1}{n_k}]$. Clearly $\Pi_q^k = (\Pi_q^k)^2$, Π_q^k sends $F^{q+1}M \otimes \mathbb{Z}[\frac{1}{n_k}]$ to zero, and maps $M \otimes \mathbb{Z}[\frac{1}{n_k}]$ into $F^qM \otimes \mathbb{Z}[\frac{1}{n_k}]$, hence the projector Π_q^k defines a map $s_q^k : \mathrm{gr}^qM \otimes \mathbb{Z}[\frac{1}{n_k}] \to F^qM \otimes \mathbb{Z}[\frac{1}{n_k}]$ which is ψ_l-equivariant for all l. It is easy to see that s_q^k is a splitting of the quotient map $F^qM \otimes \mathbb{Z}[\frac{1}{n_k}] \to \mathrm{gr}^qM \otimes \mathbb{Z}[\frac{1}{n_k}]$, hence

$$x = \sum_{q=a}^{b} \Pi_q^k(x), \;\; \psi_l(\Pi_q^k(x)) = l^q x$$

for all $x \in M \otimes \mathbb{Z}[\frac{1}{n_k}]$ and all $l = 2, 3, \ldots$. From this, we see that

$$(M \otimes \mathbb{Z}[\tfrac{1}{n_k}])^{(q)} = s_q^k(\mathrm{gr}^qM \otimes \mathbb{Z}[\tfrac{1}{n_k}]); \quad a \leq q \leq b. \tag{14.1}$$

For $x \in M \otimes \mathbb{Z}[\frac{1}{n_k}, \frac{1}{n_l}]$, we have

$$\psi_l(\Pi_q^k(x)) = l^q x, \;\; \psi_k(\Pi_q^l(x)) = k^q x,$$

from which it follows that

$$\Pi_q^l(x) = \Pi_q^k(x).$$

From this and (14.1), we see that $(M \otimes \mathbb{Z}[\frac{1}{n_k}])^{(q)}$ and $(M \otimes \mathbb{Z}[\frac{1}{n_l}])^{(q)}$ have the same image in $M \otimes \mathbb{Z}[\frac{1}{n_k}, \frac{1}{n_l}]$, namely $(M \otimes \mathbb{Z}[\frac{1}{n_k}, \frac{1}{n_l}])^{(q)}$.

This compatibility of the subspaces $(M \otimes \mathbb{Z}[\frac{1}{n_k}])^{(q)}$ for different k implies that

$$M^{(q)} \otimes \mathbb{Z}[\tfrac{1}{n_k}] = (M \otimes \mathbb{Z}[\tfrac{1}{n_k}])^{(q)}$$

for each k. This implies that

$$M = \oplus_{q=a}^{b} M^{(q)},$$

as desired.

For (2), the map $n_k^q \Pi_k^q$ sends $F_1^{q+1}M$ to zero and maps M into F_2^qM. The map $n_k^q s_k^q$ gives a quasi-splitting of $F_q^1M \to \mathrm{gr}_1^qM$, i.e., the composition

$$\mathrm{gr}_1^qM \xrightarrow{n_k^q s_k^q} F_1^qM \to \mathrm{gr}_1^qM$$

is $n_k^q\mathrm{id}$. Assume by induction that

$$(n_k^{\geq q+1})F_1^{q+1} \subset F_2^{q+1},$$

and take $x \in F_1^qM$. Then $n_k^q x - (n_k^q \Pi_k^q)(x)$ is in $F_1^{q+1}M$, hence

$$(n_k^{\geq q})F_1^q \subset F_2^q,$$

and the induction goes through. Since this holds for all $k \geq 2$, we have

$$n^{\geq q}(a,b)F_1^qM \subset F_2^qM,$$

as desired. $\square$

We let $N_{q,r}$ be the gcd of the integers $k^q(k^r-1)$ $k=2,3,\ldots$. It is easy to see that $N_{q,r}$ divides $[(r+1)!]^q$; in particular, for $s \geq r+1$, $\gcd(N_{q,r}, s!)$ involves only primes $\leq r+1$.

Theorem 14.7. *Let X be a regular B-scheme with $\dim X = d$. Then (with reference to the spectral sequence (2.9)):*

(1) For each $r \geq 1$, $N_{q,r} d_{2r+1}^{p,q} = 0$.

(2) Writing as usual $E_\infty^{p,-2q} = Z_\infty^{p,-2q}/B_\infty^{p,-2q}$, both $E_2^{p,-2q}/Z_\infty^{p-2q}$ and $B_\infty^{p,-2q}$ have finite exponent. Letting $N_1^{p,q}$ be the exponent of $E_2^{p,-2q}/Z_\infty^{p-2q}$ and $N_2^{p,q}$ the exponent of $B_\infty^{p,-2q}$, we have

$$N_1^{p,q} | \prod_{r=1}^{\dim X - p+q-1} N_{q,r}, \quad N_2^{p,q} | \prod_{r=1}^{q-2} N_{q-r,r}.$$

For $p=2q$, we have $Z_\infty^{p,-2q} = E_2^{p,-2q}$. In particular, $E_2^{p,-2q} = E_\infty^{p,-2q}$ after inverting $(q-1)!(\dim X - p+q)!$ (or inverting $(q-1)!$ if $p=2q$).

(3) After inverting $(\dim X + n-1)!$, the group $K_n(X)$ is a direct sum of the k^q eigenspaces for ψ_k (for $n \geq 1$):

$$K_n(X)[\tfrac{1}{(\dim X+n-1)!}] = \oplus_{q=0}^{\dim X+n} K_n(X)^{(q)}[\tfrac{1}{(\dim X+n-1)!}].$$

The filtration $F^*_{\mathrm{HC}} K_n(X)$ induced by the spectral sequence (2.9) is given by

$$F^m_{\mathrm{HC}} K_n(X)[\tfrac{1}{(\dim X+n-1)!}] = \oplus_{q=m}^{\dim X+n} K_n(X)^{(q)}[\tfrac{1}{(\dim X+n-1)!}].$$

For $n=0$, the same holds after inverting $\dim X!$

(4) Suppose that B is semi-local. For $n \geq 1$, we have

$$n^{\geq q}(2, \dim X + n) F^q_{\mathrm{HC}} K_n(X) \subset F^q_\gamma K_n(X) \subset F^q_{\mathrm{HC}} K_n(X)$$

for all $q \geq 3$; for $n \geq 2$ we have

$$F^2_{\mathrm{HC}} K_n(X) = F^2_\gamma K_n(X) = K_n(X),$$

and for $n=1$,

$$F^2_{\mathrm{HC}} K_1(X) = F^2_\gamma K_1(X),\ F^1_{\mathrm{HC}} K_1(X) = F^1_\gamma K_1(X) = K_1(X).$$

For $n=0$, we have

$$n^{\geq q}(1, \dim X + n) F^q_{\mathrm{HC}} K_0(X) \subset F^q_\gamma K_0(X) \subset F^q_{\mathrm{HC}} K_0(X)$$

for all $q \geq 1$.

(5) For $n \geq 1$, we have isomorphisms

$$\mathrm{CH}^q(X,n)[\tfrac{1}{(\dim X+n-1)!}] \cong K_n(X)^{(q)}[\tfrac{1}{(\dim X+n-1)!}].$$

For $n=0$, we have isomorphisms

$$\mathrm{CH}^q(X)[\tfrac{1}{(q-1)!}] \cong \mathrm{gr}^q_{\mathrm{HC}} K_0(X)[\tfrac{1}{(q-1)!}],$$

and

$$\mathrm{CH}^q(X)[\tfrac{1}{\dim X!}] \cong K_0(X)^{(q)}[\tfrac{1}{\dim X!}].$$

Proof. By Theorem 12.12 we have

$$\begin{aligned} k^{q+r} d_{2r+1}^{p,q} &= \psi_k \circ d_{2r+1}^{p,-2q} \\ &= d_{2r+1}^{p,-2q} \circ \psi_k \\ &= k^q d_{2r+1}^{p,q}, \end{aligned}$$

i.e., $k^q(k^r-1)d_{2r+1}^{p,q} = 0$, proving (1). (2) follows from (1) and Proposition 14.4(2), noting that $E_2^{p,-2q} = E_\infty^{p,-2q}$ for $q = 0, 1$.

For (3), the filtration $F_{\mathrm{HC}}^* K_n(X)$ on $K_n(X)$ admits Adams operations, with ψ_k acting on $\mathrm{gr}_{\mathrm{HC}}^q K_n(X) = E_\infty^{-n+2q,-2q}$ by $k^q\mathrm{id}$ (Theorem 12.1). By Theorem 14.5, we have (for $n \geq 2$) $\mathrm{gr}_{\mathrm{HC}}^q K_n(X) = 0$ for $q < 2$ and for $q > \dim X + n$; Lemma 14.6(1) gives the desired splitting of $K_n(X)$ after inverting $(\dim X + n - 1)!$. For $n = 1$, we may split off the term $E_2^{1,-2} = \Gamma(X, \mathcal{O}_X^*)$ by the determinant mapping, and apply the same argument to $F_{\mathrm{HC}}^2 K_1(X)$. For $n = 0$, we may split off the factor $E_2^{0,0} = H^0(X, \mathbb{Z})$ by the rank homomorphism, and use the same argument to split $F_{\mathrm{HC}}^1 K_0(X)$.

For (4), the theory of λ-rings tells us that ψ_k acts by $k^q\mathrm{id}$ on $\mathrm{gr}_\gamma^q K_n(X)$ (see e.g. [1, Proposition 5.3]). From [29, §5.2, Théorèm 4(iv)], we have the same splittings for $\mathrm{gr}_\gamma^1 K_1$ and $\mathrm{gr}_\gamma^0 K_0$ as described above for $\mathrm{gr}_{\mathrm{HC}}^*$; this with Lemma 14.6(2) proves (4).

The assertion (5) follows from (2) and (3). □

Theorem 14.8. *Let X be a finite type B-scheme which is a closed subscheme of a regular B-scheme of finite type (e.g., X quasi-projective over B). Then the spectral sequence* (2.4) *degenerates at E^1, after tensoring with $\mathbb{Q}$.*

Proof. Suppose X is a closed subscheme of a regular finite type B-scheme $f : Y \to B$. We may suppose Y is irreducible; let $d = \dim Y$. Let $Y_X^{(p,q)}$ be the subset of $Y^{(p,q)}$ consisting of those W which are subsets of $X \times \Delta^p \subset Y \times \Delta^p$. We have the evident identity

$$X_{(p,q)} = Y_X^{(p,\dim Y - q)}.$$

Let $G_X^{(q)}(Y,p)$ be the limit of the spectra $G_W(Y,p)$, as W runs over finite unions of elements of $Y_X^{(p,q)}$. The exact functor

$$(i_X \times \mathrm{id})_* : \mathcal{M}_X(p) \to \mathcal{M}_Y(p)$$

induces the natural map

$$i_{X*}(p) : G_{(q)}(X,p) \to G_X^{(\dim Y - q)}(Y,p),$$

which, by [20, Proposition 7.7], is a weak equivalence, naturally in p. Thus, letting $G_X^{(\dim Y - q)}(Y,-)$ be the simplicial spectrum $p \mapsto G_X^{(\dim Y-q)}(Y,p)$, we have the weak equivalence

$$i_{X*} : G_{(q)}(X,-) \to G_X^{(\dim Y - q)}(Y,-),$$

functorial with respect to q.

Replacing G-theory with K-theory, we form the simplicial spectra $K_X^{(q)}(Y,-)$; since Y is regular, Quillen's resolution theorem [24, §4, Corollary 1] tells us that the evident map

$$K_X^{(q)}(Y,-) \to G_X^{(q)}(Y,-)$$

is a weak equivalence. Forming the presheaves $\mathcal{G}_X^{(q)}(Y,-)$ and $\mathcal{K}_X^{(q)}(Y,-)$ on Y in the evident fashion, we have the weak equivalences in $\mathbf{Hot}(Y)$

$$i_{X*}\mathcal{G}_{(\dim Y-q)}(X,-) \to \mathcal{G}_X^{(q)}(Y,-) \to \mathcal{K}_X^{(q)}(Y,-).$$

Thus, the spectral sequence (2.4) for X is isomorphic to the spectral sequence associated to the tower

$$\ldots \to f_*\mathcal{K}_X^{(p+1)}(Y,-) \to f_*\mathcal{K}_X^{(p)}(Y,-) \to \ldots \to f_*\mathcal{K}_X^{(0)}(Y,-) \sim f_*\mathcal{G}(X).$$

Here $f : Y \to B$ is the structure morphism.

Let $z_X^q(Y,p)$ be the subgroup of $z^q(Y,p)$ consisting of those cycles supported on $X \times \Delta^p \subset Y \times \Delta^p$, forming the simplicial abelian group $z_X^q(Y,-)$, and the presheaf of simplicial abelian groups $\mathcal{Z}_X^q(Y,-)$. As above, we have

$$i_{X*}\mathcal{Z}_q(X,-) = \mathcal{Z}_X^{\dim Y-q}(Y,-),$$

and we have the weak equivalence of the cofiber $f_*\mathcal{K}_X^{(p)}(Y,-)/f_*\mathcal{K}_X^{(p+1)}(Y,-)$ with the sheaf of simplicial abelian groups $f_*\mathcal{Z}_X^p(Y,-)$ on B.

The special $K_0(B)$-λ-algebra structure given by Theorem 12.1, together with Remark 12.11, gives Adams operations ψ_k for the spectral sequence

$$E^1_{p,q}(Y_X) = \pi_{p+q}(B; f_*\mathcal{Z}_X^p(Y,-)) \Longrightarrow \pi_{p+q}(B; f_*\mathcal{K}_X^{(0)}(Y,-)) \cong G_{p+q}(X).$$

As in the proof of Theorem 12.12, we can compute the action of ψ_k on the E^1 terms through the action of ψ_k on $K_0^{(p/p-1)}(Y \times \Delta^{p+q})$, where the action is known to be multiplication by k^p by the Adams-Riemann-Roch theorem [8, Theorem 6.3]. Taking $k = 2$, we see that the differentials $d_r^{p,q}$ on $E^r_{p,q}(Y_X) \otimes \mathbb{Q}$ are zero for all p, q and r. This completes the proof. □

Remark 14.9. As in Theorem 14.7, if one has a bound on the embedding dimension of X, one gets more precise information on the primes one needs to invert to force the various differentials in (2.4) to vanish.

Corollary 14.10. *(1) Let $f : X \to B$ be a finite type B scheme which is embeddable as a closed subscheme of a regular finite type B-scheme, and suppose* $\dim X \leq d$. *Let $m \leq d$ be an integer. Then the map*

$$\pi_n(B; f_*\mathcal{G}_{(m)}(X,-)) \otimes \mathbb{Q} \to \pi_n(B; f_*\mathcal{G}_{(d)}(X,-)) \otimes \mathbb{Q} \cong G_n(X) \otimes \mathbb{Q}$$

induced by the tower (2.1) *is injective for all n.*

(2) Let $f : Y \to B$ be a regular B-scheme of finite type, $m \geq 0$ an integer. Then the inclusion

$$\pi_n(B; f_*\mathcal{G}^{(m)}(Y,-)) \otimes \mathbb{Q} \to G_n(Y) \otimes \mathbb{Q} = K_n(Y) \otimes \mathbb{Q}$$

identifies $\pi_n(B; f_*\mathcal{G}^{(m)}(Y,-)) \otimes \mathbb{Q}$ with the subgroup $\oplus_{q \geq m} K_n(Y)^{(q)}_{\mathbb{Q}}$ of $K_n(Y) \otimes \mathbb{Q}$, where $K_n(Y)^{(q)}_{\mathbb{Q}}$ is the k^q-eigenspace of ψ_k on $K_n(Y) \otimes \mathbb{Q}$.

(3) Let Y be as in (2). The map

$$\pi_n(B; f_*\mathcal{G}^{(2)}(Y,-)) \to G_n(Y) = K_n(Y)$$

is injective for $n = 0$, split injective for $n = 1$, and an isomorphism for $n \geq 2$. Similarly, the map

$$\pi_n(B; f_*\mathcal{G}^{(1)}(Y,-)) \to G_n(Y) = K_n(Y)$$

is split injective for $n = 0$ and an isomorphism for $n \geq 1$. The cokernels are as follows:

(1) $m = 2$, $n = 0$, $\mathrm{coker} = \mathrm{Pic}(Y) \oplus z^0(Y)$.
(2) $m = 2$, $n = 1$, $\mathrm{coker} = \Gamma(Y, \mathcal{O}_Y^*)$.
(3) $m = 1$, $n = 0$, $\mathrm{coker} = z^0(Y)$.

Proof. For (1), we may truncate the tower (2.1) to end at $q = m$, giving the spectral sequence

$$E^1_{p,q}(X, m) = \pi_{p+q}(B; f_*\mathcal{Z}_p(X, -)) \Longrightarrow \pi_{p+q}(B; f_*\mathcal{G}_{(m)}(X, -));\ p \leq m,$$

which is evidently a sub-spectral sequence of the spectral sequence $E(X, d)$. Since, by Theorem 14.8, the spectral sequence $E(X, d)$ degenerates at E^1 after tensoring with $\mathbb{Q}$, the same holds for the truncated spectral sequence $E(X, m)$, whence (1).

The proof of (2) is similar, using the spectral sequence (2.9) together with the fact that ψ_k acts by k^p on $E_2^{p,q}$ (Theorem 12.12).

For (3), first take the case $m = 2$; we may assume that Y is irreducible. We use the spectral sequence (2.9), which we denote by $E^{**}(Y)$, and the truncation $E^{**}(Y, 2)$. The only E_2 terms in the full spectral sequence $E^{**}(Y)$ which don't appear in the truncation $E^{**}(Y, 2)$ are

$$E_2^{0,0}(Y) = H^0(Y, \mathbb{Z}(0)),\ E_2^{1,-2}(Y) = H^1(Y, \mathbb{Z}(1)),\ E_2^{2,-2}(Y) = H^2(Y, \mathbb{Z}(1)).$$

Since d_r maps $E_r^{p,q}$ to $E_r^{p+r,q-r+1}$, there are no differentials in $E^{**}(Y)$ which involve these terms. Thus $E_\infty^{p,q}(Y) = E_\infty^{p,q}(Y, 2)$ for all (p, q) which occur in $E^{**}(Y, 2)$, which proves the injectivity, and identifies the image of $\pi_n(B; f_*\mathcal{G}^{(2)}(Y, -))$ in $K_n(Y)$ with the subgroup $F^2K_n(Y)$, where F^* is the filtration induced by the spectral sequence $E^{**}(Y)$. This also shows that $\pi_n(B; f_*\mathcal{G}^{(2)}(Y, -)) \to K_n(Y)$ is an isomorphism for $n \geq 2$.

Similarly, the map $\pi_n(B; f_*\mathcal{G}^{(1)}(Y, -)) \to K_n(Y)$ is injective, and an isomorphism for $n \geq 1$.

Sending $n \in \mathbb{N}$ to the free sheaf $\mathcal{O}_Y^n$ defines a splitting to the rank homomorphism $K_0(Y) \to \mathbb{Z}$, which gives the splitting to the injection $\pi_0(B; f_*\mathcal{G}^{(1)}(Y, -)) \to K_0(Y)$; this also gives the splitting of $K_0(Y)/F^2K_0(Y)$ as $\mathrm{Pic}(Y) \oplus \mathbb{Z}$. Similarly, sending a unit u to the class in $K_1(Y)$ given by the automorphism $\times u$ of $\mathcal{O}_Y$ gives a splitting to the determinant mapping

$$K_1(Y) \to H^0(Y, \mathcal{K}_1(Y)) \cong H^0(Y, \mathcal{O}_Y^*),$$

and gives the splitting of the injection $\pi_1(B; f_*\mathcal{G}^{(2)}(Y, -)) \to K_1(Y)$. $\square$

14.11. **Codimension one.** We have already seen that the codimension one case is somewhat simpler; we continue with this theme.

Lemma 14.12. *Let $f : X \to B$ be a B-scheme of finite type, with $\dim X = d$. Then the natural map*

$$\pi_n(G_{(d-1)}(X, -)) \to \pi_n(B, f_*\mathcal{G}_{(d-1)}(X, -))$$

is an isomorphism for all n. Similarly, the natural map

$$H_n(z_{d-1}(X, *)) \to \mathbb{H}^{-n}(B; f_*\mathcal{Z}_{d-1}(X, *))$$

is an isomorphism for all n.

Proof. We first consider the G-theory. It suffices to show that $G_{(d-1)}(X,-)$ has the localization property, i.e., for each closed subset Z of X, the sequence

$$G_{(d-1)}(Z,-) \xrightarrow{i_{Z*}} G_{(d-1)}(X,-) \xrightarrow{j_U^*} G_{(d-1)}(U,-) \tag{14.2}$$

is a homotopy fiber sequence, where $U = X \setminus Z$. For this, note that, if $W \subset U \times \Delta^p$ is in $U_{(p,\leq d-1)}$, then the closure $\bar{W}$ of W in $X \times \Delta^p$ is in $X_{(p,\leq d-1)}$. Indeed, an irreducible subset W of $U \times \Delta^p$ is in $U_{(p,\leq d-1)}$ if and only if W contains no subscheme of the form $U \times v$, v a vertex of Δ^p. Since this property is clearly inherited by $\bar{W}$, our assertion is verified. It then follows from Quillen's localization theorem [24, §7, Proposition 3.1] and [20, Proposition 7.7] that the sequence

$$G_{(d-1)}(Z,p) \xrightarrow{i_{Z*}} G_{(d-1)}(X,p) \xrightarrow{j_U^*} G_{(d-1)}(U,p)$$

is a homotopy fiber sequence for each p. This implies that (14.2) is a homotopy fiber sequence.

The proof for the cycle complexes is similar: The discussion above shows that the restriction $X_{(p,\leq d-1)} \to U_{(p,\leq d-1)}$ is surjective, giving the exact sequence

$$0 \to z_{(d-1)}(Z,p) \xrightarrow{i_{Z*}} z_{(d-1)}(X,p) \xrightarrow{j_U^*} z_{(d-1)}(U,p) \to 0,$$

and the distinguished triangle

$$z_{(d-1)}(Z,*) \xrightarrow{i_{Z*}} z_{(d-1)}(X,*) \xrightarrow{j_U^*} z_{(d-1)}(U,*).$$

□

Let $X \to B$ be a smooth B scheme. We consider the spectrum $G^{(1)}(X/B,-)$. Let $u \neq 1$ be a global section of $\mathcal{O}_X^*$, and let $Z(u) \subset X \times \Delta^1$ be the graph of the rational map

$$(\frac{1}{1-u}, \frac{u}{u-1}) : X \to \Delta^1.$$

It is clear that $Z(u) \cap (X \times (1,0)) = Z(u) \cap (X \times (0,1)) = \emptyset$, hence the sheaf $\mathcal{O}_{Z(u)}$ determines a point $\mathcal{O}_{Z(u)}$ of $G^{(1)}(X/B,1)$, with

$$\delta_0^*(\mathcal{O}_{Z(u)}) = \delta_1^*(\mathcal{O}_{Z(u)}) = *.$$

This gives us the canonical map

$$\mathcal{O}^1_{Z(u)} : (S^1,*) \to |G^{(1)}(X/B,-)|,$$

and a corresponding class $[\mathcal{O}_{Z(u)}]$ in $\pi_1(G^{(1)}(X/B,-))$.

Lemma 14.13. *Let $X \to B$ be a smooth B-scheme.*
(1) Sending u to $[\mathcal{O}_{Z(u)}]$ defines a group homomorphism

$$\gamma_X : \Gamma(X,\mathcal{O}_X^*) \to \pi_1(G^{(1)}(X/B,-)),$$

natural in X.
(2) The composition

$$\Gamma(X,\mathcal{O}_X^*) \xrightarrow{\gamma_X} \pi_1(G^{(1)}(X/B,-)) \to \pi_1(G^{(1)}(X,-)) \cong K_1(X)$$

is the canonical map $\Gamma(X,\mathcal{O}_X^*) \to K_1(X)$.
(3) Suppose $\dim X \leq 1$ *and that B is either affine, or quasi-projective over a field. Then the composition*

$$\pi_2(G^{(1)}(X/B,-)) \to \pi_2(G^{(1)}(X,-)) \to G_2(X) = K_2(X)$$

is surjective.

Proof. For (1), let u and v be units on X. Let $l(u,v) \subset X \times \Delta^2$ subscheme defined by $uvt_0 + ut_1 + t_2 = 0$. One checks directly that

$$\delta_0^{-1}(l(u,v)) = Z(u),\ \delta_1^{-1}(l(u,v)) = Z(uv),\ \delta_2^{-1}(l(u,v)) = Z(v),$$

giving a homotopy of the composition $\mathcal{O}^1_{Z(v)} * \mathcal{O}^1_{Z(u)}$ with $\mathcal{O}^1_{Z(uv)}$. This proves (1).

For (2), the assertion is natural in $X \to B$, so it suffices to take $B = \operatorname{Spec}\mathbb{Z}$, $X = \operatorname{Spec}\mathbb{Z}[u,u^{-1}]$. In this case, the map $\Gamma(X, \mathcal{O}_X^*) \to K_1(X)$ is an isomorphism, with inverse the determinant map $K_1(X) \to \Gamma(X, \mathcal{O}_X^*)$.

Identifying $(\Delta^1, (1,0), (0,1))$ with $(\mathbb{P}^1 \setminus \{1\}, 0, \infty)$ via the map $(1-t, t) \mapsto (t-1 : t)$, $Z(u)$ is sent to the graph $\Gamma(u)$ of the morphism $(1 : u) : X \to \mathbb{P}^1$, restricted to $\mathbb{P}^1 \setminus \{1\}$. Let $\mathbb{A} = \mathbb{P}^1 \setminus \{1\}$. We have the resolution

$$0 \to \mathcal{I} \to \mathcal{O}_{X\times\mathbb{A}} \to \mathcal{O}_{\Gamma(u)} \to 0;$$

the map

$$\times(uX_0 - X_1/X_0 - X_1) : \mathcal{O}_{X\times\mathbb{A}} \to \mathcal{I} \tag{14.3}$$

is an isomorphism. On $X \times 0$ and $X \times \infty$ the map $\mathcal{I} \to \mathcal{O}$ is an isomorphism; on $X \times \infty$ this agrees with the restriction of (14.3) and on $X \times 0$ the two isomorphisms differ by the automorphism $\times u$. Tracing through the weak equivalence $K(X) \to G(X, -)$ gives (2).

For (3), since X is smooth over B, the assumption $\dim X \le 1$ implies that either B is a point, or that $X \to B$ is étale. If the first case, we have

$$G^{(1)}(X/B, -) = G^{(1)}(X, -)$$

and the result follows from Corollary 14.10 and Lemma 14.12; in the second, we may replace B with X.

Let W be in $B^{(p,2)}$. Then either $W \to B$ is equi-dimensional, or W maps to a closed point b of B. In the second case, W is an element of $b^{(p,1)}$. Thus, if F is a face of Δ^p, and W' is an irreducible component of $W \cap (B \times F)$, then either W' is equi-dimensional over B, or W' is codimension one on $b \times F$ for some closed point b. Since B is either affine or quasi-projective over a field, W is contained in a $W^* \in B^{(p,1)}$. Thus, the map $G^{(2)}(B, -) \to G^{(1)}(B, -)$ factors through $G^{(1)}(B/B, -)$, hence, we need only show that $\pi_2(G^{(2)}(B, -)) \to K_2(B)$ is surjective.

Let η be the generic point of B, and let $\eta_B^{(p,2)}$ be the subset of $\eta^{(p,2)}$ consisting of those W whose closure in $B \times \Delta^p$ are in $B^{(p,2)}$. We let $G^{(2)}(\eta_B, p)$ be the limit of the spectra $G_W(\eta, p)$, as W runs over finite unions of elements of $\eta_B^{(p,2)}$. Similarly, for $U \subset B$ open, let $U_B^{(p,2)}$ be the subset of $U^{(p,2)}$ consisting of those W whose closure in $B \times \Delta^p$ are in $B^{(p,2)}$, giving the spectrum $G^{(2)}(U_B, p)$.

Using Quillen's localization theorem [24, §7, Proposition 3.1], we have the homotopy fiber sequence

$$\prod_{b\in B\setminus U} G^{(1)}(b, p) \to G^{(2)}(B, p) \to G^{(2)}(U_B, p),$$

Forming the simplicial spectrum $G^{(2)}(U_B,-)$ in the evident manner, we have the long exact sequence

$$\begin{aligned}\ldots \to \oplus_{b\in B\setminus U}\pi_2(G^{(1)}(b,-)) &\to \pi_2(G^{(2)}(B,-))\\ &\to \pi_2(G^{(2)}(U_B,-)) \to \oplus_b \pi_1(G^{(1)}(b,-)) \to \ldots\ .\end{aligned}$$

Taking the limit over U gives the long exact sequence

$$\begin{aligned}\ldots \to \oplus_b\pi_2(G^{(1)}(b,-)) &\to \pi_2(G^{(2)}(B,-))\\ &\to \pi_2(G^{(2)}(\eta_B,-)) \to \oplus_b \pi_1(G^{(1)}(b,-)) \to \ldots\ ,\end{aligned} \tag{14.4}$$

where the sum is over the closed points of B. We have a similar sequence for the sheafs on B

$$\begin{aligned}\ldots \to \oplus_b\pi_2(G^{(1)}(b,-)) &\to \pi_2(B;\mathrm{id}_*G^{(2)}(B,-))\\ &\to \pi_2(G^{(2)}(\eta,-)) \to \oplus_b \pi_1(G^{(1)}(b,-)) \to \ldots\end{aligned} \tag{14.5}$$

and the evident map of (14.4) to(14.5).

We consider the truncated version of (2.9)

$$E_2^{p,q} = H^p(\eta,\mathbb{Z}(-q/2)) \Longrightarrow \pi_{-p-q}(G^{(2)}(\eta,-)); q \le -4.$$

The only term with $-p-q=2$ is $H^2(B,\mathbb{Z}(2))$, giving the isomorphism

$$\pi_2(G^{(2)}(\eta,-)) \cong H^2(B,\mathbb{Z}(2)) = \mathrm{CH}^2(k,2).$$

By [23] and [36], the map

$$\lambda(u,v) = (\frac{1}{1-u},\frac{v-u}{(u-1)(v-1)},\frac{v}{v-1}) - (\frac{1}{1-v},\frac{u-v}{(u-1)(v-1)},\frac{u}{u-1})$$

gives an isomorphism of $K_2^M(k)$ with $\mathrm{CH}^2(k,2)$, and the composition

$$K_2^M(k(\eta)) \xrightarrow{\lambda} \mathrm{CH}^2(\eta,2) \cong \pi_2(G^{(1)}(\eta,-)) \to K_2(\eta)$$

is the usual isomorphism induced by cup product $K_1(\eta)\otimes K_1(\eta)\to K_2(\eta)$.

For $u,v\in k(\eta)^*$, let $s_+(u,v)\subset B\times\Delta^2$ be the closure of the graph of the rational map

$$(\frac{1}{1-u},\frac{v-u}{(u-1)(v-1)},\frac{v}{v-1}) : B \to \Delta^2,$$

and let $s_-(u,v)\subset B\times\Delta^2$ be the closure of the graph of the rational map

$$(\frac{1}{1-v},\frac{u-v}{(u-1)(v-1)},\frac{u}{u-1}) : B \to \Delta^2.$$

One checks that $s_+(u,v)$ and $s_-(u,v)$ are in $B^{(2,2)}$ if $u\neq v$ and $\mathrm{div}(u)$ and $\mathrm{div}(v)$ have disjoint support in B.

Using the bilinearity of symbols $\{u,v\}\in K_2^M(k(\eta))$ and the Steinberg relation, one shows (see e.g. the result of Tate [22, Lemma 13.7]) each element $x\in K_2^M(k(\eta))$ can be written as a sum of symbols,

$$x = \sum_{i=1}^r \{u_i,v_i\};\quad u_i,v_i\in k(\eta)^*,$$

such that $\mathrm{div}(u_i)$ and $\mathrm{div}(v_i)$ have disjoint support for each $i = 1, \ldots, r$. Since $\{u, u\} = \{u, -1\}$ and $\{-1, -1\} = \{t, -1\} - \{-t, -1\}$ for all $t \in k(\eta)^*$, we may assume that $u_i \neq v_i$ for all i. Thus, the map

$$\pi_2(G^{(2)}(\eta_B, -)) \to \pi_2(G^{(2)}(\eta, -)) \cong K_2^M(k(\eta))$$

is surjective. Using the map map of (14.4) to(14.5) described above, we see that

$$\pi_2(G^{(2)}(B, -)) \to \pi_2(B; \mathcal{G}^{(2)}(B, -))$$

is surjective. By Corollary 14.10(3), the map $\pi_2(B; \mathcal{G}^{(2)}(B, -)) \to K_2(B)$ is surjective, which completes the proof. $\square$

Appendix A. Basic concepts

A.1. **Spectra and related notions.** We set out some basic notations; for details on the fundamental constructions, we refer the reader to [34] and [14].

Let **Ord** denote the category of finite non-empty ordered sets. We write $[n]$ for the set $\{0, \ldots, n\}$ with the standard ordering; **Ord** is equivalent to the full subcategory with objects $[n]$, $n = 0, 1, \ldots$. Let $\mathcal{S}$ denote the category of simplicial sets, and **Sp** the category of spectra. Let T be a topological space, or more generally, a Grothendieck site. We have the category $\mathbf{PreSp}(T)$ of presheaves of spectra on T. If P is a property of spectra, we say that P holds for a given presheaf of spectra $\mathcal{G}$ if P holds for the sections $\mathcal{G}(U)$ for all open U in T, and similarly for maps or diagrams of presheaves. For example, a map $f : \mathcal{G} \to \mathcal{H}$ is a weak equivalence of presheaves if the map on sections over U is a weak equivalence for each U. We say that P holds stalk-wise if P holds for the stalks $\mathcal{G}_x$ for all $x \in T$, and similarly for morphisms or diagrams. We let $\mathbf{Hot}(T)$ denote the homotopy category of $\mathbf{PreSp}(T)$, localized with respect to stalk-wise weak equivalences. By [14], $\mathbf{Hot}(T)$ has the structure of a closed model category.

We have the hypercohomology functor $\mathbb{H}^n(T; -)$ on $\mathbf{Hot}(T)$. If $\mathcal{G}$ is a presheaf of spectra on T, then $\mathbb{H}^n(T, \mathcal{G})$ is defined by taking a globally fibrant model $\mathcal{G} \to \tilde{\mathcal{G}}$ of $\mathcal{G}$, and then setting $\mathbb{H}^n(T, \mathcal{G}) := \pi_{-n}(\tilde{\mathcal{G}}(T))$. We sometimes write $\pi_n(T, -)$ for $\mathbb{H}^{-n}(T, -)$. Similarly, if $g : T \to T'$ is a map of topological spaces (or sites), we have the object $\mathbb{R}g_*\mathcal{G}$ in $\mathbf{Hot}(T')$ defined as the image of $g_*\tilde{\mathcal{G}}$. This yields the natural isomorphism $\mathbb{H}^n(T', \mathbb{R}g_*\mathcal{G}) \cong \mathbb{H}^n(T, \mathcal{G})$, and the natural map $g_*\mathcal{G} \to \mathbb{R}g_*\mathcal{G}$.

A *simplicial spectrum* is a simplicial object in the category of spectra. This yields the notion of a presheaf of simplicial spectra on a topological space T. The geometric realization functor $\mathcal{G}(-) \mapsto |\mathcal{G}(-)|$ sends simplicial spectra to spectra. We say that a property P holds for a simplicial spectrum $\mathcal{G}(-)$ if P holds for the geometric realization $|\mathcal{G}(-)|$, and similarly for maps and diagrams. These notions extend as above to presheaves of simplicial spectra. In addition, we define the hypercohomology functor on the category of presheaves of simplicial spectra by taking the hypercohomology of the presheaf of geomtric realizations.

The functor sending an abelian group to the associated Eilenberg-Maclane spectrum extends to a functor from the derived category of abelian groups to the homotopy category $\mathbf{Hot}(\mathrm{pt.})$; we have a similar functor from the derived category of sheaves of abelian groups on T to $\mathbf{Hot}(T)$. Via this functor, the two notions of hypercohomology agree.

We conclude this section by recalling a special case of result of Thomason on presheaves of spectra which satisfy the Mayer-Vietoris property.

Proposition A.2 (*cf.* [35])**.** *Let T be a noetherian topological space of finite Krull dimension, let $\mathcal{G}$ be a presheaf of spectra on T, and let $\mathcal{G} \to \tilde{\mathcal{G}}$ be a globally fibrant model. Suppose that $\mathcal{G}$* satisfies Mayer-Vietoris*, that is, for open subsets U, V of T, the diagram*

$$\begin{array}{ccc} \mathcal{G}(U \cup V) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{G}(V) & \longrightarrow & \mathcal{G}(U \cap V) \end{array}$$

is homotopy cartesian. Then $\mathcal{G} \to \tilde{\mathcal{G}}$ is a weak equivalence of presheaves.

Remark A.3. (1) As a consequence of Thomason's Mayer-Vietoris theorem, we have the following statement: Let $\mathcal{G}$ be a sheaf of spectra on T which satisfies Mayer-Vietoris. Then, for all open $U \subset T$, the natural map $\pi_n(\mathcal{G}(U)) \to \mathbb{H}^{-n}(U, \mathcal{G})$ is an isomorphism.

(2) Thomason's theorem has a relative version as well. Let $f : T \to S$ be a map of topological spaces. For $s \in S$, we have the *stalk* of T over s, $j_s : T_s \to T$, defined by

$$T_s := \lim_{\substack{\longleftarrow \\ U \supset f^{-1}(s)}} U.$$

For a presheaf $\mathcal{G}$ on T, let $\mathcal{G}_s = j_s^* \mathcal{G}$. Suppose that, for each $s \in S$, the presheaf $\mathcal{G}_s$ on T_s satisfies Mayer-Vietoris. Then the natural map $f_* \mathcal{G} \to \mathbb{R} f_* \mathcal{G}$ is an isomorphism in $\mathbf{Hot}(S)$. Indeed, if $\mathcal{G} \to \tilde{\mathcal{G}}$ is a globally fibrant model of $\mathcal{G}$, then $j_s^* \mathcal{G} \to j_s^* \tilde{\mathcal{G}}$ is a globally fibrant model of $j_s^* \mathcal{G}$. By Thomason's theorem, $j_s^* \mathcal{G} \to j_s^* \tilde{\mathcal{G}}$ is a weak equivalence of presheaves on T_s. Since $f_* \mathcal{G}_s = j_s^* \mathcal{G}(T_s)$ and $f_* \tilde{\mathcal{G}}_s = j_s^* \tilde{\mathcal{G}}(T_s)$, the map $f_* \mathcal{G} \to f_* \tilde{\mathcal{G}}$ is a stalk-wise weak equivalence on S.

Appendix B. G-theory and K-theory of cosimplicial schemes

Definition B.1. Let $Y : \mathbf{Ord} \to \mathbf{Sch}$ be a cosimplicial scheme. We say that Y is of *finite Tor-dimension* if

(1) For each injective map $g : [r] \to [n]$ in **Ord**, the map $Y(g) : Y^r \to Y^n$ has finite Tor-dimension.
(2) For each surjective map $g : [r] \to [n]$ in **Ord** the map $Y(g) : Y^r \to Y^n$ is flat.

If $f : Y \to X$ is a morphism of cosimplicial schemes, and P is a property of morphisms of schemes (e.g. flat, quasi-projective) we say that f has the property P if $f^p : Y^p \to X^p$ has the property P for each p. We often consider a scheme B as a constant cosimplicial scheme, so we may speak of a cosimplicial B-scheme $Y \to B$.

B.2. **G-theory of cosimplicial schemes.** For a scheme U, let $\mathcal{M}_U$ denote the category of coherent sheaves on U, and $\mathcal{P}_U$ the full subcategory of locally free sheaves. For an exact category $\mathcal{E}$, we have the K-theory spectrum $K(\mathcal{E})$, defined in degree $n+1$ via Waldhousen's multiple Q-construction: $K(\mathcal{E})_{n+1} = \mathcal{N}^n Q^n(\mathcal{E})$.

Let Y be a cosimplicial scheme, and let $U \subset Y^n$ be an open subset. We let $\mathcal{M}_U(\partial)$ be the full subcategory of $\mathcal{M}_U$ with objects the coherent sheafs $\mathcal{F}$ which are $Y(g)$-flat for all morphisms g in **Ord**.

Remark B.3. For a morphism of schemes $f : Y \to Z$, we have the pull-back functor $f^* : \mathcal{M}_Z \to \mathcal{M}_Y$. Given a second morphism $g : Z \to W$, there is a canonical natural isomorphism $\theta_{f,g} : f^* \circ g^* \to (g \circ f)^*$, which makes the assignment $Y \mapsto \mathcal{M}_Y$ into a pseudo-functor. By a standard method (see e.g. [24]), one may transform this pseudo-functor into a functor (on any given small subcategory of schemes) by replacing the categories $\mathcal{M}_Y$ with suitable equivalent categories. We will perform this transformation without explicit mention, allowing us to assume that $Y \mapsto \mathcal{M}_Y$ is a functor. In particular, if Y is a cosimplicial scheme, then for each $g : [r] \to [n]$ in **Ord**, the functor $Y(g)^* : \mathcal{M}_{Y^n}(\partial) \to \mathcal{M}_{Y^r}(\partial)$ is exact, hence the assignment $p \mapsto \mathcal{M}_{Y^p}(\partial)$ extends to a simplicial exact category $\mathcal{M}_Y(\partial)$. Similarly, the assigment $p \mapsto \mathcal{P}_{Y^p}$ extends to a simplicial exact category $\mathcal{P}_Y$.

Definition B.4. Let Y be a cosimplicial scheme. We let $K(Y)$ denote the simplicial spectrum $p \mapsto K(\mathcal{P}(Y^p))$, and $G(Y)$ the simplicial spectrum $p \mapsto K(\mathcal{M}_{Y^p}(\partial))$.

As for schemes, the spectrum $G(Y)$ is contravariantly functorial for flat morphisms of cosimplicial schemes, and $K(Y)$ is contravariantly functorial for arbitrary morphisms of cosimplicial schemes.

We now extend these definitions to theories with supports.

Definition B.5. Let Y be an N-truncated cosimplicial scheme, $0 \leq N \leq \infty$, and let $W_p \subset Y^p$ be a closed subset, for each p, $0 \leq p \leq N$. We call W a *cosimplicial closed subset of* Y if the collection of complements $U^p := Y^p \setminus W_p$, form an open N-truncated cosimplicial subscheme of Y.

Clearly the intersection of a family of cosimplicial closed subsets is a cosimplicial closed subset. Thus, if Y is an N-truncated cosimplicial scheme, and we are given closed subsets $W_p \subset Y_p$, there is a unique minimal cosimplicial closed subset $\bar{W}$ of Y containing all the W_p; we call $\bar{W}$ the cosimplicial closed subset generated by the collection $\{W_p\}$. One can rephrase the condition that a collection of closed subsets $\{W_p \subset Y^p\}$ form a cosimplicial closed subset of Y as: for each morphism $g : [p] \to [q]$ in **Ord**, we have

$$W_p \supset Y(g)^{-1}(W_q).$$

Lemma B.6. *Let Y be an N-truncated cosimplicial scheme, $0 \leq N \leq \infty$, and for each p, let $\mathcal{P}_p$ be a subset of the set of irreducible closed subsets of Y_p. Suppose that, for each $g : [p] \to [q]$ in* **Ord***, $p, q \leq N$, and each $C \in \mathcal{P}_p$, each irreducible component of $Y(g)^{-1}(C)$ is in $\mathcal{P}_q$. Let $\{W_p \subset Y_p,\ p = 0, \ldots, M < \infty\}$ be a collection of closed subsets such that each W_p is a finite union of elements of $\mathcal{P}_p$. Let $\bar{W}$ be the cosimplicial closed subset of Y generated by the $\{W_p\}$. Then $\bar{W}_p$ is a finite union of elements of $\mathcal{P}_p$.*

Proof. Clearly $\bar{W}_p$ is the union of the $Y(g)^{-1}(W_q)$, as $g : [p] \to [q]$ runs over all maps in **Ord** with $q \leq M$. Since this set of maps is finite for each p, the assumption on the sets $\mathcal{P}_q$ and W_q implies the result. □

Definition B.7. Let Y be a cosimplicial scheme, and let W be a cosimplicial closed subset of Y, with open complement $j : U \to Y$. We let $G_W(Y)$ denote the homtopy fiber of $j^* : G(Y) \to G(U)$, and we call $G_W(Y)$ the G-theory spectrum with supports in W. Similarly, we define the K-theory spectrum of Y with supports in W, $K^W(Y)$, as the homotopy fiber of $j^* : K(Y) \to K(U)$.

Let $f : Y \to X$ be a morphism of cosimplicial schemes and W a cosimplicial closed subset of X. The maps $f^* : K(X) \to K(Y)$ and $f^* : K(X \setminus W) \to K(Y \setminus f^{-1}(W))$ give the map of spectra $f^* : K^W(X) \to K^{f^{-1}(W)}(Y)$. Similarly, if f is flat, we have the map $f^* : G_W(X) \to G_{f^{-1}(W)}(Y)$. These pull-back maps satisfy the functoriality $(g \circ f)^* = f^* \circ g^*$, when defined.

Using the contravariant functoriality for flat morphisms, we may form the presheaves on Y^0_{Zar}:

$$\begin{aligned} \mathcal{K}^W(Y) : &\quad U \mapsto K^{W \cap U}(U) \\ \mathcal{G}_W(Y) : &\quad U \mapsto G_{W \cap U}(U). \end{aligned}$$

We conclude this section with some comparison results.

Lemma B.8. *Let Y be a cosimplicial scheme of finite Tor-dimension, such that each Y^r is quasi-projective over Y^0. Let $U \subset Y^r$ be an open subscheme. Then the inclusion $\mathcal{M}_U(\partial) \to \mathcal{M}_U$ induces a weak equivalence $K(\mathcal{M}_U(\partial)) \to K(\mathcal{M}_U)$.*

Proof. Let Y^n_{inj} be the disjoint union:

$$Y^n_{\mathrm{inj}} := \coprod_{g:[r]\to[n]} Y^r$$

where the union is over all injective order-preserving maps g. The maps $Y(g) : Y^r \to Y^n$ define the map $\iota_n : Y^n_{\mathrm{inj}} \to Y^n$. Since each map in **Ord** has a factorization $g_1 \circ g_2$, with g_2 surjective and g_1 injective, it follows that $\mathcal{F}$ is in $\mathcal{M}_U(\partial)$ if and only if $\mathcal{F}$ is flat with respect to the projection $\iota_U : Y^n_{\mathrm{inj}} \times_{Y^n} U \to U$. By assumption, ι_U has finite Tor-dimension, say Tor-dimension $\leq d$.

Since Y^n is quasi-projective over Y^0, Y^n is isomorphic to a locally closed subscheme of $\mathbb{P}^N_{Y^0}$ for some N. Let $\bar{Y}^n \supset Y^n$ be the closure of Y^n in $\mathbb{P}^N_{Y^0}$, let $\bar{\mathcal{F}}$ be the extension of $\mathcal{F}$ to a coherent sheaf on $\bar{Y}^n$, and let $q : \bar{Y}^n \to Y^0$ denote the projection.

For M sufficiently large, the natural map $q^* q_* \bar{\mathcal{F}}(M) \to \bar{\mathcal{F}}(M)$ is surjective. Iterating, we form a resolution of $\bar{\mathcal{F}}$:

$$0 \to \mathcal{P}_d \to \mathcal{P}_{d-1} \to \ldots \to \mathcal{P}_0 \to \bar{\mathcal{F}} \to 0$$

with $\mathcal{P}_i$ of the form $q^*\mathcal{Q}_i$ for some coherent sheaf $\mathcal{Q}_i$ on Y^0, for $i = 0, \ldots, d-1$, and $\mathcal{P}_d$ the kernel of $\mathcal{P}_{d-1} \to \mathcal{P}_{d-2}$. As the unique map $[r] \to [0]$ is surjective, Y^r is flat over Y^0 for all r. Thus, the sheaves $\mathcal{P}_i$ are ι_U-flat for each $i = 0, \ldots, d-1$. Since ι_U has Tor-dimension $\leq d$, it follows that $\mathcal{P}_d$ is also ι_U-flat. Restricting the resolution $\mathcal{P}_*$ to U thus gives a finite resolution of $\mathcal{F}$ by objects in $\mathcal{M}_U(\partial)$; applying Quillen's resolution theorem completes the proof. $\square$

Remark B.9. Most of the properties of the K-theory and G-theory of schemes extend immediately to cosimplicial schemes, using the fact that a map of simplicial spectra which is a term-wise weak equivalence is a weak equivalence on the geometric realizations. For instance:

1. Let Y be a regular cosimplicial scheme (all Y^p are regular). Then the natural map $K(Y) \to G(Y)$ is a weak equivalence.
2. Let $f : E \to Y$ be a flat map of cosimplicial schemes such that each fiber of $f^p : E^p \to Y^p$ is an affine space. Then $f^* : G(Y) \to G(E)$ is a weak equivalence.

Indeed, for (1), since Y^p is regular, it admits an ample family of line bundles. Thus, each coherent sheaf on Y^p admits a finite resolution by locally free coherent sheaves. It follows from Quillen's resolution theorem [24, §4, Corollary 1] that the inclusion $\mathcal{P}_{Y^p} \to \mathcal{M}_{Y^p}$ induces a weak equivalence $K(\mathcal{P}_Y) \to K(\mathcal{M}_Y)$. The result follows from this and Lemma B.8.

The proof of (2) is similar, using the homotopy property to conclude that $G(Y^p) \to G(E^p)$ is a weak equivalence for each p, and then using Lemma B.8.

B.10. **Projective push-forward.** Let $f : Y \to X$ be a projective morphism of cosimplicial schemes. For an open subscheme U of Y^p with complement T, let $V = X^p \setminus f^p(T)$, let $U' = (f^p)^{-1}(V)$ and let $j : U' \to U$ be the inclusion. We define $f^p_* : \mathcal{M}_U \to \mathcal{M}_V$ as the composition

$$\mathcal{M}_U \xrightarrow{j^*} \mathcal{M}_{U'} \xrightarrow{f^p_*} \mathcal{M}_V.$$

We let $\mathcal{M}_U(\partial)_f$ be the full subcategory of $\mathcal{M}_U(\partial)$ consisting of coherent sheaves $\mathcal{F}$ with $R^j f^p_* \mathcal{F} = 0$.

Definition B.11. Let $f : Y \to X$ be a morphism of cosimplicial schemes. We call f *Tor-independent* if, for each $g : [p] \to [q]$ in **Ord**, the diagram

$$\begin{tikzcd} Y^q \arrow[r, "Y(g)"] \arrow[d, "f^q"'] & Y^p \arrow[d, "f^p"] \\ X^q \arrow[r, "X(g)"'] & X^p \end{tikzcd}$$

is cartesian, and $\mathrm{Tor}_j^{\mathcal{O}_{X^p}}(\mathcal{O}_{X^q}, \mathcal{O}_{Y^p}) = 0$ for $j > 0$.

Lemma B.12. *Let $f : Y \to X$ be a projective morphism of cosimplicial schemes of finite Tor-dimension. We suppose that X is quasi-projective over X^0 and that f is Tor-independent.*

(1) *Let $U \subset Y^p$ be an open subscheme. Then the inclusion $\mathcal{M}_U(\partial)_f \to \mathcal{M}_U(\partial)$ induces a weak equivalence $K(\mathcal{M}_U(\partial)_f) \to K(\mathcal{M}_U(\partial))$.*
(2) *Let $U' \subset Y^q$ be an open subset with $U' \subset Y(g)^{-1}(U)$. Then $Y(g)^* : \mathcal{M}_U(\partial) \to \mathcal{M}_{U'}(\partial)$ sends $\mathcal{M}_U(\partial)_f$ to $\mathcal{M}_{U'}(\partial)_f$*
(3) *Let $U \subset Y^p$ be an open subscheme with complement C, and let $V = X^p \setminus f(C)$. Then f^p_* sends $\mathcal{M}_U(\partial)_f$ to $\mathcal{M}_V(\partial)$.*

Proof. Take $\mathcal{F}$ be in $\mathcal{M}_U(\partial)_f$, and take $g : [q] \to [p]$ in **Ord**. If $\mathcal{E}^* \to \mathcal{O}_{X^q} \to 0$ is a finite resolution of $\mathcal{O}_{X^q}$ by locally free $\mathcal{O}_{X^p}$-modules, then $f^{p*}\mathcal{E}^* \to \mathcal{O}_{Y^q} \to 0$ is a finite resolution of $\mathcal{O}_{Y^q}$ by locally free $\mathcal{O}_{Y^p}$-modules. Thus, we may compute $\mathrm{Tor}_*^{\mathcal{O}_{Y^p}}(\mathcal{O}_{Y^q}, \mathcal{F})$ as the sheaf homology of $\mathcal{F} \otimes f^{p*}\mathcal{E}^*$; as $\mathcal{F}$ is $Y(g)$-flat, this complex is a finite resolution of $\mathcal{F} \otimes \mathcal{O}_{Y^q}$ by elements of $\mathcal{M}_U(\partial)_f$. This implies that $Y(g)^*\mathcal{F}$ is in $\mathcal{M}_{U'}(\partial)_f$ for all g, proving (2).

Furthermore, we may apply f^p_* to the acyclic complex $\mathcal{F} \otimes f^{p*}\mathcal{E}^* \to \mathcal{F} \otimes \mathcal{O}_{Y^q} \to 0$, yielding the acyclic complex $f^p_*(\mathcal{F} \otimes f^{p*}\mathcal{E}^*) \to f^p_*(\mathcal{F} \otimes \mathcal{O}_{Y^q}) \to 0$. Since the natural map

$$f^p_*\mathcal{F} \otimes \mathcal{E}^* \to f^p_*(\mathcal{F} \otimes f^{p*}\mathcal{E}^*)$$

is an isomorphism, the natural map

$$f^p_*\mathcal{F} \otimes \mathcal{O}_{X^q} \to f^p_*(\mathcal{F} \otimes \mathcal{O}_{Y^q})$$

is an isomorphism as well; since $f^p_*(\mathcal{F}\otimes f^{p*}\mathcal{E}^*)\to f^p_*(\mathcal{F}\otimes\mathcal{O}_{Y^q})\to 0$ is acyclic, this shows that $f^p_*\mathcal{F}$ is $Y(g)$-flat for all g, proving (3).

For (1), f is projective, so, for some N, we can factor f^p as a closed embedding $Y^p\to X^p\times\mathbb{P}^N$ followed by the projection $X^p\times\mathbb{P}^N\to X^p$. Let $\mathcal{O}_{Y^p}(1)$ be the restriction of $\mathcal{O}(1)$ to Y^p, and let $\mathcal{F}$ be in $\mathcal{M}_U(\partial)$. Since the pull-back $\mathcal{O}_U(1)$ of $\mathcal{O}(1)$ to $(f^p)^{-1}(V)$ is f^p-ample, and j^* is exact, a sufficently high twist $\mathcal{F}(n)$ of $\mathcal{F}$ is in $\mathcal{M}_U(\partial)_f$. On the other hand, we have the free coherent sheaf $\mathcal{E}:=p_{1*}\mathcal{O}(n)$ on X^p and the surjection $p_1^*\mathcal{E}\to\mathcal{O}(n)$. Taking the sheaf-Hom into $\mathcal{O}(n)$ and restricting to U gives the exact sequence

$$0\to\mathcal{O}_U\to\mathcal{O}_U^M(n)\to\mathcal{K}\to 0,$$

with $\mathcal{K}$ locally free (M is the rank of $\mathcal{E}$). Tensoring this sequence with $\mathcal{F}$ yields the exact sequence in $\mathcal{M}_U(\partial)$,

$$0\to\mathcal{F}\to\mathcal{F}(n)^M\to\mathcal{F}_1\to 0.$$

If there is an integer $L>1$ such that $R^jf^p_*(\mathcal{F})=0$ for $j\geq L$, then clearly $R^jf^p_*(\mathcal{F}_1)=0$ for $q\geq L-1$. Since $R^jf^p_*(\mathcal{G})=0$ for $q>N$, and for all coherent sheaves $\mathcal{G}$ on U, it follows that $\mathcal{F}$ admits a resolution

$$0\to\mathcal{F}\to\mathcal{G}_0\to\ldots\to\mathcal{G}_N\to 0$$

with each $\mathcal{G}_i$ in $\mathcal{M}_U(\partial)_f$. The lemma thus follows from Quillen's resolution theorem [24, §4, Corollary 1]. $\square$

Let $f:Y\to X$ be a projective Tor-independent morphism. By Lemma B.12(2), we may form the simplicial exact category $p\mapsto\mathcal{M}_{Y^p}(\partial)_f$. Taking the K-theory spectrum yields the simplicial spectrum $G(Y)_f$. By Lemma B.12(1), the natural map $G(Y)_f\to G(Y)$ is a weak equivalence. By Lemma B.12(3), the functors $f^p_*:\mathcal{M}_{Y^p}(\partial)_f\to\mathcal{M}_{X^p}(\partial)$ define an exact functor of simplicial exact categories $f_*:\mathcal{M}_Y(\partial)_f\to\mathcal{M}_X$, giving the map $f_*:G(Y)_f\to G(X)$.

More generally, if W is a cosimplicial closed subset of Y with complement U, and if $V\subset X$ is the complement of $f(W)$, the same costruction as above gives the commutative diagram

$$\begin{tikzcd}
G(Y) \arrow[r] & G(U) \\
G(Y)_f \arrow[u] \arrow[r] \arrow[d, "f_*"'] & G(U)_f \arrow[u] \arrow[d, "f_*"] \\
G(X) \arrow[r] & G(V)
\end{tikzcd}$$

The top vertical arrows are weak equivalences, and $G(U)_f$ is the K-theory spectrum of the simplicial exact category $p\mapsto\mathcal{M}_{U^p}(\partial)_f$. Letting $G_W(Y)_f$ denote the homotopy fiber of $G(Y)_f\to G(U)_f$, we have the diagram

$$G_W(Y)\xleftarrow{\iota}G_W(Y)_f\xrightarrow{f_*}G_{f(W)}(X).$$

As ι is a weak equivalence, this diagram defines the map $f_*:G_W(Y)\to G_{f(W)}(X)$ in $\mathbf{Hot}(\text{pt.})$. Taking the associated presheaves of spectra on X^0_{Zar}, we have the map

$$f_*:f^0_*\mathcal{G}_W(Y)\to\mathcal{G}_{f(W)}(X)$$

in $\mathbf{Hot}(X^0)$.

Lemma B.13. *Let $g : Z \to Y$, $f : Y \to X$ be projective Tor-independent morphisms of cosimplicial schemes, let W be a cosimplicial closed subset of Z, and let $W'' = (f \circ g)(W)$. Suppose that X, Y and Z are all of finite Tor-dimension, and that X is quasi-projective over X^0. Then $(f \circ g)_* = f_* \circ g_*$ as maps $(f^0 \circ g^0)_* \mathcal{G}_W(Z) \to \mathcal{G}_{W''}(X)$ in $\mathbf{Hot}(X^0)$.*

Proof. It is evident that $f \circ g$ is projective and Tor-independent, so the statement of the lemma makes sense. Let $W' = g(W)$, let $U = Z \setminus W$, $V = Y \setminus W'$. Let $\mathcal{M}_{U^p}(\partial)_{g,f\circ g}$ be the intersection of $\mathcal{M}_{U^p}(\partial)_{f\circ g}$ and $\mathcal{M}_{U^p}(\partial)_g$ in $\mathcal{M}_{U^p}(\partial)$, and define $\mathcal{M}_{Z^p}(\partial)_{g,f\circ g}$. The same argument as for Lemma B.12(2) implies that the $\mathcal{M}_{U^p}(\partial)_{g,f\circ g}$ define a simplicial exact subcategory of $\mathcal{M}_{U^p}(\partial)$, and similarly for the $\mathcal{M}_{Z^p}(\partial)_{g,f\circ g}$. Letting $G(U)_{g,f\circ g}$ and $G(Z)_{g,f\circ g}$ be the associated K-theory simplicial spectra, and $G_W(Z)_{g,f\circ g}$ the homotopy fiber of $G(Z)_{g,f\circ g} \to G(U)_{g,f\circ g}$, the same argument as for Lemma B.12(1) shows that $G_W(Z)_{g,f\circ g} \to G_W(Z)$ is a weak equivalence.

Additionally, the map g^p_* sends $\mathcal{M}_{U^p}(\partial)_{g,f\circ g}$ to $\mathcal{M}_{V^p}(\partial)_f$ and $\mathcal{M}_{Z^p}(\partial)_{g,f\circ g}$ to $\mathcal{M}_{Y^p}(\partial)_f$, giving the map $g_* : G_W(Z)_{g,f\circ g} \to G_{W'}(Y)_g$, evidently compatible with the map $g_* : G_W(Z)_g \to G_{W'}(Y)$.

Consider the diagram

$$\begin{tikzcd}
G_W(Z)_{g,f\circ g} \arrow[r, "g_*"] \arrow[dr, "(f\circ g)_*"'] & G_{W'}(Y)_g \arrow[d, "f_*"] \\
& G_{W''}(X)
\end{tikzcd}.$$

We have the canonical isomorphisms of functors $\theta^p : f^p_* \circ g^p_* \to (f^p \circ g^p)_*$. These isomorphisms, via [24, §2], give a homotopy of $f_* \circ g_*$ with $(f\circ g)_*$, and this homotopy is natural in the base-scheme X^0, with respect to flat morphisms. This gives us the desired identity $(f \circ g)_* = f_* \circ g_*$ in $\mathbf{Hot}(X^0)$. □

Remark B.14. The notions and constructions described above extend without trouble to multi-cosimplicial schemes. For example, if Y is a bi-simplicial scheme, we have the full subcategory $\mathcal{M}_{Y^{p,q}}(\partial)$ of $\mathcal{M}_{Y^{p,q}}$ consisting of coherent sheaves $\mathcal{F}$ which are $Y(g)$-flat for all $g : [a] \times [b] \to [p] \times [q]$. Via the functors $Y(g)^*$, the assignment $(p, q) \mapsto \mathcal{M}_{Y^{p,q}}(\partial)$ forms a bisimplicial exact catgory; taking the K-theory spectrum yields the bisimplicial spectrum $G(Y)$: $(p, q) \mapsto K(\mathcal{M}_{Y^{p,q}}(\partial))$. If Y is of finite Tor-dimension and $Y^{p,q}$ is quasi-projective over $Y^{0,0}$ for all p, q, then, as above, the map $K(\mathcal{M}_{Y^{p,q}}(\partial)) \to K(\mathcal{M}_{Y^{p,q}})$ is a weak equivalence for all p, q.

B.15. **Compatibilities.** We discuss the compatibility of pull-back and projective pushforward.

Let $f : Y \to X$ be a morphism of cosimplicial schemes. For $U \subset X^p$, let $\mathcal{M}^f_U(\partial)$ be the full subcategory of $\mathcal{M}_U(\partial)$ with objects the f^p-flat coherent sheaves in $\mathcal{M}_U(\partial)$. Clearly $\mathcal{M}^f_U(\partial)$ contains $\mathcal{P}_U$.

Lemma B.16. *Let $f : Y \to X$ be a Tor-independent morphism of cosimplicial schemes of finite Tor-dimension. Suppose that X is quasi-projective over X^0.*

1. *Let U be an open subscheme of X^p for some p. Suppose that X^p is regular. Then the inclusion $\mathcal{P}_U \to \mathcal{M}^f_U(\partial)$ induces a weak equivalence $K(\mathcal{P}_U) \to K(\mathcal{M}^f_U(\partial))$.*

(2) *Let $U \subset X$ be an open simplicial subscheme. Then the categories $\mathcal{M}^f_{U^p}(\partial)$ form a simplicial exact subcategory of $\mathcal{M}_X(\partial)$.*

(3) *Let $U \subset X$ be an open simplicial subscheme and let $V = f^{-1}(U)$. For $\mathcal{F}$ in $\mathcal{M}^f_{U^p}(\partial)$, $f^{p*}\mathcal{F}$ is in $\mathcal{M}^f_{V^p}(\partial)$.*

Proof. (1) follows from Quillen's resolution theorem, as in the proof of Lemma B.8. The proof of (2) is the same as for Lemma B.12(2). For (3), let $g : [q] \to [p]$ be a map in **Ord**. Since $\mathcal{F}$ is f^p-flat and f is Tor-independent, we have the identity

$$\mathrm{Tor}_j^{\mathcal{V}^p}(f^{p*}\mathcal{F}, \mathcal{O}_{V^q}) = f^{p*}\mathrm{Tor}_j^{\mathcal{O}_{U^p}}(\mathcal{F}, \mathcal{O}_{U^q}) = 0,$$

which proves (3). □

Proposition B.17. *Let*

$$\begin{array}{ccc} T & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & X \end{array}$$

be a cartesian square of finite type cosimplicial B-schemes, with X regular, and with f projective. Suppose that f and g are both Tor-independent, that X is quasi-projective over X^0 and that Z is quasi-projective over Z^0. Let W be a cosimplicial closed subset of Y, and let $W' = g^{-1}(f(W))$. Suppose that, for each p, we have

$$\mathrm{Tor}_j^{\mathcal{O}_{X^p}}(\mathcal{O}_{Y^p}, \mathcal{O}_{Z^p}) = 0; \quad j > 0.$$

Then $g^*f_* = f'_*g'^*$, as maps $f^0_*\mathcal{K}^W(Y) \to g^0_*\mathcal{G}^{W'}(Z)$ in $\mathbf{Hot}(X^0)$.

Proof. To clarify the statement of the proposition, the composition g^*f_* is defined by the diagram

$$f^0_*\mathcal{K}^W(Y) \to f^0_*\mathcal{G}_W(Y) \xrightarrow{f_*} g^0_*\mathcal{G}_{f(W)}(X) \xleftarrow{\sim} g^0_*\mathcal{K}^{f(W)}(X) \xrightarrow{g^*} g^0_*\mathcal{K}^{W'}(Z) \to g^0_*\mathcal{G}_{W'}(Z),$$

and $f'_*g'^*$ is defined by the diagram

$$f^0_*\mathcal{K}^W(Y) \xrightarrow{g'^*} (f^0 \circ g^0)_*\mathcal{K}^{g'^{-1}(W)}(T) \to (f^0 \circ g^0)_*\mathcal{G}_{g'^{-1}(W)}(T)] \xrightarrow{f'_*} g^0_*\mathcal{G}_{W'}(Z).$$

For an open subscheme U of Y^p U, let $\mathcal{P}_{U,f} = \mathcal{M}_U(\partial)_f \cap \mathcal{P}_U$. We note that:

(1) Let $U \subset Y$ be an open cosimplicial subscheme. Then, for each $g : [p] \to [q]$, the pull-back $Y(g)^*$ maps $\mathcal{P}_{U^q,f}$ to $\mathcal{P}_{U^p,f}$.

(2) Let U be an open subscheme of Y^p with complement C, and let $V = X \setminus f^p(C)$. The functor f^p_* sends $\mathcal{P}_{U,f}$ to $\mathcal{M}^g_V(\partial)$.

(3) The inclusion $\mathcal{P}_{U,f} \to \mathcal{P}_U$ induces a weak equivalence $K(\mathcal{P}_{U,f}) \to K(\mathcal{P}_U)$.

(4) Let $V' = (g'^p)^{-1}(U)$. Then the functor $(g'^p)^*$ sends $\mathcal{P}_{U,f}$ to $\mathcal{M}_{V'}(\partial)_{f'}$.

(5) Let U be an open subscheme of Y^p and let $\mathcal{F}$ be in $\mathcal{P}_{U,f}$. Then the natural map $g^{p*}f^p_*\mathcal{F} \to f'^p_*g'^{p*}\mathcal{F}$ is an isomorphism.

(1) follows from Lemma B.12, as does the fact that f^p_* maps $\mathcal{M}_U(\partial)_f$, *a fortiori* $\mathcal{P}_{U,f}$, into $\mathcal{M}_V(\partial)$. For the rest of (2), we have the identity

$$\mathrm{Tor}_q^{\mathcal{O}_V}(f^p_*\mathcal{F}, \mathcal{G}) = \mathrm{Tor}_q^{\mathcal{O}_U}(\mathcal{F}, f^{p*}\mathcal{G}) \tag{B.1}$$

for all coherent sheaves $\mathcal{F}$ on U with $R^qf^p_*\mathcal{F} = 0$, $q > 0$, and all quasi-coherent sheaves $\mathcal{G}$ on V with $\mathrm{Tor}_q^{\mathcal{O}_V}(\mathcal{O}_U, \mathcal{G}) = 0$, $q > 0$. Since $\mathcal{O}_{Y^p}$ and $\mathcal{O}_{Z^p}$ are Tor-independent over $\mathcal{O}_{X^p}$, it follows from the above identity that $f^p_*\mathcal{F}$ is g^p-flat for $\mathcal{F}$ in $\mathcal{P}_{U,f}$, finishing (2).

The proof of (3) is similar to the proof of Lemma B.12(1). The proof of (4), it follows from our assumptions that, for each $g:[q]\to[p]$ in **Ord**, $\mathcal{O}_{Z^q}$ and $\mathcal{O}_{Y^p}$ are Tor-independent over $\mathcal{O}_{X^p}$ (using $(g^p\circ Y(q))^*$ to make $\mathcal{O}_{Z^q}$ an $\mathcal{O}_{X^p}$-module). The proof of (4) and (5) is essentially the same as the proof of Lemma B.12(2): compare $g^{p*}f^p_*\mathcal{F}$ and $f'^p_*g'^{p*}\mathcal{F}$ by taking a finite locally free (quasi-coherent) resolution $\mathcal{E}_*\to\mathcal{O}_{Z^p}$ of $\mathcal{O}_{Z^p}$ as a $\mathcal{O}_{X^p}$-module and comparing the resolution $f^p_*(f^{p*}\mathcal{E}_*\otimes\mathcal{F})\to f'^p_*g'^{p*}\mathcal{F}$ of $f'^p_*g'^{p*}\mathcal{F}$ with the resolution $\mathcal{E}_*\otimes f^p_*\mathcal{F}\to g^{p*}f^p_*\mathcal{F}$ of $g^{p*}f^p_*\mathcal{F}$.

Thus, if W is a cosimplicial closed subset of Y with complement U, we have the simplicial exact category $p\mapsto K(\mathcal{P}_{U,f}$. Let $K^W(Y)_f$ denote the homotopy fiber of $K(\mathcal{P}_{Y,f})\to K(\mathcal{P}_{U,f})$, giving the weak equivalence $K^W(Y)_f\to K^W(Y)$. Similarly, for $V=X\setminus f(W)$, we let $G_{f(W)}(X)^g$ denote the homotopy fiber of $K(\mathcal{M}^g_X(\partial))\to K(\mathcal{M}^g_V(\partial))$, and we have the weak equivalence $G_{f(W)}(X)^g\to G_{f(W)}(X)$.

By (1-4), we have the diagram

$$\begin{tikzcd} K^W(Y)_f \arrow[r, "g^*"] \arrow[d, "f_*"'] & K^{g^{-1}(W)}(T)_{f'} \arrow[d, "f'_*"] \\ G_{f(W)}(X)^g \arrow[r, "g^*"'] & G_{W'}(Z), \end{tikzcd}$$

which commutes up to the natural homotopy given by the natural isomorphism $g^*f_*\to f'_*g'^*$ of (5). This, together with the weak equivalences we have already noted, completes the proof. □

Appendix C. Products

C.1. **Products in K-theory.** We recall Waldhausen's construction [39] of products for the K-theory spectra.

Let $\mathcal{E}$ be an exact category. One may interate the Q-construction on $\mathcal{E}$, forming the k-category $Q^k(\mathcal{E})$. The nerve of $Q^k(\mathcal{E})$ is naturally a k-simplicial set. Waldhausen has constructed a natural weak equivalence $\mathcal{N}Q^k(\mathcal{E})\to\Omega\mathcal{N}Q^{k+1}(\mathcal{E})$.

A bi-exact functor $\cup:\mathcal{E}_1\otimes\mathcal{E}_2\to\mathcal{E}_3$ induces the map of $a+b$-simplicial sets

$$\cup:\mathcal{N}Q^a(\mathcal{E}_1)\times\mathcal{N}Q^b(\mathcal{E}_2)\to\mathcal{N}Q^{a+b}(\mathcal{E}_3).$$

Since $0\otimes M$ and $N\otimes 0$ are canonically isomorphic to the zero object of $\mathcal{E}_3$, we have the natural homotopy equivalence of the restriction of $\cup$ to $Q^a(\mathcal{E}_1)\times 0\vee 0\times Q^b(\mathcal{E}_2)$ to the 0-map. In fact, we may replace the $\mathcal{E}_i$ with equivalent categories having a unique zero-object, in which case the map $\cup$ uniquely factors through

$$\cup:\mathcal{N}Q^a(\mathcal{E}_1)\wedge\mathcal{N}Q^b(\mathcal{E}_2)\to\mathcal{N}Q^{a+b}(\mathcal{E}_3). \tag{C.1}$$

In particular, the diagram

$$K(\mathcal{E}_1)\wedge K(\mathcal{E}_2)=\Omega Q(\mathcal{E}_1)\wedge\Omega Q(\mathcal{E}_2)\xrightarrow{\cup}\Omega^2Q^2(\mathcal{E}_3)\leftarrow K(\mathcal{E}_3) \tag{C.2}$$

defines a product map

$$K(\mathcal{E}_1)\wedge K(\mathcal{E}_2)\xrightarrow{\cup}K(\mathcal{E}_3) \tag{C.3}$$

in the homotopy category of spaces. Doing the same for the iterated Q-construction gives the product in **Hot**(pt.).

C.2. **External products in G-theory.** Let $X \to B$, $Y \to B$ be finite type B-schemes, with Y flat over B, and admitting a B-ample family of line bundles. Let $\mathcal{M}_{Y/B}$ be the exact category of coherent sheaves on Y which are flat over B, and let $G(Y/B)$ denote the K-theory spectrum $K(\mathcal{M}_{Y/B})$. Since Y is flat and finite type over B, and admits an ample family of line bundles, each coherent sheaf on Y admits a finite resolution by sheaves in $\mathcal{M}_{Y/B}$. Thus, the natural map $G(Y/B) \to G(Y)$ is a weak equivalence, by Quillen's resolution theorem [24, §4, Corollary 1].

Since the tensor product

$$\otimes : \mathcal{M}_X \otimes \mathcal{M}_{Y/B} \to \mathcal{M}_{X\times_B Y}$$

is bi-exact, the construction of products in §C.1 gives the natural product of spectra

$$\cup_{X,Y/B} : G(X) \wedge G(Y/B) \to G(X \times_B Y).$$

More generally, let $f : Z \to X$, $g : W \to Y$ be flat morphisms of finite type B-schemes, with W flat over B, and admitting an ample family of line bundles. We have the commutative diagrams

$$\begin{tikzcd}
\Omega^{-1}G(X) \wedge \Omega^{-1}G(Y/B) \arrow[r, "f^*\wedge \mathrm{id}"] \arrow[d, "\mathrm{id}\wedge g^*"'] & \Omega^{-1}G(Z) \wedge \Omega^{-1}G(Y/B) \arrow[d, "\mathrm{id}\wedge g^*"] \\
\Omega^{-1}G(X) \wedge \Omega^{-1}G(W/B) \arrow[r, "f^*\wedge \mathrm{id}"] & \Omega^{-1}G(Z) \wedge \Omega^{-1}G(W/B)
\end{tikzcd} \tag{C.4}$$

$$\begin{tikzcd}
\Omega^{-2}G(X \times_B Y) \arrow[r, "(f\times \mathrm{id})^*"] \arrow[d, "(\mathrm{id}\times g)^*"'] & \Omega^{-2}G(Z \times_B Y) \arrow[d, "(\mathrm{id}\times g)^*"'] \\
\Omega^{-2}G(X \times_B W) \arrow[r, "(f\times \mathrm{id})^*"] & \Omega^{-2}G(Z \times_B W)
\end{tikzcd} \tag{C.5}$$

The natural maps (C.1) give the map of diagram (C.4) to diagram (C.5).

Let D_1 be the diagram (C.4), together with maps of the one-point space $*$ to the three terms other than $\Omega^{-1}G(X) \wedge \Omega^{-1}G(Y)$. Define D_2 to be the similar construction for the diagram (C.5). Let $\mathrm{Fib}(f)$, $\mathrm{Fib}(g)$ denote the homotopy fibers of f and g. We have the natural map

$$\tau : \mathrm{Fib}(f) \wedge \mathrm{Fib}(g) \to \underset{\leftarrow}{\mathrm{holim}}\, D_1.$$

Composing τ with the map

$$\rho : \underset{\leftarrow}{\mathrm{holim}}\, D_1 \to \underset{\leftarrow}{\mathrm{holim}}\, D_2$$

gives the natural map

$$\underset{\leftarrow}{\mathrm{holim}} \cup : \mathrm{Fib}(f) \wedge \mathrm{Fib}(g) \to \underset{\leftarrow}{\mathrm{holim}}\, D_2.$$

As an example, let $C \subset X$, $C' \subset Y$ be closed subsets, $j : U \to X$, $i : V \to Y$ the respective complements. The Mayer-Vietoris property for G-theory gives the natural weak equivalence

$$\sigma : \mathrm{Fib}(\Omega^{-2}G(X \times_B Y) \to \Omega^{-2}G(U \times Y \cup X \times V)) \to \underset{\leftarrow}{\mathrm{holim}}\, D_2.$$

The above construction thus gives the natural map (in the homotopy category of spectra)

$$\Omega^{-2}\cup^{C,C'}_{X,Y/B} : \Omega^{-1}G_C(X) \wedge \Omega^{-1}G_{C'}(Y/B) \to \Omega^{-2}G_{C\times C'}(X \times_B Y). \tag{C.6}$$

Taking Ω^2 thus gives the gives the natural map (in the homotopy category of spectra)

$$\cup_{X,Y/B}^{C,C'} : G_C(X) \wedge G_{C'}(Y/B) \to G_{C\times C'}(X \times_B Y). \tag{C.7}$$

C.3. **Products for cosimplicial schemes.** The map (C.7) has a stronger functoriality than that of a functor to the homotopy category. In fact, the map (C.6) is defined via a "zig-zag diagram" in the category of bi-simplicial spectra, with the wrong-way morphisms being weak equivalences. Each of the terms and morphisms in this diagram is functorial in the tuple (X, C, Y, C'), so the product (C.7) extends to define products on the appropriate homotopy limit or colimit, in case we replace (X, C) and (Y, C') with functors from some small category to pairs of schemes and closed subsets.

Specifically, let X and Y be cosimplicial B-schemes. Suppose that:
(C.8)

1. X and Y are locally of finite-Tor dimension.
2. X^p is quasi-projective over X^0 and Y^p is quasi-projective over Y^0, for all p.
3. Y^p is flat over B, for all p.
4. Y^0 admits a B-ample family of line bundles.

Let $\mathcal{M}_{Y^p/B}(\partial)$ be the full subcategory of $\mathcal{M}_{Y^p}(\partial)$ consisting of the B-flat sheaves. As above, the inclusion $\mathcal{M}_{Y^p/B}(\partial) \to \mathcal{M}_{Y^p}(\partial)$ induces a weak equivalence on the K-theory spectra. We form the simplicial simplicial spectrum $G(Y/B)$: $p \mapsto K(\mathcal{M}_{Y^p/B}(\partial))$.

Following Remark B.14, we have the bi-simplicial spectrum $G(X\times_B Y)$: $(p, p') \mapsto K(\mathcal{M}_{X^p\times_B Y^q}(\partial))$. The bi-exact pairing of exact categories

$$\otimes : \mathcal{M}_{X^p}(\partial) \times \mathcal{M}_{Y^q/B}(\partial) \to \mathcal{M}_{X^p\times_B Y^q}(\partial)$$

yields the map of bisimplicial spectra

$$\cup_{X,Y/B} : G(X) \wedge G(Y/B) \to G(X \times_B Y).$$

Under our assumptions on X and Y, it follows from Remark B.14 that the inclusions $\mathcal{M}_{X^p}(\partial) \to \mathcal{M}_{X^p}$, $\mathcal{M}_{Y^q/B} \to \mathcal{M}_{Y^q}$ and $\mathcal{M}_{X^p\times_B Y^q}(\partial) \to \mathcal{M}_{X^p\times_B Y^q}$ all induce weak equivalences on the associated K-theory spectra. Thus, given cosimplicial closed subsets $C \subset X$, $C' \subset Y$, we have the natural map of bisimplicial spectra

$$\cup : G_C(X) \wedge_\delta G_{C'}(Y/B) \to G_{C\times C'}(X \times_B Y).$$

Taking the associated diagonal simplicial spectra yield the map of simplicial spectra

$$\cup_{X,Y}^{C,C'} : G_C(X) \wedge_\delta G_{C'}(Y/B) \to G_{C\times C'}(X \times_B Y). \tag{C.9}$$

The associativity and commutativity of the tensor product similarly implies that the product (C.9) is associative and commutative in **Hot**(pt.).

Lemma C.4. *Let X and Y cosimplicial B-schemes satisfying* (C.8). *Let $C_1 \subset C_2 \subset X$, $C' \subset Y$ be cosimplicial closed subsets. Let $U = X \setminus C_1$. Then the products*

(C.9) *define a map of distinguished triangles*

$$\begin{aligned}&\big(G_{C_1}(X)\wedge_\delta G_{C'}(Y)\to G_{C_2}(X)\wedge_\delta G_{C'}(Y)\to G_{C_2\cap U}(U)\wedge_\delta G_{C'}(Y)\\&\qquad\qquad\to \Sigma G_{C_1}(X)\wedge_\delta G_{C'}(Y)\big)\\&\to\big(G_{C_1\times C'}(X\times_B Y)\to G_{C_2\times C'}(X\times_B Y)\to G_{(C_2\cap U)\times C'}(U\times Y)\\&\qquad\qquad\to\Sigma G_{C_1\times C'}(X\times_B Y)\big).\end{aligned}$$

Similarly, if $C\subset X$ and $C_1'\subset C_2'\subset Y$ are cosimplicial closed subsets, let $V=Y\setminus C_1'$. Then the products (C.9) *define a map of distinguished triangles*

$$\begin{aligned}&\big(G_C(X)\wedge_\delta G_{C_1'}(Y)\to G_C(X)\wedge_\delta G_{C_2'}(Y)\to G_C(X)\wedge_\delta G_{C_2'\cap V}(V)\\&\qquad\qquad\to\Sigma G_C(X)\wedge_\delta G_{C_1'}(Y)\big)\\&\to\big(G_{C\times C_1'}(X\times_B Y)\to G_{C\times C_2'}(X\times_B Y)\to G_{C\times(C_2'\cap V)}(X\times V)\\&\qquad\qquad\to\Sigma G_{C\times C_1'}(X\times_B Y)\big).\end{aligned}$$

Proof. The argument for the second map of distinguished triangles is similar to that for the first, and is left to the reader. To prove the result, we may replace the simplicial spaces $*\wedge_\delta *$ with the bisimplicial spaces $*\wedge *$, and the simplicial spaces $G_{A\times C}(Z\times_B W)$ with the bi-simplicial spaces $G_{A\times C}(Z\times_B W)$. The necessary commutativities, except for the commutativity of

$$\begin{tikzcd} G_{C_2\cap U}(U)\wedge G_{C'}(Y) \arrow[r, "\partial\wedge\mathrm{id}"] \arrow[d, "\cup"'] & \Sigma G_{C_1}(X)\wedge G_{C'}(Y) \arrow[d, "\cup"] \\ G_{(C_2\cap U)\times C'}(U\times Y) \arrow[r, "\partial"'] & \Sigma G_{C_1\times C'}(X\times_B Y), \end{tikzcd}$$

then follow directly from the naturality of the products in K-theory. We proceed to check this last commutativity.

The boundary map $\partial: G_{C_2\cap U}(U)\to\Sigma G_{C_1}(X)$ may be described as the composition

$$\begin{aligned}G_{C_2\cap U}(U)=\mathrm{Fib}\,(G(U)\to G(U\setminus C_2))\to G(U)\to\mathrm{Cofib}\,(G(X)\to G(U))\\ \sim\Sigma\mathrm{Fib}\,(G(X)\to G(U))=\Sigma G_{C_1}(X).\end{aligned}$$

The boundary map $\partial: G_{(C_2\cap U)\times C'}(U\times Y)\to\Sigma G_{C_1\times C'}(X\times_B Y)$ has a similar description. Thus, we need only check the commutativity of the two diagrams

$$\begin{tikzcd} \mathrm{Fib}\,(G(U)\to G(U\setminus C_2))\wedge G_{C'}(Y) \arrow[r] \arrow[d, "\cup"'] & G(U)\wedge G_{C'}(Y) \arrow[d, "\cup"] \\ \mathrm{Fib}\,\big(G_{U\times C'}(U\times Y)\to G_{(U\setminus C_2)\times C'}((U\setminus C_2)\times Y)\big) \arrow[r] & G_{U\times C'}(U\times Y) \end{tikzcd}$$

and

$$\begin{tikzcd} G(U)\wedge G_{C'}(Y) \arrow[r] \arrow[d, "\cup"'] & \mathrm{Cofib}\,(G(X)\wedge G_{C'}(Y)\to G(U)\wedge G_{C'}(Y)) \arrow[d, "\cup"] \\ G_{U\times C'}(U\times Y) \arrow[r] & \mathrm{Cofib}\,(G_{X\times C'}(X\times Y)\to G_{U\times C'}(U\times Y)) \end{tikzcd}$$

where the products on the homotopy fiber and cofiber are the canonical ones induced by the products on the individual terms, using the naturality of the product in K-theory. The commutativity follows directly from this definition of the product. $\square$

C.5. **Variations.** We have concentrated on the product in G-theory, as this requires the most care. Replacing the categories $\mathcal{M}_{-}(\partial)$ and $\mathcal{M}_{-/B}(\partial)$ with $\mathcal{P}_{-}$, one constructs external products for K-theory of cosimplicial schemes. Due to a lack of Mayer-Vietoris for K-theory in general, one needs to restrict to regular schemes in order to have external products for the K-theory of cosimplicial schemes with support.

Suppose we have projective morphisms $f : X \to X'$, $g : Y \to Y'$. We have the full subcategories

$$\begin{aligned}
&\mathcal{M}_{X^p}(\partial)_f \subset \mathcal{M}_{X^p}(\partial),\\
&\mathcal{M}_{Y^q/B}(\partial)_g \subset \mathcal{M}_{Y^q/B}(\partial),\\
&\mathcal{M}_{X^p\times_B Y^q}(\partial)_{f\times g} \subset \mathcal{M}_{X^p\times_B Y^q}(\partial),
\end{aligned}$$

respectively defined by requiring that $R^j f_* = 0$, $R^j g_* = 0$ or $R^j(f\times g)_* = 0$ for $j > 0$. Making this replacement, we have the maps of simplicial spectra $G_C(X)_f \to G_C(X)$, $G_{C'}(Y/B)_f \to G_{C'}(Y/B)$ and the map of bisimplicial spectra $G_{C\times C'}(X\times Y)_{f\times g} \to G_{C\times C'}(X\times Y)$, which are all term-wise weak equivalences if X and Y satisfy the conditions (C.8).

C.6. **Naturality.** We conclude with a discussion of the naturality of the external products.

Proposition C.7. *Let X and Y, X' and Y' be cosimplicial B-schemes which satisfy the conditions* (C.8)*. Let C be a cosimplicial closed subset of X and D a cosimplicial closed subset of Y.*

(1) *Let $f : X \to X'$ and $g : Y \to Y'$ be projective morphisms and let $C' = f(C)$, $D' = g(D)$. Then the diagram*

$$\begin{tikzcd}
G_C(X)_f \wedge G_D(Y/B)_g \arrow[r, "\cup"] \arrow[d, "f_*\wedge g_*"'] & G_{C\times D}(X\times_B Y)_{f\times g} \arrow[d, "(f\times g)_*"] \\
G_{C'}(X') \wedge G_{D'}(Y'/B) \arrow[r, "\cup"] & G_{C'\times D'}(X'\times_B Y')
\end{tikzcd}$$

commutes (in the homotopy category).

(2) *Let $f : X' \to X$ and $g : Y' \to Y$ be flat morphisms, and let $C' = f^{-1}(C)$, $D' = g^{-1}(D)$. Then the diagram*

$$\begin{tikzcd}
G_C(X) \wedge G_D(Y/B) \arrow[r, "\cup"] \arrow[d, "f^*\wedge g^*"'] & G_{C\times D}(X\times_B Y) \arrow[d, "(f\times g)_*"] \\
G_{C'}(X') \wedge G_{D'}(Y'/B) \arrow[r, "\cup"] & G_{C'\times D'}(X'\times_B Y')
\end{tikzcd}$$

commutes (in the homotopy category).

(3) *Let $f : X' \to X$ and $g : Y' \to Y$ be morphisms, and let $C' = f^{-1}(C)$, $D' = g^{-1}(D)$. Suppose that X^p and Y^q are regular for all p and q. Then*

the diagram

$$\begin{tikzcd} K_C(X) \wedge K_D(Y/B) \arrow[r, "\cup"] \arrow[d, "f^* \wedge g^*"'] & K_{C\times D}(X \times_B Y) \arrow[d, "(f\times g)_*"] \\ G_{C'}(X') \wedge G_{D'}(Y'/B) \arrow[r, "\cup"] & G_{C'\times D'}(X' \times_B Y') \end{tikzcd}$$

commutes (in the homotopy category). If X' and Y' are regular, the same holds with G theory replaced with K-theory.

Proof. (2) and (3) follow easily from the naturality of the products for the K-theory spectra of exact categories, and the existence of the canonical isomorphism $f_1^*\mathcal{F}_1 \boxtimes f_2^*\mathcal{G} \cong (f_1 \times f_2)^*(\mathcal{F} \boxtimes \mathcal{G})$, for morphisms of schemes $f_i : X_i \to Y_i$, and coherent sheaves $\mathcal{F}_i$ on Y_i, $i = 1, 2$. For (1), we first note that the map $g_* : G_D(Y/B) \to G_{D'}(Y'/B)$ is really defined. To see this, just note that a sheaf $\mathcal{F}$ on a B-scheme $p : T \to B$ is flat over B if and only if $p_*\mathcal{F}$ is a torsion free sheaf on B, since B is regular and has Krull dimension at most one. Thus, the functor $g_* : \mathcal{M}_{Y^p}(\partial)_g \to \mathcal{M}_{Y'^p}(\partial)$ sends $\mathcal{M}_{Y^p/B}(\partial)_g$ to $\mathcal{M}_{Y'^p/B}(\partial)$. (1) then follows as above from the natural isomorphism $f_{1*}\mathcal{F}_1 \boxtimes f_{2*}\mathcal{F}_2 \cong (f_1 \times f_2)_*(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ for morphisms of schemes $f_i : X_i \to Y_i$, and coherent sheaves $\mathcal{F}_i$ on X_i, $i = 1, 2$. □

Appendix D. Spectral sequences

In this appendix, we briefly recall the construction of the spectral sequence associated to a tower of spectra

$$X_* := \ldots \to X_p \to \ldots \to X_{N-1} \to X_N, \tag{D.1}$$

and describe how a multiplicative structure on the tower leads to a multiplicative structure on the spectral sequence

D.1. **The spectral sequence.** Given a tower of spectra (D.1) and integers $b \leq a \leq 0$, let $X_{a/b}$ denote the cofiber (in the category of spectra) of the map $X_b \to X_a$. For $b \leq b' \leq a \leq a'$, we have the evident map $p_{a/b,a'/b'} : X_{a/b} \to X_{a'/b'}$. For each integer $r \geq 1$, define

$$E^r_{p,q} := \mathrm{Im}\big(\pi_{p+q}X_{p/p-r} \to \pi_{p+q}X_{p+r-1/p-1}\big).$$

The cofibration sequences $X_{p-1/p-r-1} \to X_{p/p-r-1} \to X_{p/p-1}$ and $X_{p-1/p-2} \to X_{p+r-1/p-2} \to X_{p+r-1/p-1}$ give rise to the commutative diagram

$$\begin{tikzcd} & \pi_{p+q}X_{p/p-r} \arrow[d] & \\ \pi_{p+q}X_{p/p-r-1} \arrow[r] & \pi_{p+q}X_{p/p-1} \arrow[r, "\partial^r_{p,q}"] \arrow[d] & \pi_{p+q-1}X_{p-1/p-r-1} \arrow[d] \\ & \pi_{p+q}X_{p+r-1/p-1} \arrow[r, "\partial"'] & \pi_{p+q-1}X_{p-1/p-2} \arrow[d] \\ & & \pi_{p+q-1}X_{p+r-2/p-2} \end{tikzcd} \tag{D.2}$$

Defining $\delta : \pi_{p+q}X_{p/p-r} \to \pi_{p+q-1}X_{p+r-2/p-2}$ to be the evident composition, the commutativity of the diagram shows that δ descends to a map $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$.

The standard arguments used in the construction of a spectral sequence of a filtered complex (see e.g. [30]) are easily modified to show that the data $\{d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}\}$ defines a spectral sequence of homological type

(D.3) $$E^1_{p,q}(X_*) = \pi_{p+q}X_{p/p-1} \Longrightarrow \pi_{p+q}X_N.$$

which we will refer to as the *spectral sequence of the tower* X_*. The E^∞ term is given by

$$E^\infty_{p,q} = \mathrm{gr}^F_p \pi_{p+q}X_N,$$

where $F_p\pi_nX_N$ is the image of π_nX_p in π_nX_N.

D.2. **Convergence.** In general, the spectral sequence (D.3) is not convergent. However, it is easy to see that (D.3) is strongly convergent if for each n, there is an M with $\pi_nX_p = 0$ for $p < M$.

D.3. **Multiplicative structure.** Recall that a *pairing* of spectral sequences $E = \{E^r_{p,q}, d^r_{p,q}\}$, $E' = \{E'^r_{p,q}, d^r_{p,q}\}$ into a spectral sequence $E'' = \{E''^r_{p,q}, d^r_{p,q}\}$ is given by maps

$$\cup^r : E^r_{p,q} \otimes E'^r_{p',q'} \to E''^r_{p+p',q+q'}$$

with $d^r(a \cup^r b) = d^r(a) \cup b + (-1)^{p+q} a \cup^r d^r(b)$. In case $E = E' = E''$, we call the pairing a *multiplicative structure* on E; we call a multiplicative structure associative if the evident associative holds for the various products $E^r_{p.q} \otimes E^r_{p',q'} \otimes E^r_{p'',q''} \to E^r_{p+p'+p'',q+q'+q''}$.

Definition D.4. Given towers X_*, X'_* and $''_*$, a *pairing* of X_*, X'_* into X''_*, written

$$\cup : X_* \wedge X'_* \to X''_*,$$

is given by a collection of maps in the stable homotopy category

$$\cup_{a/b,a'/b'} : X_{a/b} \wedge X'_{a'/b'} \to X''_{a+a'/b+b'}$$

satisfying

(1) The maps $\cup_{*/*,*/*}$ are compatible with the change-of-index maps, $X_{a/b} \to X_{c/d}$, $X'_{a'/b'} \to X'_{c'/d'}$, in the stable homotopy category,
(2) For each set of indices $a \ge b \ge c$, $a' \ge b' \ge c'$, the maps $\cup_{*/*,*/*}$ define a map of distinguished triangles

$$\begin{aligned}(X_{b/c} \wedge X'_{a'/b'} \to X_{a/c} \wedge X'_{a'/b'} \to X_{a/b} \wedge X'_{a'/b'} \to \Sigma X_{b/c} \wedge X'_{a'/b'})\\ \to (X''_{a'+b/b'+c} \to X''_{a'+a/b'+c} \to X''_{a+a'/b+b'} \to \Sigma X''_{a'+b/b'+c})\end{aligned}$$

and

$$\begin{aligned}(X_{a/b} \wedge X'_{b'/c'} \to X_{a/b} \wedge X'_{a'/c'} \to X_{a/b} \wedge X'_{a'/b'} \to \Sigma X_{a/b} \wedge X'_{b'/c'})\\ \to (X''_{a+b'/b+c'} \to X''_{a'+a/b+c'} \to X''_{a+a'/b+b'} \to \Sigma X''_{a+b'/b+c'})\end{aligned}$$

We call a pairing $\cup : X_* \wedge X_* \to X_*$ a *multiplicative structure* on the tower X_*. We say that a multiplicative structure $\cup : X_* \wedge X_* \to X_*$ is *associative* if the evident associative holds for all the various double products $X_{a/b} \wedge X_{a'/b'} \wedge X_{a''/b''} \to X_{a+a'+a''/b+b'+b''}$.

Remark D.5. The maps in the distinguished triangles above are $\partial \wedge \mathrm{id} : X_{a/b} \wedge X'_{a'/b'} \to \Sigma X_{b/c} \wedge X'_{a'/b'}$ and $\tau \circ (\mathrm{id} \wedge \partial') \circ \tau^{-1} : X_{a/b} \wedge X'_{a'/b'} \to \Sigma X_{a/b} \wedge X'_{b'/c'}$, where $\partial : X_{a/b} \to \Sigma X_{b/c}$ and $\partial' : X'_{a'/b'} \to \Sigma X'_{b'/c'}$ are the maps in the distinguished triangles

$$X_{b/c} \to X_{a/c} \to X_{a/b} \to \Sigma X_{b/c}$$
$$X'_{b'/c'} \to X_{a'/c'} \to X_{a'/b'} \to \Sigma X_{b'/c'}$$

and $\tau : X_{a/b} \wedge \Sigma X'_{b'/c'} \to \Sigma X_{a/b} \wedge X'_{b'/c'}$ is the canonical isomorphism.

In particular, a pairing of towers $\cup : X_* \wedge X'_* \to X''_*$ induces maps for the associated spectral sequences

$$\cup^r : E^r_{p,q} \otimes E'^r_{p',q'} \to E''^r_{p+p',q+q'}.$$

Lemma D.6. *Let $\cup : X_* \wedge X'_* \to X''_*$ be a pairing of towers. Then the products $\cup^r : E^r_{p,q} \otimes E'^r_{p',q'} \to E''^r_{p+p',q+q'}$ induced by the pairing $\cup$ define a pairing of spectral sequence $E \otimes E' \to E''$. In particular, a (associative) multiplicative structure $\cup : X_* \wedge X_* \to X_*$ defines a (associative) multiplicative structure on the spectral sequence $\{E^r_{p,q}\}$.*

Proof. The pairing of towers $\cup : X_* \wedge X'_* \to X''_*$ gives us maps

$$\cup : \pi_n X_{p/p-1} \otimes \pi_m X'_{p'/p'-s} \to \pi_{n+m} X''_{p+p'/p+p'-1},$$
$$\cup : \pi_n X_{p/p-s} \otimes \pi_m X'_{p'/p'-1} \to \pi_{n+m} X''_{p+p'/p+p'-1},$$
$$\cup : \pi_n X_{p/p-s} \otimes \pi_m X'_{p'/p'-s} \to \pi_{n+m} X''_{p+p'/p+p'-s},$$

for each $s \geq 1$.

From the definition of the maps $\cup^r$, we see that it suffices to show

$$\partial''^r_{p+p',q+q'}(a \cup b) = \partial^r_{p,q}(a) \cup b + (-1)^{p+q} a \cup \partial'^r_{p',q'}(b)$$

for $a \in \pi_{p+q}(X_{p/p-1})$, $b \in \pi_{p+q}(X_{p'/p'-1})$, where the maps ∂^r are those in the diagram (D.2), and the maps ∂' and ∂'' are defined similarly with respect to the towers X'_* and X''_*.

Let $E = X_{p/p-r-1}$, $B = X_{p/p-1}$, $F = X_{p-1/p-r-1}$, $E' = X'_{p'/p'-r-1}$, $B' = X'_{p'/p'-1}$, $F' = X'_{p'-1/p'-r-1}$, $E'' = X''_{p+p'/p+p'-r-1}$, $B'' = X''_{p+p'/p+p'-1}$, $F'' = X''_{p+p'-1/p+p'-r-1}$. The various products define a map Φ of the tower

$$F \wedge F' \to X \wedge F' \times F \wedge X' \to\to X \wedge X \to B \wedge B'$$

to the tower

$$* \to F'' \to X'' \to B'',$$

in the homotopy category. Since $\cup$ is a pairing of towers, Φ defines the map of distinguished triangles formed by taking the appropriate cofibers with respect to $F \wedge F'$ in the first tower,

(D.4)

$$\begin{tikzcd}
F \wedge B' \times B \wedge F' \arrow[r] \arrow[d] & X \wedge X'/F \wedge F' \arrow[r] \arrow[d] & B \wedge B' \arrow[r, "\partial *"] \arrow[d, "\cup"'] & \Sigma F \wedge B' \times \Sigma B \wedge F' \arrow[d, "\Sigma\cup\circ p_1 + \Sigma\cup\circ p_2"'] \\
F'' \arrow[r] & X'' \arrow[r] & B'' \arrow[r, "\partial''"'] & \Sigma F''.
\end{tikzcd}$$

Following Remark D.5, we see that

$$\partial * = (\partial \wedge \mathrm{id}, \tau \circ (\mathrm{id} \wedge \partial) \circ \tau^{-1}).$$

Since the exchange of factors $S^{n+m} \cong S^n \wedge S^m \to S^m \wedge S^n \cong S^{m+n}$ induces multiplication by $(-1)^{n+m}$ on $\pi_{n+m}(S^{n+m})$, the lemma follows from the commmutativity of the diagram (D.4). $\square$

Remark D.7. Let X be a scheme. Since all the weak equivalences used in this section arise from finite functorial zig-zag diagrams, all the constructions and results of this section extend without change to towers of presheaves of spectra on X.

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