SUMMER COURSE IN MOTIVIC HOMOTOPY THEORY

MARC LEVINE

Contents

0. Introduction

In this note, we give some algebraic geometry background needed for the construction and understanding of the triangulated category of motives and $\mathbb{A}^1$ homotopy theory. All of this material is well-known and excellently discussed in numerous texts; our goal is to collect the main facts to give the reader a convenient first introduction and quick reference. For this reason, many of the proofs will be only sketched or completely omitted. For further details, we suggest the reader take a

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look at [4], [18], [29] (for commutative algebra), [14], [19] (for algebraic geometry) [9] (for algebraic cycles) and [1], [2] and [3] (for sheaves and Grothendieck topologies).

1. The category of schemes

1.1. **The spectrum of a commutative ring.** Let A be a commutative ring. Recall that an ideal $\mathfrak{p} \subset A$ is a *prime* ideal if

$$ab \in \mathfrak{p}, a \notin \mathfrak{p} \Longrightarrow b \in \mathfrak{p}.$$

This property easily extends from elements to ideals: If I and J are ideals of A, we let IJ be the ideal generated by products ab with $a \in I$, $b \in J$. Then, if $\mathfrak{p}$ is a prime ideal, we have

$$IJ \subset \mathfrak{p}, I \not\subset \mathfrak{p} \Longrightarrow J \subset \mathfrak{p}.$$

Since $IJ \subset I \cap J$, we have as well

$$I \cap J \subset \mathfrak{p}, I \not\subset \mathfrak{p} \Longrightarrow J \subset \mathfrak{p}.$$

In addition to product and intersection, we have the operation of sum: if $\{I_\alpha \mid \alpha \in \mathcal{A}\}$ is a set of ideals of A, we let $\sum_\alpha I_\alpha$ be the smallest ideal of A containing all the ideals I_α. One easily sees that

$$\sum_\alpha I_\alpha = \{\sum_\alpha x_\alpha \mid x_\alpha \in I_\alpha \text{ and almost all } x_\alpha = 0\}.$$

We let $\mathrm{Spec}\,(A)$ denote the set of proper prime ideals of A:

$$\mathrm{Spec}\,(A) := \{\mathfrak{p} \subset A \mid \mathfrak{p} \text{ is prime, } \mathfrak{p} \neq A\}.$$

For a subset S of A, we set

$$V(S) = \{\mathfrak{p} \in \mathrm{Spec}\,(A) \mid \mathfrak{p} \supset S\}.$$

We note the following properties of the operation V:

(1.1.1)

1. Let S be a subset of A, and let $(S) \subset A$ be the ideal generated by S. Then $V(S) = V((S))$.
2. Let $\{I_\alpha \mid \alpha \in \mathcal{A}\}$ be a set of ideals of A. Then $V(\sum_\alpha I_\alpha) = \cap_\alpha V(I_\alpha)$.
3. Let $I_1, \ldots, I_N$ be ideals of A. Then $V(\cap_{j=1}^N I_j) = \cup_{j=1}^N V(I_j)$.
4. $V(0) = \mathrm{Spec}\,(A)$, $V(A) = \emptyset$.

Definition 1.1.1. The *Zariski topology* on $\mathrm{Spec}\,(A)$ is the topology for which the closed subsets are exactly the subsets of the form $V(I)$, I an ideal of A.

It follows from the properties (1.1.1) that this really does define a topology on the set $\mathrm{Spec}\,(A)$.

Examples 1.1.2. (1) Let F be a field. Then (0) is the unique proper ideal in F, and is prime since F is an integral domain ($ab = 0$, $a \neq 0$ implies $b = 0$). Thus $\mathrm{Spec}\,(F)$ is the one-point space $\{(0)\}$.

(2) Again, let F be a field, and let $A = F[t]$ be a polynomial ring in one variable t. A is an integral domain, so (0) is a prime ideal. A is a unique factorization domain (UFD), which means that the ideal (f) generated by an irreducible polynomial f is a prime ideal. In fact, each non-zero proper prime ideal in A is of the form (f), $f \neq 0$, f irreducible. If we take f monic, then different f's give different ideals, so we have

$$\mathrm{Spec}\,(A) = \{(0)\} \cup \{(f) \mid f \in A \text{ a monic irreducible polynomial}\}.$$

Clearly $(0) \subset (f)$ for all such f; there are no other containment relations as the ideal (f) is *maximal* (i.e., not contained in any other non-zero ideal) if f is irreducible, $f \neq 0$. Thus the closure of (0) is all of $\mathrm{Spec}\,(A)$, and the other points $(f) \in \mathrm{Spec}\,(A)$, f is irreducible, $f \neq 0$, are closed points.

(3) Suppose F is algebraically closed, e.g., $F = \mathbb{C}$. Then an irreducible monic polynomial $f \in F[t]$ is necessarily linear, hence $f = t - a$ for some $a \in F$. We thus have a 1-1 correspondence between the closed points of $\mathrm{Spec}\,(F[t])$ and the set F. For this reason, $\mathrm{Spec}\,(F[t])$ is called the *affine line over* F (even if F is not algebraically closed), written $\mathbb{A}^1_F$.

The operation $A \mapsto \mathrm{Spec}\,(A)$ actually defines a contravariant functor from the category of commutative rings to topological spaces. In fact, let $\phi : A \to B$ be a homomorphism of commutative rings, and let $\mathfrak{p} \subset B$ be a proper prime ideal. Then it follows directly from the definition that $\phi^{-1}(\mathfrak{p})$ is a prime ideal of A, and is proper, since $1_A \in \phi^{-1}(\mathfrak{p})$ implies $1_B = \phi(1_A)$ is in $\mathfrak{p}$, which implies $\mathfrak{p} = B$. Thus, sending $\mathfrak{p}$ to $\phi^{-1}(\mathfrak{p})$ defines the map of sets

$$\hat{\phi} : \mathrm{Spec}\,(B) \to \mathrm{Spec}\,(A).$$

In addition, if I is an ideal of A, then $\phi^{-1}(\mathfrak{p}) \supset I$ for some $\mathfrak{p} \in \mathrm{Spec}\,(B)$, if and only if $\mathfrak{p} \supset \phi(I)$. Thus $\hat{\phi}^{-1}(V(I)) = V(\phi(I))$, hence $\hat{\phi}$ is continuous.

The space $\mathrm{Spec}\,(A)$ encodes lots (but not all) of the information regarding the ideals of A. For example, let $I \subset A$ be an ideal. We have the quotient ring A/I and the canonical surjective ring homomorphism $\phi_I : A \to A/I$. If $\bar{J} \subset A/I$ is an ideal, we have the inverse image ideal $J := \phi^{-1}(\bar{J})$; sending $\bar{J}$ to J is then a bijection between the ideals of A/I and the ideals J of A with $J \supset I$. We have

Lemma 1.1.3. *The map* $\hat{\phi}_I : \mathrm{Spec}\,(A/I) \to \mathrm{Spec}\,(A)$ *gives a homeomorphism of* $\mathrm{Spec}\,(A/I)$ *with the closed subspace* $V(I)$ *of* $\mathrm{Spec}\,(A)$*.*

On the other hand, one can have $V(I) = V(J)$ even if $I \neq J$. For an ideal I, the *radical* of I is the ideal

$$\sqrt{I} := \{x \in A \mid x^n \in I \text{ for some integer } n \geq 1\}.$$

It is not hard to see that $\sqrt{I}$ really is an ideal. Clearly, if $\mathfrak{p}$ is prime, and $x^n \in \mathfrak{p}$ for some $n \geq 1$, then $x \in \mathfrak{p}$, so $V(I) = V(\sqrt{I})$, but it is easy to construct examples of ideals I with $I \neq \sqrt{I}$ (e.g. $I = (t^2) \subset F[t]$, F a field). In terms of Spec , the quotient map $A/I \to A/\sqrt{I}$ induces the homeomorphism

$$\mathrm{Spec}\,(A/\sqrt{I}) \to \mathrm{Spec}\,(A/I).$$

In fact, $\sqrt{I} \supset I$ is the largest ideal with this property, since we have the formula

$$\sqrt{I} = \cap_{\mathfrak{p} \supset I} \mathfrak{p}. \tag{1.1.2}$$

The open subsets of $\mathrm{Spec}\,(A)$ have an algebraic interpretation as well. Let S be a subset of A, closed under multiplication and containing 1. Form the *localization* of A with respect to S as the ring of "fractions" a/s with $a \in A$, $s \in S$, where we identify two fraction a/s, a'/s' if there is a third $s'' \in S$ with

$$s''(s'a - sa') = 0.$$

We multiply and add the fractions by the usual rules. (If A is an integral domain, the element s'' is superfluous, but in general it is needed to make sure that the addition of fractions is well-defined). We denote this ring by $S^{-1}A$; sending a to $a/1$ defines the ring homorphism $\phi_S : A \to S^{-1}A$. If f is an element of A, we may take $S = S(f) := \{f^n \mid n = 0, 1, \ldots\}$, and write A_f for $S(f)^{-1}A$. The homomorphism ϕ_S is universal for ring homomorphisms $\psi : A \to B$ such that B is commutative and $\psi(S)$ consists of units in B.

Note that this operation allows one to invert elements one usually doesn't want to invert, for example 0 or non-zero nilpotent elements. In this extreme case, we end up with the 0-ring; in general the ring homomorphism ϕ_S is not injective. However, all is well if we don't invert *zero-divisors*, i.e., an element $a \neq 0$ such that there is a $b \neq 0$ with $ab = 0$. The most well-know case of localization is of course the formation of the quotient field of an integral domain A, where we take $S = A \setminus \{0\}$.

What does localization do to ideals? If $\mathfrak{p} \neq A$ is prime, there are two possibilities: If $\mathfrak{p} \cap S = \emptyset$, then the image of $\mathfrak{p}$ in $S^{-1}A$ generates a proper prime ideal. In fact, if $\mathfrak{q} \subset S^{-1}A$ is the ideal generated by $\phi_S(\mathfrak{p})$, then

$$\mathfrak{p} = \phi_S^{-1}(\mathfrak{q}).$$

If $\mathfrak{p} \cap S \neq \emptyset$, then clearly the image of $\mathfrak{p}$ in $S^{-1}A$ contains invertible elements, hence generates the unit ideal $S^{-1}A$. Thus,

Lemma 1.1.4. *Let $S \subset A$ be a multiplicatively closed subset containing 1. Then $\hat{\phi}_S : \mathrm{Spec}\,(S^{-1}A) \to \mathrm{Spec}\,(A)$ gives a homeomorphism of $\mathrm{Spec}\,(S^{-1}A)$ with the complement of $\cup_{g \in S} V((g))$ in $\mathrm{Spec}\,(A)$.*

In general, the subspace $\hat{\phi}_S(\mathrm{Spec}\,(S^{-1}A))$ is not open, but if $S = S(f)$ for some $f \in A$, then the image of $\hat{\phi}_S$ is the open complement of $V((f))$.

1.2. **Ringed spaces.** Let T be a topological space, and let $\mathrm{Op}(T)$ be the category with objects the open subsets of T and morphisms $V \to U$ corresponding to inclusions $V \subset U$. Recall that a *presheaf* $\mathcal{S}$ (of abelian groups) on T is a functor $\mathcal{S} : \mathrm{Op}(T)^{\mathrm{op}} \to \mathbf{Ab}$. For $V \subset U$, we often denote the homomorphism $\mathcal{S}(U) \to \mathcal{S}(V)$ by $\mathrm{res}_{V,U}$. A presheaf $\mathcal{S}$ is a *sheaf* if for each open covering $U = \cup_\alpha U_\alpha$, the sequence

$$(1.2.1)\quad 0 \to \mathcal{S}(U) \xrightarrow{\prod_\alpha \mathrm{res}_{U_\alpha,U}} \prod_\alpha \mathcal{S}(U_\alpha) \xrightarrow{\prod_{\alpha,\beta} \mathrm{res}_{U_\alpha\cap U_\beta,U_\alpha} - \mathrm{res}_{U_\alpha\cap U_\beta,U_\beta}} \prod_{\alpha,\beta} \mathcal{S}(U_\alpha \cap U_\beta)$$

is exact. Replacing $\mathbf{Ab}$ with other suitable categories, we have sheaves and presheaves of sets, rings, etc.

Let $f : T \to T'$ be a continuous map. If $\mathcal{S}$ is a presheaf on T, we have the presheaf $f_*\mathcal{S}$ on T' with sections

$$f_*\mathcal{S}(U') := \mathcal{S}(f^{-1}(U')).$$

The restriction maps are given by the obvious formula, and it is easy to see that $f_*\mathcal{S}$ is a sheaf if $\mathcal{S}$ is a sheaf.

Definition 1.2.1. A *ringed space* is a pair $(X, \mathcal{O}_X)$, where X is a topological space, and $\mathcal{O}_X$ is a sheaf of rings on X. A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a pair (f, ϕ), where $f : X \to Y$ is a continuous map, and $\phi : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a homomorphism of sheaves of rings on Y.

Explicitly, the condition that $\phi : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a homomorphism of sheaves of rings means that, for each open $V \subset Y$, we have a ring homomorphism

$$\phi(V) : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V)),$$

and for $V' \subset V$, the diagram

$$\begin{array}{ccc} \mathcal{O}_Y(V) & \xrightarrow{\phi(V)} & \mathcal{O}_X(f^{-1}(V)) \\ {\scriptstyle \mathrm{res}_{V',V}}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_{f^{-1}(V'),f^{-1}(V)}} \\ \mathcal{O}_Y(V') & \xrightarrow[\phi(V')]{} & \mathcal{O}_X(f^{-1}(V')) \end{array}$$

commutes.

We want to define a sheaf of rings $\mathcal{O}_X$ on $X = \mathrm{Spec}\,(A)$ so that our functor $A \mapsto \mathrm{Spec}\,(A)$ becomes a faithful functor to the category of ringed spaces. For this, note that the open subsets $X_f := X \setminus V((f))$ form a basis for the topology of X. Indeed, each open subset $U \subset X$ is of the form $U = X \setminus V(I)$ for some ideal I. As $V(I) = \cap_{f\in I} V((f))$, we have

$$U = \cup_{f\in I} X_f.$$

Similarly, given $\mathfrak{p} \in \mathrm{Spec}\,(A)$, the open subsets X_f, $f \notin \mathfrak{p}$ form a basis of neighborhoods of $\mathfrak{p}$ in X. Noting that $X_f \cap X_g = X_{fg}$, we see that this basis is closed under finite intersection.

We start our definition of $\mathcal{O}_X$ by setting

$$\mathcal{O}_X(X_f) := A_f.$$

Suppose that $X_g \subset X_f$. Then $\mathfrak{p} \supset (f) \implies \mathfrak{p} \supset (g)$. Thus, by Lemma 1.1.4, it follows that $\phi_g(f)$ is contained in no proper prime ideal of A_g, hence $\phi_g(f)$ is a unit in A_g. By the universal property of ϕ_f, there is a unique ring homomorphism $\phi_{g,f} : A_f \to A_g$ making the diagram

$$\begin{tikzcd} A \arrow[r, "\phi_f"] \arrow[d, "\phi_g"'] & A_f \arrow[dl, "\phi_{g,f}"] \\ A_g & \end{tikzcd}$$

By uniqueness, we have

$$\phi_{h,f} = \phi_{h,g} \circ \phi_{g,f} \tag{1.2.2}$$

in case $X_h \subset X_g \subset X_f$.

Lemma 1.2.2. *Let f be in A and let $\{g_\alpha\}$ be a set of elements of A such that $X_f = \cup_\alpha X_{fg_\alpha}$. Then the sequence*

$$0 \to A_f \xrightarrow{\prod \phi_{fg_\alpha,f}} \prod_\alpha A_{fg_\alpha} \xrightarrow{\prod \phi_{fg_\alpha g_\beta,fg_\alpha} - \phi_{fg_\alpha g_\beta,fg_\beta}} \prod_{\alpha,\beta} A_{fg_\alpha g_\beta}$$

is exact.

Proof. See [29] or [4] □

We thus have the "partially defined" sheaf $\mathcal{O}_X(X_f) = A_f$, satisfying the sheaf axiom for covers consisting of the basic open subsets. Let now $U \subset X$ be an arbitrary open subset. Let $I(U)$ be the ideal

$$I(U) := \cap_{\mathfrak{p}\notin U}\mathfrak{p},$$

so $U = X \setminus V(I(U))$. Write U as a union of basic open subsets:

$$U = \cup_{f\in I(U)} X_f,$$

and define $\mathcal{O}_X(U)$ as the kernel of the map

$$\prod_{f\in I(U)} \mathcal{O}_X(X_f) \xrightarrow{\phi_{fg,f}-\phi_{fg,g}} \prod_{f,g\in I(U)} \mathcal{O}_X(X_{fg}).$$

If we have $V \subset U$, then $I(V) \subset I(U)$, and so we have the projections

$$\pi_{V,U} : \prod_{f\in I(U)} \mathcal{O}_X(X_f) \to \prod_{f\in I(V)} \mathcal{O}_X(X_f)$$

$$\pi'_{V,U} : \prod_{f,g\in I(U)} \mathcal{O}_X(X_{fg}) \to \prod_{f,g\in I(V)} \mathcal{O}_X(X_{fg})$$

These in turn induce the map

$$\mathrm{res}_{V,U} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$$

satisfying $\mathrm{res}_{W,V} \circ \mathrm{res}_{V,U} = \mathrm{res}_{W,U}$ for $W \subset V \subset U$.

We now have two defintions of $\mathcal{O}_X(U)$ in case $U = X_f$, but by Lemma 1.2.2, these two agree. It remains to check the sheaf axiom for an arbitrary cover of an arbitrary open $U \subset X$, but this follows formally from Lemma 1.2.2. Thus, we have the sheaf of rings $\mathcal{O}_X$ on X.

Let $\psi : A \to B$ be a homomorphism of commutative rings, giving us the continuous map $\hat{\psi} : Y := \mathrm{Spec}\,(B) \to X := \mathrm{Spec}\,(A)$. Take $f \in B$, $g \in A$, and suppose that $\hat{\psi}(Y_f) \subset X_g$. This says that, if $\mathfrak{p}$ is a prime idea in B with $f \notin \mathfrak{p}$, then $\psi(g) \notin \mathfrak{p}$. This implies that $\phi_f(\psi(g))$ is in no prime ideal of B_f, hence $\phi_f(\psi(g))$ is a unit in B_f.

By the universal property of $\phi_g : A \to A_g$, there is a unique ring homomorphism $\psi_{f,g} : A_g \to B_f$ making the diagram

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\phi_g"'] & B \arrow[d, "\phi_f"] \\ A_g \arrow[r, "\psi_{f,g}"'] & B_f \end{tikzcd}$$

commute. One easily checks that the $\psi_{f,g}$ fit together to define the map of sheaves

$$\tilde{\psi} : \mathcal{O}_X \to \hat{\psi}_* \mathcal{O}_Y,$$

giving the map of ringed spaces

$$(\hat{\psi}, \tilde{\psi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_X).$$

The functoriality

$$(\widehat{\psi_1 \circ \psi_2}, \widetilde{\psi_1 \circ \psi_2}) = (\hat{\psi}_2, \tilde{\psi}_2) \circ (\hat{\psi}_1, \tilde{\psi}_1)$$

is also easy to check.

Thus, we have the contravariant functor Spec from commutative rings to ringed spaces. Since $\mathcal{O}_X(X) = A$ if $X = \operatorname{Spec} A$, Spec is clearly a faithful embedding.

1.2.3. *Local rings and stalks.* Recall that a commutative ring $\mathcal{O}$ is called a *local* ring if $\mathcal{O}$ has a unique maximal ideal $\mathfrak{m}$. The field $\mathfrak{k} := \mathcal{O}/\mathfrak{m}$ is the *residue field* of $\mathcal{O}$. A homomorphism $f : \mathcal{O} \to \mathcal{O}'$ of local rings is called a local homomorphism if $f(\mathfrak{m}) \subset \mathfrak{m}'$.

Example 1.2.4. Let A be a commutative ring, $\mathfrak{p} \subset A$ a proper prime ideal. Let $S = A \setminus \mathfrak{p}$; S is then a multiplicatively closed subset of A containing 1. We set $A_\mathfrak{p} := S^{-1}A$, and write $\mathfrak{p}A_\mathfrak{p}$ for the ideal generated by $\phi_S(\mathfrak{p})$.

We claim that $A_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}A_\mathfrak{p}$. Indeed, each proper prime ideal of $A_\mathfrak{p}$ is the ideal generated by $\phi_S(\mathfrak{q})$, for $\mathfrak{q}$ some prime ideal of A with $\mathfrak{q} \cap S = \emptyset$. As this is equivalent to $\mathfrak{q} \subset \mathfrak{p}$, we find that $\mathfrak{p}A_\mathfrak{p}$ is indeed the unique maximal ideal of $A_\mathfrak{p}$.

Definition 1.2.5. Let $\mathcal{F}$ be a sheaf (of sets, abelian, rings, etc.) on a topological space X. The *stalk* of $\mathcal{F}$ at x, written $\mathcal{F}_x$, is the direct limit

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U).$$

Note that, if $(f, \phi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of ringed spaces, and we take $y \in Y$, then $\phi : \mathcal{O}_X \to f_*\mathcal{O}_Y$ induces the homomorphism of stalks

$$\phi_y : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}.$$

Lemma 1.2.6. *Let A be a commutative ring, $(X, \mathcal{O}_X) = \mathrm{Spec}\,(A)$. Then for $\mathfrak{p} \in \mathrm{Spec}\,(A)$, we have*

$$\mathcal{O}_{X,\mathfrak{p}} = A_{\mathfrak{p}}.$$

Proof. We may use the principal open subsets X_f, $f \notin \mathfrak{p}$, to define the stalk $\mathcal{O}_{X,\mathfrak{p}}$:

$$\begin{aligned}\mathcal{O}_{X,\mathfrak{p}} &= \varinjlim_{f \notin \mathfrak{p}} \mathcal{O}_X(X_f)\\ &= \varinjlim_{f \notin \mathfrak{p}} A_f\\ &= (A \setminus \mathfrak{p})^{-1}A\\ &= A_{\mathfrak{p}}.\end{aligned}$$

□

Thus, the ringed spaces of the form $(X, \mathcal{O}_X) = \mathrm{Spec}\,(A)$ are special, in that the stalks of the sheaf $\mathcal{O}_X$ are all *local* rings. The morphisms $(\hat{\psi}, \tilde{\psi})$ coming from ring homomorphisms $\psi : A \to B$ are also special:

Lemma 1.2.7. *Let $(X, \mathcal{O}_X) = \mathrm{Spec}\,A$, $(Y, \mathcal{O}_Y) = \mathrm{Spec}\,B$, and let $\psi : A \to B$ be a ring homomorphism. Take $y \in Y$. Then*

$$\tilde{\psi}_y : \mathcal{O}_{X,\hat{\psi}(y)} \to \mathcal{O}_{Y,y}$$

is a local homomorphism.

Proof. In fact, if y is the prime ideal $\mathfrak{p} \subset B$, then $\hat{\psi}(y)$ is the prime ideal $\mathfrak{q} := \psi^{-1}(\mathfrak{p})$, and $\tilde{\psi}_y$ is just the ring homomorphism

$$\psi_{\mathfrak{p}} : A_{\mathfrak{q}} \to B_{\mathfrak{p}}$$

induced by ψ, using the universal property of localization. Since

$$\psi(\psi^{-1}(\mathfrak{p})) \subset \mathfrak{p},$$

$\psi_{\mathfrak{p}}$ is a local homomorphism. □

1.3. **Schemes.** We can now define our main object:

Definition 1.3.1. A *scheme* is a ringed space $(X, \mathcal{O}_X)$ which is locally Spec of a ring, i.e., for each point $x \in X$, there is an open neighborhood U of x and a commutative ring A such that $(U, \mathcal{O}_U)$ is isomorphic to $\mathrm{Spec}\, A$, where $\mathcal{O}_U$ denotes the restriction of $\mathcal{O}_X$ to a sheaf of rings on U.

A morphism of schemes $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces which is locally of the form $(\hat{\psi}, \tilde{\psi})$ for some homomorphism of commutative rings $\psi : A \to B$, that is, for each $x \in X$, there are neighborhoods U of x and V of $f(x)$ such that f restricts to a map $f_U : (U, \mathcal{O}_U) \to (V, \mathcal{O}_V)$, a homomorphism of commutative rings $\psi : A \to B$ and isomorphisms of ringed spaces

$$(U, \mathcal{O}_U) \xrightarrow{g} \mathrm{Spec}\, B; \quad (V, \mathcal{O}_V) \xrightarrow{h} \mathrm{Spec}\, A$$

making the diagram

$$\begin{tikzcd}
(U, \mathcal{O}_U) \arrow[r, "f_U"] \arrow[d, "g"'] & (V, \mathcal{O}_V) \arrow[d, "h"] \\
\mathrm{Spec}\, B \arrow[r, "(\hat{\psi}, \tilde{\psi})"'] & \mathrm{Spec}\, A
\end{tikzcd}$$

commute.

Note that, as the functor Spec is a faithful embedding, the isomorphism $(U, \mathcal{O}_U) \to \mathrm{Spec}\, A$ required in the first part of Definition 1.3.1 is unique up to unique isomorphism of rings $A \to A'$. Thus, the data is the second part of the definition are uniquely determined (up to unique isomorphism) once one fixes the open neighborhoods U and V.

Definition 1.3.2. A scheme $(X, \mathcal{O}_X)$ isomorphic to $\mathrm{Spec}\,(A)$ for some commutative ring A is called an *affine scheme*. An open subset $U \subset X$ such that $(U, \mathcal{O}_U)$ is an affine scheme is called an *affine open subset* of X.

If $U \subset X$ is an affine open subset, then $(U, \mathcal{O}_U) = \mathrm{Spec}\, A$, where $A = \mathcal{O}_X(U)$.

Let **Sch** denote the category of schemes, and **Aff** $\subset$ **Sch** the full subcategory of affine schemes. Sending A to $\mathrm{Spec}\, A$ thus defines the functor

$$\mathrm{Spec} : \mathbf{Rings}^{\mathrm{op}} \to \mathbf{Aff}$$

Lemma 1.3.3. Spec *is an equivalence of categories; in particular*

$$\mathrm{Hom}_{\mathbf{Rings}}(A, B) \cong \mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}\, B, \mathrm{Spec}\, A).$$

Proof. We have already seen that Spec is a faithful embedding, so it remains to see that Spec is full, i.e., a morphism of affine schemes $f : \operatorname{Spec} B \to \operatorname{Spec} A$ arises from a homomorphism of rings.

For this, write $\operatorname{Spec} A = (X, \mathcal{O}_X)$, $\operatorname{Spec} B = (Y, \mathcal{O}_Y)$. f gives us the ring homomorphism

$$f^*(Y, X) : \mathcal{O}_X(X) \to \mathcal{O}_Y(Y),$$

i.e., a ring homomorphism $\psi : A \to B$, so we need to see that $f = (\hat{\psi}, \tilde{\psi})$.

Take $y \in Y$ and let $x = f(y)$. Then as f locally of the form $(\hat{\phi}, \tilde{\phi})$, the homomorphism $f_y^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is local and $(f_y^*)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_y$, where $\mathfrak{m}_x$ and $\mathfrak{m}_y$ are the respective maximal ideals.

Suppose x is the prime ideal $\mathfrak{p} \subset A$ and y is the prime ideal $\mathfrak{q} \subset B$. We thus have $\mathcal{O}_{Y,y} = B_{\mathfrak{q}}$, $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$, and the diagram

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\phi_{\mathfrak{p}}"'] & B \arrow[d, "\phi_{\mathfrak{q}}"] \\ A_{\mathfrak{p}} \arrow[r, "f_y^*"'] & B_{\mathfrak{q}} \end{tikzcd}$$

commutes. Since f_y^* is local, it follows that $\psi^{-1}(\mathfrak{q}) \subset \mathfrak{p}$; as $\phi_{\mathfrak{p}}$ induces a bijection between the prime ideals of A contained in $\mathfrak{p}$ and the prime ideals of $A_{\mathfrak{p}}$, it follows that $\mathfrak{p} = \psi^{-1}(\mathfrak{q})$. Thus, as maps of topological spaces, f and $\hat{\psi}$ agree.

We note that the map $\psi_{\mathfrak{p},\mathfrak{q}} : A_{\mathfrak{p}} \to A_{\mathfrak{q}}$ is the unique local homomorphism ρ making the diagram

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\phi_{\mathfrak{p}}"'] & B \arrow[d, "\phi_{\mathfrak{q}}"] \\ A_{\mathfrak{p}} \arrow[r, "\rho"'] & B_{\mathfrak{q}} \end{tikzcd}$$

commute. Thus $f_y^* = \psi_{\mathfrak{p},\mathfrak{q}}$. Now, for $U \subset X$ open, the map

$$\mathcal{O}_X(U) \to \prod_{x \in U} \mathcal{O}_{X,x}$$

is injective, and similarly for open subsets V of Y. Thus $f^* = \tilde{\psi}$, completing the proof. $\square$

Remark 1.3.4. It follows from the proof of Lemma 1.3.3 that one can replace the condition in Definition 1.3.1 defining a morphism of schemes with the following:

If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are schemes, a morphism of ringed spaces $(f, \phi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of schemes if for each $x \in X$, the map $\phi_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a *local* homomorphism.

Remark 1.3.5. Let $(X, \mathcal{O}_X)$, and $(Y, \mathcal{O}_Y)$ be arbitrary schemes. We have the rings $A := \mathcal{O}_X(X)$ and $B := \mathcal{O}_Y(Y)$, and each map of schemes $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ induces the ring homomorphism $f^*(X) : A \to B$, giving us the functor

$$\Gamma : \mathbf{Sch} \to \mathbf{Rings}^{\mathrm{op}}.$$

We have seen above that $\Gamma \circ \mathrm{Spec} = \mathrm{id}_{\mathbf{Rings}}$, and that the restriction of Γ to $\mathbf{Aff}$ is the inverse to Spec. More generally, suppose that X is affine, $(X, \mathcal{O}_X) = \mathrm{Spec}\, A$. Then

$$\Gamma : \mathrm{Hom}_{\mathbf{Sch}}(Y, X) \to \mathrm{Hom}_{\mathbf{Rings}}(A, B)$$

is an isomorphism. This is *not* the case in general for non-affine X.

1.3.6. *Local/global principal.* There is an analogy between topological manifolds and schemes, in that a scheme is a locally affine ringed space, while a topological manifold is a locally Euclidean topological space. As for manifolds, one can construct a scheme by *gluing* affine schemes: Let U_α be a collection of schemes, together with open subschemes $U_{\alpha,\beta} \subset U_\alpha$, and isomorphisms

$$g_{\beta,\alpha} : U_{\alpha,\beta} \to U_{\beta,\alpha}$$

satisfying the cocycle condition

$$\begin{aligned} &g_{\beta,\alpha}^{-1}(U_{\beta,\alpha} \cap U_{\beta,\gamma}) = U_{\alpha,\beta} \cap U_{\alpha,\gamma} \\ &g_{\gamma,\alpha} = g_{\gamma,\beta} \circ g_{\beta,\alpha} \text{ on } U_{\alpha,\beta} \cap U_{\alpha,\gamma}. \end{aligned}$$

This allows one to define the underlying topological space of the glued scheme X by gluing the underlying spaces of the schemes U_α, and the structure sheaf $\mathcal{O}_X$ is constructed by gluing the structure sheaves $\mathcal{O}_{U_\alpha}$. If the U_α and the open subschemes $U_{\alpha,\beta}$ are all affine, the entire structure is defined via commutative rings and ring homomorphisms.

Remark 1.3.7. Throughout the text, we will define various properties along these lines by requiring certain conditions hold on some affine open cover of X. It is usually the case that these defining conditions then hold on *every* affine open subscheme (see below) of X, and we will use this fact without further explicit mention.

1.3.8. *Open and closed subschemes.* Let $(X, \mathcal{O}_X)$ be a scheme. An *open subscheme* of X is a scheme of the form $(U, \mathcal{O}_U)$, where $U \subset X$ is an open subspace. A morphism of schemes $j : (V, \mathcal{O}_V) \to (X, \mathcal{O}_X)$ which gives rise to an isomorphism $(V, \mathcal{O}_V) \cong (U, \mathcal{O}_U)$ for some open subscheme $(U, \mathcal{O}_U)$ of $(X, \mathcal{O}_X)$ is called an *open immersion*.

Closed subschemes are a little less straightforward. We first define the notion of a *sheaf of ideals*. Let $(X, \mathcal{O}_X)$ be a ringed space. We have the category of $\mathcal{O}_X$-modules: an $\mathcal{O}_X$-module is a sheaf of abelian groups $\mathcal{M}$ on X, together with a map of sheaves

$$\mathcal{O}_X \times \mathcal{M} \to \mathcal{M}$$

which is associative and unital (in the obvious sense). Morphisms are maps of sheaves respecting the multiplication. Now suppose that $(X, \mathcal{O}_X) = \operatorname{Spec} A$ is an affine scheme, and $I \subset A$ is an ideal. For each $f \in A$, we have the ideal $I_f \subset A_f$, being the ideal generated by $\phi_f(I)$. These patch together to form an $\mathcal{O}_X$-submodule of $\mathcal{O}_X$, called the *ideal sheaf generated by* I, and denoted $\tilde{I}$. In general, if X is a scheme, we call an $\mathcal{O}_X$-submodule $\mathcal{I}$ of $\mathcal{O}_X$ an *ideal sheaf* if $\mathcal{I}$ is locally of the form $\tilde{I}$ for some ideal $I \subset \mathcal{O}_X(U)$, $U \subset X$ affine.

Let $\mathcal{I}$ be an ideal sheaf. We may form the sheaf of rings $\mathcal{O}_X/\mathcal{I}$ on X. The *support* of $\mathcal{O}_X/\mathcal{I}$ is the (closed) subset of X consisting of those x with $(\mathcal{O}_X/\mathcal{I})_x \neq 0$. Letting $i : Z \to X$ be the inclusion of the support of $\mathcal{O}_X/\mathcal{I}$, we have the ringed space $(Z, \mathcal{O}_X/\mathcal{I})$ and the morphism of ringed spaces $(i, \pi) : (Z, \mathcal{O}_X/\mathcal{I}) \to (X, \mathcal{O}_X)$, where π is given by the surjection $\mathcal{O}_X \to i_*\mathcal{O}_X/\mathcal{I}$. If we take an affine open subscheme $U = \operatorname{Spec} A$ of X and an ideal $I \subset A$ for which $\mathcal{I}_{|U} = \tilde{I}$, then one has

$$Z \cap U = V(I).$$

We call $(Z, \mathcal{O}_X/\mathcal{I})$ a *closed subscheme* of $(X, \mathcal{O}_X)$. More generally, let $i : (W, \mathcal{O}_W) \to (X, \mathcal{O}_X)$ a morphism of schemes such that

(1) $i : W \to X$ gives a homeomorphism of W with a closed subset Z of X.
(2) $i^* : \mathcal{O}_X \to i_*\mathcal{O}_W$ is surjective, with kernel an ideal sheaf.

Then we call i a *closed embedding*.

If $\mathcal{I} \subset \mathcal{O}_X$ is the ideal sheaf associated to a closed subscheme $(Z, \mathcal{O}_Z)$, we call $\mathcal{I}$ the *ideal sheaf* of $(Z, \mathcal{O}_Z)$, and write $\mathcal{I} = \mathcal{I}_Z$.

Example 1.3.9. Let $(X, \mathcal{O}_X) = \operatorname{Spec} A$, and $I \subset A$ an ideal, giving us the ideal sheaf $\tilde{I} \subset \mathcal{O}_X$, and the closed subscheme $i : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$. Then $(Z, \mathcal{O}_Z)$ is affine, $(Z, \mathcal{O}_Z) = \operatorname{Spec} A/I$, and $i = (\hat{\pi}, \tilde{\pi})$, where $\pi : A \to A/I$ is the canonical surjection. Conversely, if $(Z, \mathcal{O}_Z)$

is a closed subscheme of $\operatorname{Spec} A$ with ideal sheaf $\mathcal{I}_Z$, then $(Z, \mathcal{O}_Z)$ is affine, $(Z, \mathcal{O}_Z) = \operatorname{Spec} A/I$ (as closed subscheme), where $I = \mathcal{I}_Z(X)$.

1.3.10. *Fiber products.* An important property of the category **Sch** is the existence of a (categorical) fiber product $Y \times_X Z$ for each pair of morphisms $f : Y \to X$, $g : Z \to X$. We sketch the construction.

First consider the case of affine $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $Z = \operatorname{Spec} C$. We thus have ring homomorphisms $f^* : A \to B$, $g^* : A \to C$ with $f = (\hat{f}^*, \tilde{f}^*)$ and similarly for g. The maps $B \to B \otimes_A C$, $C \to B \otimes_A C$ defined by $b \mapsto b \otimes 1$, $c \mapsto 1 \otimes c$ give us the commutative diagram of rings

$$\begin{tikzcd} B \otimes_A C & B \arrow[l] \\ C \arrow[u] & A. \arrow[l, "g^*"] \arrow[u, "f^*"'] \end{tikzcd}$$

It is not hard to see that this exhibits $B \otimes_A C$ as the categorical coproduct of B and C over A (in **Rings**), and thus applying Spec yields the fiber product diagram

$$\begin{tikzcd} \operatorname{Spec}(B \otimes_A C) \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"'] & X \end{tikzcd}$$

in $\mathbf{Aff} = \mathbf{Rings}^{\mathrm{op}}$. By Remark 1.3.5, this diagram is a fiber product diagram in **Sch** as well.

Now suppose that X, Y and Z are arbitrary schemes. We can cover X, Y and Z by affine open subschemes X_α, Y_α, Z_α such that the maps f and g restrict to $f_\alpha : Y_\alpha \to X_\alpha$, $g_\alpha : Z_\alpha \to X_\alpha$ (we may have $X_\alpha = X_\beta$ for $\alpha \neq \beta$). We thus have the fiber products $Y_\alpha \times_{X_\alpha} Z_\alpha$ for each α. The universal property of fiber products together with the gluing data for the individual covers $\{X_\alpha\}$, $\{Y_\alpha\}$, $\{Z_\alpha\}$ defines gluing data for the pieces $Y_\alpha \times_{X_\alpha} Z_\alpha$, which forms the desired categorical fiber product $Y \times_X Z$.

Example 1.3.11 (The fibers of a morphism). Let X be a scheme, $x \in |X|$ a point, $k(x)$ the residue field of the local ring $\mathcal{O}_{X,x}$. The inclusion $x \to |X|$ together with the residue map $\mathcal{O}_{X,x} \to k(x)$ defines the morphism of schemes $i_x : \operatorname{Spec} k(x) \to X$. If $f : Y \to X$ is a morphism, we set

$$f^{-1}(x) := \operatorname{Spec} k(x) \times_X Y.$$

Via the first projection, $f^{-1}(x)$ is a scheme over $k(x)$, called the *fiber* of f over x.

1.3.12. *Sheaves of $\mathcal{O}_X$-modules.* Let $(X, \mathcal{O}_X)$ be a ringed space. As we noted in our discussion of closed subschemes, a sheaf of abelian groups $\mathcal{M}$ equipped with a multiplication map

$$\mathcal{O}_X \times \mathcal{M} \to \mathcal{M},$$

satisfying the usual associate and unit properties, is an *$\mathcal{O}_X$-module.* Morphisms of $\mathcal{O}_X$-modules are morphisms of sheaves of abelian groups respecting the multiplication, giving us the category $\mathrm{Mod}_{\mathcal{O}_X}$ of $\mathcal{O}_X$-modules.

If $(f, \psi) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism of ringed spaces, we have the map $\hat{\psi} : f^*\mathcal{O}_X \to \mathcal{O}_Y$ adjoint to $\psi : \mathcal{O}_X \to f_*\mathcal{O}_Y$. If $\mathcal{M}$ is an $\mathcal{O}_X$-module, then $f^*\mathcal{M}$ is an $f^*\mathcal{O}_X$-module, and the tensor product $\mathcal{O}_Y \otimes_{f^*\mathcal{O}_X} f^*\mathcal{M}$ is an $\mathcal{O}_Y$-module, denoted $(f, \psi)^*(\mathcal{M})$. Thus, we have the pull-back functor

$$(f, \psi)^* : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_Y}$$

and a natural isomorphism $((f, \psi) \circ (g, \phi))^* \cong (g, \phi)^* \circ (f, \psi)^*$.

In particular, for a map of schemes $f : Y \to X$, we have the pull-back morphism $f^* : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_Y}$. As we noted in the case of ideal sheaves, the full category $\mathrm{Mod}_{\mathcal{O}_X}$ is not of central importance; we should rather consider those $\mathcal{O}_X$-modules which are locally given by modules over a ring.

To explain this, let A be a commutative ring and M an A-module. If $S \subset A$ is a multiplicatively closed subset containing 1, we have the localization $S^{-1}A$ and the $S^{-1}A$-module $S^{-1}M$. Exactly the same considerations which lead to the construction of the sheaf $\mathcal{O}_X$ on $X := \mathrm{Spec}\, A$ give us the sheaf of $\mathcal{O}_X$-modules $\tilde{M}$ on $\mathrm{Spec}\, A$, satisfying

$$\tilde{M}(X_f) = S(f)^{-1}M$$

for all $f \in A$. The ideal sheaf $\tilde{I}$ considered in (1.3.8) is a special case of this construction.

Definition 1.3.13. Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{M}$ is called *quasi-coherent* if there is an affine open cover $X = \cup_\alpha U_\alpha := \mathrm{Spec}\, A_\alpha$ and A_α-modules M_α and isomorphisms of $\mathcal{O}_{U_\alpha}$-modules $\mathcal{M}_{|U_\alpha} \cong \tilde{M}_\alpha$.

If X is noetherian and each M_α is a finitely generated A_α-module, then $\mathcal{M}$ is called a *coherent* $\mathcal{O}_X$-module. If each of the M_α is a free A_α-module of rank n, then $\mathcal{M}$ is called a *locally free* $\mathcal{O}_X$-module of rank n. A rank 1 locally free $\mathcal{O}_X$-module is called an *invertible sheaf.*

We thus have the category $\mathrm{Q.Coh}_X$ of quasi-coherent $\mathcal{O}_X$-modules as a full subcategory of $\mathrm{Mod}_{\mathcal{O}_X}$ and, for X noetherian, the full subcategory Coh_X of coherent $\mathcal{O}_X$-modules. We have as well the full subcategory

$\mathcal{P}_X$ of locally free sheaves of finite rank. Given a morphism $f : Y \to X$, the pull-back $f^* : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_Y}$ restricts to

$$f^* : \mathrm{Q.Coh}_X \to \mathrm{Q.Coh}_Y$$
$$f^* : \mathrm{Coh}_X \to \mathrm{Coh}_Y$$
$$f^* : \mathcal{P}_X \to \mathcal{P}_Y$$

The categories $\mathrm{Mod}_{\mathcal{O}_X}$, $\mathrm{Q.Coh}_X$ and Coh_X (for X noetherian) are all abelian categories, in fact abelian subcategories of the category of sheaves of abelian groups on $|X|$.

1.4. **Schemes and morphisms.** In practice, one restricts attention to various special types of schemes and morphisms. In this section, we describe the most important of these.

For a scheme X, we write $|X|$ for the underlying topological space of X, and $\mathcal{O}_X$ for the structure sheaf of rings on $|X|$. For a morphism $f : Y \to X$ of schemes, we usually write $f : |Y| \to |X|$ for the map of underlying spaces (sometimes $|f|$ if necessary) and $f^* : \mathcal{O}_X \to f_*\mathcal{O}_Y$ for the map of sheaves of rings.

1.4.1. *Noetherian schemes.* Recall that a commutative ring A is *noetherian* if the following equivalent conditions are satisfied:

(1) Every increasing sequences of ideals in A
$$I_0 \subset I_1 \subset \ldots \subset I_n \subset \ldots$$
is eventually constant.

(2) Let M be a finitely generated A-module (i.e., there exist elements $m_1, \ldots, m_n \in M$ such that each element of M is of the form $\sum_{i=1}^n a_i m_i$ with the $a_i \in A$). Then every increasing sequence of submodules of M
$$N_0 \subset N_1 \subset \ldots \subset N_n \subset \ldots$$
is eventually constant.

(3) Let M be a finitely generated A-module, $N \subset M$ a submodule. Then N is finitely generated as an A-module.

(4) Let I be an ideal in A. Then I is a finitely generated ideal.

A topological space X is called noetherian if every sequence of closed subsets
$$X \supset X_1 \supset X_2 \supset \ldots \supset X_n \supset \ldots$$
is eventually constant. Clearly, if A is noetherian, then the topological space $\mathrm{Spec}\, A$ is a noetherian topological space (but not conversely).

We call a scheme X noetherian if $|X|$ is noetherian, and X admits an affine cover, $X = \cup_\alpha \mathrm{Spec}\, A_\alpha$ with each A_α a noetherian ring. We can take the cover to have finitely many elements. If X is a noetherian scheme, so is each open or closed subscheme of X.

Examples 1.4.2. A field k is clearly a noetherian ring. More generally, if A is a principal ideal domain, then A is noetherian, in particular, $\mathbb{Z}$ is noetherian.

The *Hilbert basis theorem* states that, if A is noetherian, so is the polynomial ring $A[X]$. Thus, the polynomial rings $k[X_1, \ldots, X_n]$ and $\mathbb{Z}[X_1, \ldots, X_n]$ are noetherian. If A is noetherian, so is A/I for each ideal I, thus, every quotient of $A[X_1, \ldots, X_n]$ (that is, every A-algebra that is finitely generated as an A-algebra) is noetherian. This applies to, e.g., a ring of integers in a number field, or rings of the form $k[X_1, \ldots, X_n]/I$.

The scheme $\operatorname{Spec} A[X_1, \ldots, X_n]$ is called the *affine n-space over A*, written $\mathbb{A}^n_A$.

1.4.3. *Irreducible schemes, reduced schemes and generic points.* Let X be a topological space. X is called *irreducible* if X is not the union of two proper closed subsets; equivalently, each non-empty open subspace of X is dense. It is easy to see that a noetherian topological space X is uniquely a finite union of irreducible closed subspaces

$$X = X_0 \cup \ldots \cup X_N$$

where no X_i contains X_j for $i \neq j$. We call a scheme X irreducible if $|X|$ is an irreducible topological space.

Example 1.4.4. Let A be an integral domain. Then $\operatorname{Spec} A$ is irreducible. Indeed, since A is a domain, (0) is a prime ideal, and as every prime ideal contains (0), $\operatorname{Spec} A$ is the closure of the singleton set $\{(0)\}$, hence irreducible. The point $x_{\mathrm{gen}} \in \operatorname{Spec} A$ corresponding to the prime ideal (0) is called the *generic point* of $\operatorname{Spec} A$.

An element x in a ring A is called *nilpotent* if $x^n = 0$ for some n; the set of nilpotent elements in a commutative ring A form an ideal, $nil(A)$, called the *nil-radical* of A. A ring A is *reduced* if $nil(A) = \{0\}$; clearly $A_{\mathrm{red}} := A/nil(A)$ is the maximal reduced quotient of A. As one clearly has $nil(A) \subset \mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of A, the quotient map $A \to A_{\mathrm{red}}$ induces a homeomorphism $\operatorname{Spec} A_{\mathrm{red}} \to \operatorname{Spec} A$.

A scheme X is called *reduced* if for each open subset $U \subset |X|$, the ring $\mathcal{O}_X(U)$ contains no non-zero nilpotent elements. A reduced, irreducible scheme is called *integral*. If $X = \operatorname{Spec} A$ is affine, then X is an integral scheme if and only if A is a domain.

For a scheme X, we have the presheaf $\underline{nil}_X$ with $\underline{nil}_X(U)$ the set of nilpotent elements of $\mathcal{O}_X(U)$; we let $nil_X \subset \mathcal{O}_X$ be the associated sheaf. nil_X is a sheaf of ideals; we let X_{red} be the closed subscheme of X with structure sheaf $\mathcal{O}_X/nil_X$. For $X = \operatorname{Spec} A$, $nil_X(X) = \cap_{\mathfrak{p}\in\operatorname{Spec} A}$ (i.e. $nil_X(X) = \sqrt{(0)}$), so $X_{\mathrm{red}} \to X$ is a homeomorphism. Thus each scheme X has a canonical reduced closed subscheme X_{red} homeomorphic to X. More generally, if $Z \subset |X|$ is a closed subset, there is a unique sheaf of ideals $\mathcal{I}_Z$ such that the closed subscheme W of X defined by $\mathcal{I}_Z$ has underlying topological space Z, and W is reduced. We usually write Z for this closed subscheme, and say that we give Z the reduced subscheme structure.

Lemma 1.4.5. *Let X be a non-empty irreducible scheme. Then $|X|$ has a unique point x_{gen} with $|X|$ the closure of x_{gen}. x_{gen} is called the* generic point *of X.*

Proof. Replacing X with X_{red}, we may assume that X is integral. If $U \subset X$ is an affine open subscheme, then U is dense in X and U is integral, so $U = \operatorname{Spec} A$ with A a domain. We have already seen that $|U|$ satisfies the lemma. If $u \in |U|$ is the generic point, the the closure of u in $|X|$ is $|X|$; uniqueness of x_{gen} follows from the uniqueness of u_{gen} and denseness of U. □

If X is noetherian, then $|X|$ has finitely many irreducible components: $|X| = \cup_{i=1}^N X_i$, without containment relations among the X_i. The X_i (with the reduced subscheme structure) are called the *irreducible components* of X, and the generic points $x_1, \ldots, x_N$ of the components X_i are called the generic points of X.

Definition 1.4.6. Let X be a noetherian scheme with generic points $x_1, \ldots, x_N$. The *ring of rational functions* on X is the ring

$$k(X) := \prod_{i=1}^N \mathcal{O}_{X,x_i}.$$

If X is integral, then $k(X)$ is a field, called the field of rational functions on X.

1.4.7. *Separated schemes and morphisms.* One easily sees that, except for trivial cases, the topological space $|X|$ underlying a scheme X is not Hausdorff. So, to replace the usual separation axioms, we have the following condition: a morphism of schemes $f : X \to Y$ is called *separated* if the diagonal inclusion $X \to X \times_Y X$ has image a closed subset (with respect to the underlying topological spaces). Noting that every scheme has a unique morphism to $\operatorname{Spec} \mathbb{Z}$, we call X a separated scheme if X is separated over $\operatorname{Spec} \mathbb{Z}$, i.e., the diagonal X in $X \times_{\operatorname{Spec} \mathbb{Z}} X$ is closed.

Separation has the following basic properties (proof left to the reader):

Proposition 1.4.8. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes.*

1. *If gf is separated, then f is separated.*
2. *If f is separated and g is separated, the gf is separated.*
3. *if f is separated and $h : W \to Y$ is an arbitrary morphism, then the projection $X \times_Y W \to W$ is separated.*

In addition:

Proposition 1.4.9. *Every affine scheme is separated.*

Proof. Let $X = \operatorname{Spec} A$. The diagonal inclusion $\delta : X \to X \times_{\operatorname{Spec} \mathbb{Z}} X$ arises from the dual diagram of rings

$$\begin{tikzcd}
A & & \\
& A \otimes_{\mathbb{Z}} A \arrow[ul, "\mu"'] & A \arrow[l, "i_1"] \arrow[ull, "\mathrm{id}"'] \\
& A \arrow[u, "i_2"] \arrow[uul, "\mathrm{id}"] & \mathbb{Z} \arrow[l] \arrow[u]
\end{tikzcd}$$

where $i_1(a) = a \otimes 1$, $i_2(a) = 1 \otimes a$, and the maps $\mathbb{Z} \to A$ are the canonical ones. Thus μ is the multiplication map, hence surjective. Letting $I \subset A \otimes_{\mathbb{Z}} A$ be the kernel of μ (in fact I is the ideal generated by elements $a \otimes 1 - 1 \otimes a$), we see that δ is a closed embedding with image $\operatorname{Spec} A \otimes_{\mathbb{Z}} A/I$. ☐

The point of using separated schemes is that this forces the condition that two morphisms $f, g : X \to Y$ be the same to be a *closed* subscheme of X. Indeed, the equalizer of f and g is the pull-back of the diagonal $\delta_Y \subset Y \times Y$ via the map $(f, g) : X \to Y \times Y$, so this equalizer is closed if δ_Y is closed in $Y \times Y$. Another nice consequence is

Proposition 1.4.10. *Let X be a separated scheme, U and V affine open subschemes. Then $U \cap V$ is also affine.*

Proof. We have the fiber product diagram

$$\begin{tikzcd}
U \cap V \arrow[r, "i_{U \cap V}"] \arrow[d, "\delta'"'] & X \arrow[d, "\delta_X"] \\
U \times V \arrow[r, "i_U \times i_V"'] & X \times X
\end{tikzcd}$$

identifying $U \cap V$ with $(U \times V) \times_{X \times X} X$. If $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$, then $U \times V = \operatorname{Spec} A \otimes B$ is affine; since δ_X is a closed embedding, so is δ'. Thus $U \cap V$ is isomorphic to a closed subscheme of the affine scheme $\operatorname{Spec} A \otimes B$, hence

$$U \cap V = \operatorname{Spec} A \otimes B/I$$

for some ideal I. $\square$

1.4.11. *Finite type morphisms.* Let A be a noetherian commutative ring. A commutative A-algebra $A \to B$ is of *finite type* if B is isomorphic to quotient of a polynomial ring over A in finitely many variables:

$$B \cong A[X_1, \ldots, X_m]/I.$$

This globalizes in the evident manner: Let X be a noetherian scheme. A morphism $f : Y \to X$ is of *finite type* if X and Y admit finite affine covers $X = \cup_i U_i = \operatorname{Spec} A_i$, $Y = \cup_i V_i = \operatorname{Spec} B_i$ with $f(V_i) \subset U_i$ and $f^* : A_i \to B_i$ making B_i a finite-type A_i-algebra for each i.

In case $X = \operatorname{Spec} A$ for some noetherian ring A, we say that Y is of finite type over A. We let $\mathbf{Sch}_A$ denote the full subcategory of all schemes with objects the A-scheme of finite type which are separated over $\operatorname{Spec} A$.

Clearly the property of $f : Y \to X$ being of finite type is preserved under fiber product with an arbitrary morphism $Z \to X$ (with Z noetherian). By the Hilbert basis theorem, if $f : Y \to X$ is a finite type morphism, then Y is noetherian (X is assumed noetherian as part of the definition).

1.4.12. *Proper, finite and quasi-finite morphisms.* In topology, a proper morphism is one for which the inverse image of a compact set is compact. As above, the lack of good separation for the Zariski topology means one needs to use a somewhat different notion.

Definition 1.4.13. A morphism $f : Y \to X$ is *closed* if for each closed subset C of $|Y|$, $f(C)$ is closed in $|X|$. A morphism $f : Y \to X$ is *proper* if

(1) f is separated.
(2) f is universally closed: for each morphism $Z \to X$, the projection $Z \times_X Y \to Z$ is closed.

Since the closed subsets in the Zariski topology are essentially locii defined by polynomial equations, the condition that a morphism $f : Y \to X$ is proper implies the "principle of elimination theory": Let C be a "closed algebraic locus in $Z \times_X Y$. Then the set of $z \in |Z|$ for which there exist a $\tilde{z} \in C \subset Z \times_X Y$ is a closed subset. We will see below in the section on projective spaces how to construct examples of proper morphisms.

Definition 1.4.14. Let $f : Y \to X$ be a morphism of noetherian schemes. f is called *affine* if for each affine open subscheme $U \subset X$, $f^{-1}(U)$ is an affine open subscheme of Y. An affine morphism is called *finite* if for $U = \operatorname{Spec} A \subset X$ with $f^{-1}(U) = \operatorname{Spec} B$, the homomorphism $f^* : A \to B$ makes B into a finitely generated A-module.

Definition 1.4.15. Let $f : Y \to X$ be a morphism of noetherian schemes. f is called *quasi-finite* if for each $x \in |X|$, $|f|^{-1}(x)$ is a finite set.

The property of a morphism $f : Y \to X$ being proper, affine, finite or quasi-finite morphisms is preserved under fiber product with an arbitrary morphism $Z \to X$, and the composition of two proper (resp. affine, finite, quasi-finite) morphisms is proper (resp. affine, finite, quasi-finite). Also

Proposition 1.4.16. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of noetherian schemes. If gf is proper (resp. finite, quasi-finite) then f is proper (resp. finite, quasi-finite).*

A much deeper result is

Proposition 1.4.17. *Let $f : Y \to X$ be a finite type morphism. Then f is finite if and only if f is proper and quasi-finite.*

1.4.18. *Flat morphisms.* The condition of flatness comes from homological algebra, but has geometric content as well. Recall that for a commutative ring A and an A-module M, the operation of tensor product $\otimes_A M$ is *right exact*:

$$\begin{aligned} &N' \to N \to N'' \to 0 \text{ exact} \\ &\qquad\qquad \Longrightarrow N' \otimes_A M \to N \otimes_A M \to N'' \otimes_A M \to 0 \text{ exact.} \end{aligned}$$

M is a *flat* A-module if $\otimes_A M$ is exact, i.e., left-exact:

$$0 \to N' \to N \text{ exact } \Longrightarrow 0 \to N' \otimes_A M \to N \otimes_A M \text{ exact.}$$

A ring homomorphism $A \to B$ is called flat if B is flat as an A-module.

Definition 1.4.19. Let $f : Y \to X$ be a morphism of schemes. f is *flat* if for each $y \in Y$, the homomorphism $f_y^* : \mathcal{O}_{X,f(x)} \to \mathcal{O}_{Y,y}$ is flat.

It follows from the "cancellation formula" $(N \otimes_A B) \otimes_B C \cong N \otimes_A C$ that a composition of flat morphisms is flat.

1.4.20. *Valuative criteria.* In the study of metric spaces, sequences and limits play a central role. In algebraic geometry, this is replaced by using the spectrum of *discrete valuation rings.*

Recall that a noetherian local domain $(\mathcal{O}, \mathfrak{m})$ is a discrete valuation ring (DVR for short) if $\mathfrak{m}$ is a principal ideal, $\mathfrak{m} = (t)$. Let $\mathcal{O}$ have quotient field F and residue field k. It is not hard to see that $\operatorname{Spec} \mathcal{O}$ consists of two points: the generic point $\eta : \operatorname{Spec} F \to \operatorname{Spec} \mathcal{O}$ and the single closed point $\operatorname{Spec} k \to \operatorname{Spec} \mathcal{O}$. In terms of our sequence/limit analogy, $\operatorname{Spec} F$ is the sequence and $\operatorname{Spec} k$ is the limit. This should motivate the follow result which characterizes separated and proper morphisms.

Proposition 1.4.21. *Let $f : Y \to X$ be a morphism of finite type.*

(1) *f is separated if and only if, for each DVR $\mathcal{O}$ (with quotient field F) and each commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} F \arrow[r] \arrow[d, "\eta"'] & Y \arrow[d, "f"] \\ \operatorname{Spec} \mathcal{O} \arrow[r] & X \end{tikzcd}$$

there exists at most one lifting $\operatorname{Spec} \mathcal{O} \to Y$.

(2) *f is proper if and only if, for each DVR $\mathcal{O}$ (with quotient field F) and each commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} F \arrow[r] \arrow[d, "\eta"'] & Y \arrow[d, "f"] \\ \operatorname{Spec} \mathcal{O} \arrow[r] & X \end{tikzcd}$$

there exists a unique lifting $\operatorname{Spec} \mathcal{O} \to Y$.

1.5. **The category $\mathbf{Sch}_k$.** In this section we fix a field k. The schemes of most interest to us in many applications are the separated k-schemes of finite type; in this section we examine a number of concepts which one can describe quite concretely for such schemes.

1.5.1. *R-valued points.* The use of R-valued points allows one to recover the classical notion of "solutions of a system of equations" within the theory of schemes.

Definition 1.5.2. Let X be a scheme, R a ring. The set of R-valued points $X(R)$ is by definition the Hom-set $\operatorname{Hom}_{\mathbf{Sch}}(\operatorname{Spec} R, X)$. If we fix a base-ring A, and X is a scheme over A and R is an A-algebra, we set

$$X_A(R) := \operatorname{Hom}_{\mathbf{Sch}_A}(\operatorname{Spec} R, X).$$

We often leave off the subscript A if the context makes the meaning clear.

Example 1.5.3. Let $X = \operatorname{Spec} k[X_1, \ldots, X_n]/(f_1, \ldots, f_r)$, and let F/k be an extension field of k. Then $X_k(F)$ is the set of maps $\operatorname{Spec} F \to \operatorname{Spec} k[X_1, \ldots, X_n]/(f_1, \ldots, f_r)$ over k, i.e., the set of k-algebra homomorphisms

$$\psi : k[X_1, \ldots, X_n]/(f_1, \ldots, f_r) \to F.$$

Clearly ψ is determined by the values $\psi(X_i)$, $i = 1, \ldots, n$; conversely, given elements $x_1, \ldots, x_n \in F$, we have the unique k-algebra homomorphism

$$\tilde{\psi} : k[X_1 \ldots, X_n] \to F$$

sending X_i to x_i. As $\tilde{\psi}(f_j) = f_j(x_1, \ldots, x_n)$, $\tilde{\psi}$ descends to an F-valued point ψ of X if and only if $f_j(x_1, \ldots, x_n) = 0$ for all j. Thus, we have identified the F-valued points of X with the set of solutions in F of the polynomial equations $f_1 = \ldots, f_r = 0$. This example explains the connection of the machinery of schemes with the basic problem of understanding the solutions of polynomial equations.

As a special case, take $X = \mathbb{A}^n_k$. Then $X_k(F) = F^n$ for all F.

1.5.4. *Group-schemes and bundles.* Just as in topology, we have the notion of a locally trivial bundle $E \to B$ with base B, fiber F and group G. The group G is an *algebraic group-scheme over* k, which is just a group-object in $\mathbf{Sch}_k$. Concretely, we have a multiplication $\mu : G \times_k G \to G$, inverse $\iota : G \to G$ and unit $e : \operatorname{Spec} k \to G$, satisfying the usual identities, interpreted as identities of morphisms.

Example 1.5.5. Let $M_n = \operatorname{Spec} k[\{X_{ij} \mid 1 \leq i, j \leq n\}] \cong \mathbb{A}^{n^2}_k$. The formula for matrix multiplication

$$\mu^*(X_{ij}) = \sum_k X_{ik} \otimes X_{kj}$$

defines the ring homomorphism

$$\mu^* : k[\ldots X_{ij} \ldots] \to k[\ldots X_{ij} \ldots] \otimes_k k[\ldots X_{ij} \ldots],$$

hence the morphism

$$\mu : M_n \times_k M_n \to M_n.$$

Let GL_n be the open subset $(M_n)_{\det}$, where det is the determinant of the $n \times n$ matrix (X_{ij}), i.e.

$$\mathrm{GL}_n := \operatorname{Spec} k[\ldots X_{ij} \ldots, \frac{1}{\det}].$$

Then μ restricts to

$$\mu : \mathrm{GL}_n \times_k \mathrm{GL}_n \to \mathrm{GL}_n,$$

and the usual formula for matrix inverse defines the inverse morphism $\iota : \mathrm{GL}_n \to \mathrm{GL}_n$. The unit is given by $e^*(X_{ij}) = \delta_{ij}$.

If G is an algebraic group-scheme over k, and F a finite type k-scheme, an action of G on F is just a morphism $\rho : G \times_k F \to F$, satisfying the usual associativity and unit conditions, again as identities of morphisms in $\mathbf{Sch}_k$. Now we can just mimic the usual definition of a fiber-bundle with fiber F and group G: $p : E \to B$ is required to have local trivializations,

$$B = \cup_i U_i,$$

with the U_i open subschemes, and there are isomorphisms over U_i,

$$\psi_i : p^{-1}(U_i) \to U_i \times_k F.$$

In addition, for each i, j, there is a morphism $g_{ij} : U_i \cap U_j \to G$ such that the isomorphism

$$\psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times_k F$$

is given by the composition

$$(U_i \cap U_j) \times_k F \xrightarrow{(p_1, g_{ij} \circ p_1, p_2)} (U_i \cap U_j) \times_k \times_k GF \xrightarrow{\mathrm{id} \times \rho} (U_i \cap U_j) \times_k F.$$

An isomorphism of bundles $f : (E \to B) \to (E' \to B)$ is given by a B-morphism $f : E \to E'$ such that, with respect to a common local trivialization, f is locally of the form

$$U \times_k F \xrightarrow{(p_1, g \circ p_1, p_2)} U \times_k \times_k GF \xrightarrow{\mathrm{id} \times \rho} U \times_k F$$

for some morphism $g : U \to G$.

For example, using $G = \mathrm{GL}_n$, $F = \mathbb{A}^n_k$, $\rho : \mathrm{GL}_n \times_k \mathbb{A}^n_k \to \mathbb{A}^n_k$ the map with

$$\rho^*(Y_i) = \sum_j X_{ij} \otimes Y_j,$$

we have the notion of an *algebraic vector bundle* of rank n.

1.5.6. *Dimension.* Let A be a commutative ring. The *Krull dimension* of A is the maximal length n of a chain of distinct prime ideals in A:

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$$

Let F be a finitely generated field extension of k. A set of elements of F are *transcendentally independent* if they satisfy no non-trivial polynomial identity with coefficients in k. A *transcendence basis* of F over k is a transcendentally independent set $\{x_\alpha \in F\}$ of elements of F such that F is algebraic over the subfield $k(\{x_\alpha\})$ generated by the x_α, i.e., each element $x \in F$ satisfies some non-trivial polynomial identity with coefficients in $k(\{x_\alpha\})$. One shows that each F admits a transcendence basis over k and that each two transcendence bases of F over k have the same cardinality, which is called the *transcendence dimension* of F over k, $\mathrm{tr.dim}_k F$. Clearly if F is finitely generated over k, then $\mathrm{tr.dim}_k F$ is finite. In particular, if X is an integral k-scheme of finite type over k, then the function field $k(X)$ has finite transcendence dimension over k.

Definition 1.5.7. Let X be an irreducible separated k-scheme of finite type. The dimension of X over k is defined by

$$\dim_k X := \mathrm{tr.dim}_k k(X_{\mathrm{red}}).$$

In general, if X is a separated k-scheme of finite type with reduced irreducible components $X_1, \ldots, X_n$, we write

$$\dim_k X \leq d$$

if $\dim_k X_i \leq d$ for all i. We say that X is *equi-dimenisonal over k* of dimension d if $\dim_k X_i = d$ for all i.

Remark 1.5.8. We can make the notion of dimension have a local character as follows: Let X be a separated finite type k-scheme and $x \in |X|$ a point. We say $\dim_k(X, x) \leq d$ if there is some neighborhood U of x in X with $\dim_k U \leq d$. We similarly say that X is equi-dimensional over k of dimension d at x if there is a U with $\dim_k U = d$. We say X is *locally equi-dimensional* over k if X is equi-dimensional over k at x for each $x \in |X|$.

If X is locally equi-dimensional over k, then the local dimension function $\dim_k(X, x)$ is constant on connected components of $|X|$. Thus, if $W \subset X$ is an integral closed subscheme, then the local dimension function $\dim_k(X, w)$ is constant over W. We set $\mathrm{codim}_X W = \dim_k(X, w) - \dim_k W$. If w is the generic point of W and $\mathrm{codim}_X W = d$, we call w a *codimension d* point of X.

We thus have two possible definitions of the dimension of $\mathrm{Spec}\, A$ for A a domain which is a finitely generated k-algebra, $\dim_k \mathrm{Spec}\, A$ and the Krull dimension of A. Fortunately, these are the same:

Theorem 1.5.9 (Krull)**.** *Let A be a domain which is a finitely generated k-algebra. Then* $\dim_k \mathrm{Spec}\, A$ *equals the Krull dimension of A.*

This result follows from the *principal ideal theorem* of Krull :

Theorem 1.5.10 (Krull)**.** *Let A be a a domain which is a finitely generated k-algebra, let f be a non-zero element of A and $\mathfrak{p} \supset (f)$ a minimal prime ideal containing (f). Then $\dim_k \operatorname{Spec} A = \dim_k \operatorname{Spec} A/\mathfrak{p}+1$.*

the *Hilbert Nullstellensatz*:

Theorem 1.5.11. *Let A be a finitely generated k-algebra which is a field. Then $k \to A$ is a finite field extension.*

and induction.

There is a relation of flatness to dimension: Let $f : Y \to X$ be a finite type morphism of noetherian schemes. Then for each $x \in |X|$, $f^{-1}(x)$ is a scheme of finite type over the field $k(x)$, so one can ask if $f^{-1}(x)$ is equi-dimensional over $k(x)$ and if so, what is the dimension. If X and Y are irreducible and f is flat, then there is an integer $d \geq 0$ such that for each $x \in |X|$, either $f^{-1}(x)$ is empty or $f^{-1}(x)$ is an equi-dimensional $k(x)$-scheme of dimension d over $k(x)$. If $f : Y \to X$ is a flat morphism in $\mathbf{Sch}_k$, with X and Y equi-dimensional over k, then each non-empty fiber $f^{-1}(x)$ is equi-dimensional over $k(x)$, and

$$\dim_k Y = \dim_k X + \dim_{k(x)} f^{-1}(x).$$

1.5.12. *Hilbert's Nullstellensatz.* Having introduced Hilbert's Nullstellensatz (Theorem 1.5.11) above, we take this opportunity to mention some important consequences:

Corollary 1.5.13. *Let X be a scheme of finite type over k, $x \in |X|$ a closed point. Then $k(x)$ is a finite field extension of k.*

Corollary 1.5.14. *Suppose k is algebraically closed. Let m be a maximal ideal in $k[X_1, \ldots, X_n]$. Then $m = (X_1 - a_1, \ldots, X_n - a_n)$ for (a unique) $(a_1, \ldots, a_n) \in k^n$.*

Proof. Take $X = \operatorname{Spec} k[X_1, \ldots, X_n]$ and $x = m \in |X|$ in Corollary 1.5.13. Since k is algebraically closed the inclusion $k \to k(x)$ is an isomorphism, so we have the exact sequence

$$0 \to m \to k[X_1, \ldots, X_n] \xrightarrow{\pi} k(x) = k \to 0.$$

If $\pi(X_i) = a_i$, then m is generated by the elements $X_i - a_i$. $\square$

.

Finally, the Nullstellensatz allows one to relate the commutative algebra encoded in Spec to the more concrete notion of solutions of polynomial equations, at least if k is algebraically closed. For this, let $I \subset k[X_1, \ldots, X_n]$ be an ideal, and let $V_k(I) \subset k^n$ be the set of

$a := (a_1, \ldots, a_n) \in k^n$ such that $f(a) = 0$ for all $f \in I$. Similarly, if S is a subset of k^n, we let

$$I(S) = \{f \in k[X_1, \ldots, X_n] \mid f(a) = 0 \text{ for all } a \in S\}.$$

The subsets of k^n of the form $V_k(I)$ for an ideal I are called the *algebraic* subsets of k^n. Clearly $V_k(I) = V_k(\sqrt{I})$ for each ideal I, so the algebraic subsets do not distinguish between I and $\sqrt{I}$.

Corollary 1.5.15. *Let k be and algebraically closed field. Let $I \subset k[X_1, \ldots, X_n]$ be an ideal. Then $I(V_k(I)) = \sqrt{I}$.*

Proof. Clearly $\sqrt{I} \subset I(V_k(I))$. Take $f \in I(V_k(I))$, let

$$A = k[X_1, \ldots, X_n]/\sqrt{I}$$

and let $\bar{f}$ be the image of f in A. If f is not in $\sqrt{I}$, then $\bar{f}$ is not nilpotent. Thus $A_{\bar{f}}$ is not the zero-ring, and hence has a maximal ideal m. Also, we have the isomorphism

$$A_{\bar{f}} \cong k[X_1, \ldots, X_n, X_{n+1}]/(\sqrt{I}, fX_{n+1} - 1),$$

so m lifts to a maximal ideal $M := (X_1 - a_1, \ldots, X_n - a_n, X_{n+1} - a_{n+1})$ in $k[X_1, \ldots, X_n, X_{n+1}]$. But then $(a_1, \ldots, a_n)$ is in $V_k(I)$, hence $f(a_1, \ldots, a_n) = 0$, contradicting the relation

$$a_{n+1} f(a_1, \ldots, a_n) - 1 = 0.$$

□

Thus, we have a 1-1 correspondence between radical ideals in the ring $k[X_1, \ldots, X_n]$ (i.e reduced closed subschemes of $\mathbb{A}^n_k$) and algebraic subsets of k^n, assuming k algebraically closed. This allows one to base the entire theory of reduced schemes of finite type over k on algebraic subsets of $\mathbb{A}^n_k$ rather than on affine schemes; for a first approach to algebraic geometry, this pre-Grothendieck simplification gives a good approximation to the entire theory, at least if one works over an algebraically closed base-field.

1.5.16. *Smooth morphisms and étale morphisms.* Let $\phi : A \to B$ be a ring homomorphism, and M a B-module. Recall that a *derivation* of B into M over A is an A-module homomorphism $\partial : B \to M$ satisfying the Leibniz rule

$$\partial(bb') = b\partial(b') + b'\partial(b).$$

Note that the condition of A-linearity is equivalent to $\partial(\phi(a)) = 0$ for $a \in A$.

The module of *Kähler differentials*, $\Omega_{B/A}$, is a B-module equipped with a universal derivation over A, $d : B \to \Omega_{B/A}$, i.e., for each derivation $\partial : B \to M$ as above, there is a unique B-module homomorphism $\psi : \Omega_{B/A} \to M$ with $\partial(b) = \psi(db)$. It is easy to construct $\Omega_{B/A}$: take the quotient of the free B-module on symbols db, $b \in B$, by the B-submodule generated by elements of the form

$$\begin{aligned}&d(\phi(a)) \text{ for } a \in A\\ &d(b+b') - db - db' \text{ for } b, b' \in B\\ &d(bb') - bdb' - b'db \text{ for } b, b' \in B.\end{aligned}$$

Let $\bar{B} = B/I$ for some ideal I. We have the *fundamental exact sequence*:

$$I/I^2 \to \Omega_{B/A} \otimes_B \bar{B} \to \Omega_{\bar{B}/A} \to 0, \tag{1.5.1}$$

where the first map is induced by the map $f \mapsto df$.

Example 1.5.17. Let k be a field, $B := k[X_1, \ldots, X_n]$ the polynomial ring over k. Then $\Omega_{B/k}$ is the free B-module on $dX_1, \ldots, dX_n$. If $B = k[X_1, \ldots, X_n]/I$ then the fundamental sequence shows that $\Omega_{B/k}$ is the quotient of $\oplus_{i=1}^n B \cdot dX_i$ by the submodule generated by $df = \sum_i \frac{\partial f}{\partial X_i} dX_i$ for $f \in I$.

Definition 1.5.18. Let $f : Y \to X$ be a morphism of finite type. f is *smooth* if

(1) f is separated.

(2) f is flat
(3) Each non-empty fiber f^{-1} is locally equi-dimensional over $k(x)$.

(4) Let y be in $|Y|$, let $x = f(y)$, let $d_y = \dim_{k(x)}(f^{-1}(x), y)$ and let $B_y = \mathcal{O}_{f^{-1}(x),y}$. Then $\Omega_{B_y/k(x)}$ is a free B_y-module of rank d_y.

The map f is *étale* if f is smooth and $d_y = 0$ for all y.

Remarks 1.5.19. (1) If X and Y are integral schemes, or if X and Y are in $\mathbf{Sch}_k$ and are both locally equi-dimensional over k, then the flatness of f implies the condition (3).
(2) Smooth (resp. étale) morphisms are stable under base-change: if $f : Y \to X$ is smooth (resp. étale) and $Z \to X$ is an arbitrary morphism, then the projection $Z \times_X Y \to Z$ is smooth (resp. étale). There is a converse: A morphism $g : Z \to X$ is *faithfully flat* if g is flat and $|g| : |Z| \to |X|$ is surjective. A morphism $f : Y \to X$ is smooth (resp. étale) if and only if the projection $Z \times_X Y \to Z$ is smooth (resp. étale) for some faithfully flat $Z \to X$.

Examples 1.5.20. (1) Let $k \to L$ be a finite extension of fields. Then $\operatorname{Spec} L \to \operatorname{Spec} k$ is smooth if and only if $\operatorname{Spec} L \to \operatorname{Spec} k$ is étale if and only if $k \to L$ is separable.

(2) Let x be a point of a scheme X, $m_{X,x} \subset \mathcal{O}_{X,x}$ the maximal ideal. The *completion* $\hat{\mathcal{O}}_{X,x}$ of $\mathcal{O}_{X,x}$ with respect to $m_{X,x}$ is the limit

$$\hat{\mathcal{O}}_{X,x} := \varprojlim_n \mathcal{O}_{X,x}/m^n_{X,x}.$$

For example, if x is the point $(X_1, \ldots, X_n)$ in $X := \operatorname{Spec} k[X_1, \ldots, X_n]$, then $\hat{\mathcal{O}}_{X,x}$ is the ring of formal power series $k[[X_1, \ldots, X_n]]$.

Suppose k is algebraically closed and $f : Y \to X$ is a morphism of finite type k-schemes. Then f is étale if and only if for each closed point $y \in Y$, the map $f^*_y : \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ induces an isomorphism on the completions $\hat{f}^*_y : \hat{\mathcal{O}}_{X,f(y)} \to \hat{\mathcal{O}}_{Y,y}$. In particular, if $X = \mathbb{A}^n_k$, then each $\hat{\mathcal{O}}_{Y,y}$ is isomorphic to a formal power series ring $k[[X_1, \ldots, X_n]]$.

Definition 1.5.21. X in $\mathbf{Sch}_k$ is called a *smooth k-scheme* if the structure morphism $X \to \operatorname{Spec} k$ is smooth. We let $\mathbf{Sm}_k$ denote the full subcategory of $\mathbf{Sch}_k$ consisting of the smooth k-schemes. If X is in $\mathbf{Sch}_k$, we call a point $x \in |X|$ a *smooth point* if there is exists an open neighborhood U of x in X which is smooth over $\operatorname{Spec} k$.

Let $\mathcal{O}$ be a noetherian local ring with maximal ideal $\mathfrak{m}$. Recall that a sequence of elements $t_1, \ldots, t_r$ in $\mathfrak{m}$ is a *regular sequence* if for each $i = 1, \ldots, r$ the image $\bar{t}_i$ in $\mathcal{O}/(t_1, \ldots, t_{i-1})$ is not a zero-divisor. It turns out that if $t_1, \ldots, t_r$ is a regular sequence, then so is each reordering of the sequence.

The local ring $\mathcal{O}$ is called *regular* if the maximal ideal is generated by a regular sequence. If $\mathcal{O} = \mathcal{O}_{X,x}$ for some point x on a scheme X, a choice $(t_1, \ldots, t_n)$ of a regular sequence generating $m_{X,x}$ is called a *system of local parameters* for X at x.

From the standpoint of homological algebra, the regular local rings are characterized by the theorem of Auslander-Buchsbaum as those for which the residue field $\mathcal{O}/m$ admits a finite free resolution, or equivalently, those for which *every* finitely generated $\mathcal{O}$-module admits a finite free resolution. The relation with smooth points is

Proposition 1.5.22. *Take $X \in \mathbf{Sch}_k$ and $x \in |X|$. If x is a smooth point, then $\mathcal{O}_{X,x}$ is a regular local ring, and the maximal ideal is generated by a regular sequence $(t_1, \ldots, t_n)$ with $n = \operatorname{codim}_X x$. Conversely, if k is perfect (k has characteristic 0 or k has characteristic $p > 0$ and*

$k^p = k$), and $\mathcal{O}_{X,x}$ is a regular local ring for some $x \in X$, then x is a smooth point of X.

Similarly, one can view a system of parameters as follows:

Proposition 1.5.23. *Let X be in $\mathbf{Sch}_k$, $x \in X$ a closed point. Suppose X is equi-dimensional over k at x, and let $n = \dim_k(X, x)$. Take $t_1, \ldots, t_n$ in $m_{X,x}$ and let U be an open neighborhood of x in X with $t_i \in \mathcal{O}_X(U)$ for each i, giving the morphism*

$$f := (t_1, \ldots, t_n) : U \to \mathbb{A}^n_k$$

with $f(x) = 0$.

1. *If f is étale at x, then x is a smooth point of X and $t_1, \ldots, t_n$ is a system of parameters for $\mathcal{O}_{X,x}$*
2. *If k is perfect, then f is étale at x if and only if x is a smooth point of X and $t_1, \ldots, t_n$ is a system of parameters for $\mathcal{O}_{X,x}$.*
3. *If x is a smooth point of X, then f is étale at x if and only if $t_1, \ldots, t_n$ is a system of parameters for $\mathcal{O}_{X,x}$.*

Another important property of smooth points on $X \in \mathbf{Sch}_k$ is that the set of smooth points forms an open subset of $|X|$. In particular $X \in \mathbf{Sch}_k$ is smooth over k if and only if each closed point of X is a smooth point. The closed subset of non-smooth (singular) points of X is denoted X_{sing}.

1.5.24. *The Jacobian criterion.* The definition of a smooth point on an affine k-scheme $X \subset \mathbb{A}^n_k$ can be given in terms of the familiar criterion from differential topology. Suppose that X is defined by an ideal $I \subset k[X_1, \ldots, X_n]$, $I = (f_1, \ldots, f_r)$. Let x be a closed point of X. For $g \in k[X_1, \ldots, X_n]$, we have the "value" $g(x) \in k(x)$, where $g(x)$ is just the image of g under the residue homomorphism $k[X_1, \ldots, X_n] \to k(x)$. In particular, we can evaluate the Jacobian matrix of the f_i's at x forming the matrix

$$\mathrm{Jac}(x) := \left(\frac{\partial f_i}{\partial X_j}\right)(x) \in M_{n \times n}(k(x)).$$

Proposition 1.5.25. *Let $X := \operatorname{Spec} k[X_1, \ldots, X_n]/(f_1, \ldots, f_r)$. Then $x \in X$ is a smooth point if and only if X is equi-dimensional over k at x and*

$$\mathrm{rank}(Jac(x)) = n - \dim_k(X, x).$$

The proof follows by considering the fundamental exact sequence (1.5.1).

1.6. **Projective schemes.** Classical algebraic geometry deals with algebraic subsets of projective space. In this section, we describe the modern machinery for constructing closed subschemes of projective spaces and, more generally, projective morphisms.

1.6.1. *The functor* Proj . The functor Spec is the basic operation going from rings to schemes. We describe a related operation Proj from graded rings to schemes.

Recall that a (non-negatively) graded ring is a ring R whose underlying additive group is a direct sum, $R = \oplus_{n=0}^{\infty} R_n$, such that the multiplication respects the grading:

$$R_n \cdot R_m \subset R_{n+m}.$$

We assume all our rings are commutative, so R is automatically an R_0-algebra.

An element of R_n is said to be homogeneous of degree n. An ideal $I \subset R$ is called *homogeneous* if $I = \sum_{n=0}^{\infty} I \cap R_n$; we often write I_n for $R_n \cap I$. Note that I is homogeneous if and only if the following condition holds:

(1.6.1)

If f is in I, and we write $f = \sum_n f_n$ with $f_n \in R_n$, then each f_n is also in I.

If R is a graded ring and $I \subset R$ a homogeneous ideal, then R/I is also graded, $R/I = \oplus_{n=0}^{\infty} R_n/I_n$.

Example 1.6.2. Fix a ring A, and let $R = A[X_0, \ldots, X_m]$, where we give each X_i degree 1. Then R has the structure of a graded ring, with $R_0 = A$, and R_n the free A-module with basis the monomials $X_0^{d_0} \cdot \ldots \cdot X_m^{d_m}$ of total degree $n = \sum_i d_i$. Unless we make explicit mention to the contrary, we will always use this structure of a graded ring on $A[X_0, \ldots, X_m]$.

Fix a ring A. We consider graded A-algebras $R = \oplus_{n=0}^{\infty} R_n$ such that

(1) $R_0 = A \cdot 1$, i.e. R_0 is generated as an A-module by 1,
(2) R is generated as an A-algebra by R_1, and R_1 is finitely generated as an A-module.

Equivalently, if R_1 is generated over A by elements $r_0, \ldots, r_m$, then sending X_i to r_i exhibits R as a (graded) quotient of $A[X_0, \ldots, X_m]$. Letting I be the kernel of the surjection $A[X_0, \ldots, X_m] \to R$, we see that I is a graded ideal, so there are homogeneous polynomials $f_1, \ldots, f_r \in A[X_0, \ldots, X_m]$ with $I = (f_1, \ldots, f_r)$.

Let R be a graded A-algebra satisfying (1) and (2). We let $R^+ \subset R$ be the ideal $\oplus_{n\geq 1} R_n$. Define the set $\operatorname{Proj} R$ to be the set of homogeneous prime ideals $\mathfrak{p} \subset R$ such that $\mathfrak{p}$ does not contain R^+. For a homogeneous ideal I, we let

$$V_h(I) \ = \{\mathfrak{p} \in \operatorname{Proj} R \mid \mathfrak{p} \supset I\},$$

The operation V_h has properties analogous to the properties (1.1.1) for V, so we can define a topology on $\operatorname{Proj} R$ for which the closed subsets are exactly those of the form $V_h(I)$, for I a homogeneous ideal.

We now define a sheaf of rings on $\operatorname{Proj} R$. For this, we use a homogeneous version of localization. Let S be a subset of R. If S is a multiplicatively closed subset of R, containing 1, we define $S_h^{-1}R$ to be the ring of fractions f/s with $s \in S_n := S \cap R_n$, $f \in R_n$, $n = 0, 1, \ldots$, modulo the usual relation

$$f/s = f'/s' \text{ if } s''(s'f - sf') = 0 \text{ for some } s'' \in S_{n''}.$$

Note that $S_h^{-1}R$ is just a commutative ring, we have lost the grading.

Let $Y = \operatorname{Proj} R$. For $f \in R_n$, we have the open subset $Y_f := Y \setminus V_h((f))$. Let $S(f) = \{1, f, f^2, \ldots\}$ and set $\mathcal{O}_Y(Y_f) := S(f)_h^{-1}R$. This forms the "partially defined" sheaf on the principal open subsets Y_f. If $U = Y \setminus V_h(I)$ is now an arbitrary open subset of $\operatorname{Proj} R$, we set

$$\mathcal{O}_Y(U) := \ker\Big(\prod_{\substack{f\in I \\ f \text{ homogeneous}}} \mathcal{O}_Y(Y_f) \to \prod_{\substack{f,g\in I \\ f,g \text{ homogeneous}}} \mathcal{O}_Y(Y_{fg})\Big)$$

where the map is the difference of the two restriction maps. Just as for affine scheme, this defines a sheaf of rings $\mathcal{O}_Y$ on Y with the desired value $\mathcal{O}_Y(Y_f) = S(f)_h^{-1}R$ on the principal open subsets Y_f.

Lemma 1.6.3. *Let f be in R_n. Then $(Y_f, (\mathcal{O}_Y)_{|Y_f}) \cong \operatorname{Spec} S(f)_h^{-1}R$ as ringed spaces.*

sketch of proof. Let $Z = \operatorname{Spec} S(f)_h^{-1}R$. Let $J \subset R$ be a homogeneous ideal. Form the ideal $J_f \subset S(f)_h^{-1}R$ as the set of elements g/f^m, $g \in J_{nm}$. Conversely, let $I \subset S(f)_h^{-1}R$ be an ideal. Let $I^h \subset R$ be the set of elements of the form g, with $g \in R_{nm}$ and $g/f^m \in I$. Then I^h is a homogeneous ideal in R.

One checks the relations:

$$(I^h)_f = I; \quad (J_f)^h \supset J.$$

In addition, the operations $I \mapsto I^h$, $J \mapsto J_f$ send prime ideals to prime ideals, and if $\mathfrak{q} \subset R$ is a homogeneous prime, $\mathfrak{q} \not\supset (f)$, then $(\mathfrak{q}_f)^h = \mathfrak{q}$. Thus, we have the bijection between Y_f and Z, which one easily sees is a homeomorphism. Under this homeomorphism, the open subset

Y_{fg}, $g \in R_{nm}$, corresponds to the open subset Z_{g/f^m}. Similarly, the isomorphism

$$\mathcal{O}_Z(Z_{g/f^m}) = S(g/f^m)^{-1}(S(f)_h^{-1}R) \cong S(fg)_h^{-1}R = \mathcal{O}_Y(Y_{fg})$$

shows that we can extend our homeomorphism to an isomorphism of ringed spaces $Y_f \cong Z$. □

Now take $\mathfrak{p} \in Y = \operatorname{Proj} R$, and take some element $f \in R_1 \setminus \mathfrak{p}_1$. Then $\mathfrak{p}$ is in Y_f; by the Lemma above, this gives us an affine open neighborhood of $\mathfrak{p}$. Thus $\operatorname{Proj} R$ is a scheme.

Sending $a \in A$ to $a/f^0 \in \mathcal{O}_Y(Y_f)$ gives the ring homomorphism $p^* : A \to \mathcal{O}_Y(Y)$, and hence the *structure morphism* $p : \operatorname{Proj} R \to \operatorname{Spec} A$.

Example 1.6.4. We take the most basic example, namely

$$R = k[X_0, \dots, X_n],$$

k a field. The scheme $\operatorname{Proj} R$ is then the *projective n-space over k*, $\mathbb{P}^n_k \to \operatorname{Spec} k$. We have the affine open cover $\mathbb{P}^n_k = \cup_{i=0}^n U_i$, where $U_i = (\operatorname{Proj} R)_{X_i} = \operatorname{Spec} k[X_0/X_i, \dots, X_n/X_i]$. As $k[X_0/X_i, \dots, X_n/X_i]$ is clearly a polynomial ring over k in variables X_j/X_i, $j \neq i$, we have the isomorphisms $U_i \cong \mathbb{A}^n_k$. The change of coordinates in passing from U_i to U_j is just

$$(X_m/X_j) = (X_m/X_i)(X_i/X_j),$$

which is the same as the standard patching data for the complex or real projective spaces.

We have a similar description of the F-valued points of $\mathbb{P}^n_k$, for F/k an extension field. Indeed, if $f : \operatorname{Spec} F \to \mathbb{P}^n_k$ is a morphism over $\operatorname{Spec} k$, then, as $|\operatorname{Spec} F|$ is a single point, f must factor through some $U_i \subset \mathbb{P}^n_k$. Thus, we have the F-valued point of $\operatorname{Spec} k[X_0/X_i, \dots, X_n/X_i]$, i.e., a homomorphism $\psi : k[X_0/X_i, \dots, X_n/X_i] \to F$, which is given by the values $\psi(X_m/X_i) = x_m^{(i)}$ for $m \neq i$, $\psi(X_i/X_i) = x_i^{(i)} = 1$. If we make a different choice of affine open U_j, we have the point $(x_0^{(j)}, \dots, x_n^{(j)})$ with $x_m^{(j)} = x_m^{(i)}/x_j^{(i)}$ for $m = 0, \dots, n$. Thus, we have the familiar description of $\mathbb{P}^n_k(F)$ as

$$\mathbb{P}^n_k(F) = \big\{x = (x_0, \dots, x_n) \in F^{n+1} \setminus \{0\}\big\}/x \sim \lambda \cdot x,\ \lambda \in F \setminus \{0\}.$$

We denote the equivalence class of a point $(x_0, \dots, x_n)$ by $(x_0 : \ldots : x_n)$.

It is not hard to see that a (graded) surjection of graded A-algebras $R \to \bar{R}$ gives rise to a closed embedding $\operatorname{Proj} \bar{R} \to \operatorname{Proj} R$, and this in turn identifies the collection of closed subschemes of $\operatorname{Proj} R$ with the collection of homogeneous ideals $J \subset R$, where we identify two such ideals J and J' if the localizations J_x, J'_x agree for all $x \in R_1$. For example, if we have homogeneous polynomials $f_1, \ldots, f_r \in k[X_0, \ldots, X_n]$, these generate a homogeneous ideal $J = (f_1, \ldots, f_r)$, a give us the closed subscheme $Y := \mathrm{Proj}\, k[X_0, \ldots, X_n]/J$ of $\mathbb{P}^n_k$. The F-valued points of Y are exactly the F-valued points $(x_0 : \ldots : x_n)$ with $f_j(x_0, \ldots, x_n) = 0$ for all j.

More generally, if $R = \oplus_{n\geq 0} R_n$ is a graded A algebra satisfying our conditions (1) and (2), choosing A-module generators $r_0, \ldots, r_n$ for R_1 defines the surjection of graded A-algebras $\pi : A[X_0, \ldots, X_n] \to R$ and thus identifies $\mathrm{Proj}\, R$ with the closed subscheme of $\mathbb{P}^n_A$ defined by the homogeneous ideal $\ker \pi$.

1.6.5. *Properness.* The main utility of Proj is that it gives a direct means of constructing proper morphisms without going to the trouble of explicitly gluing affine schemes.

Proposition 1.6.6. *Let R be a graded A-algebra satisfying (1) and (2) above. Then the structure morphism $p : \mathrm{Proj}\, R \to \mathrm{Spec}\, A$ is a proper morphism of finite type.*

Proof. If $f_1, \ldots, f_s$ generate R_1 over A, then the finite affine open cover

$$\mathrm{Proj}\, R = \cup_{i=1}^s (\mathrm{Proj}\, R)_{f_i}$$

exhibits p as a morphism of finite type. To check that p is proper, we use the valuative criterion of Proposition 1.4.21.

So, let $\mathcal{O}$ be a DVR with quotient field F and maximal ideal (t), and suppose we have a commutative diagram

$$\begin{tikzcd} \mathrm{Spec}\, F \arrow[r, "f"] \arrow[d, "\eta"'] & \mathrm{Proj}\, R \arrow[d, "p"] \\ \mathrm{Spec}\, \mathcal{O} \arrow[r, "g"'] & \mathrm{Spec}\, A \end{tikzcd}$$

Replacing A with $\mathcal{O}$ and R with $R \otimes_A \mathcal{O}$, we may assume that $g = \mathrm{id}$. R is a quotient of $\mathcal{O}[X_0, \ldots, X_m]$ for some m, so we may replace $\mathrm{Proj}\, R$ with $\mathbb{P}^m_{\mathcal{O}}$.

One can extend our characterization of the F-valued points of $\mathbb{P}^m_{\mathcal{O}}$ to the $\mathcal{O}$-valued points as follows: The $\mathcal{O}$-valued points of $\mathbb{P}^m_{\mathcal{O}}$ are $n+1$-tuples $(r_0, \ldots, r_m)$ of elements of $\mathcal{O}$ with not all r_i in (t), modulo multiplication by units in $\mathcal{O}$.

The F-valued point f of $\mathbb{P}^n_{\mathcal{O}}$ consists of an $n+1$-tuple $(f_0, \ldots, f_m)$, $f_i \in F$, with not all $f_i = 0$, modulo scalar multiplication by $F^\times$. Write each f_i as

$$f_i = u_i t^{n_i}$$

where (t) is the maximal ideal of $\mathcal{O}$, the u_i are units and the n_i integers. Letting n be the minimum of the n_i, $(u_0t^{n_0-n}, \ldots, u_mt^{n_m-n})$ gives the same F-valued point as f and all the coordinates are in $\mathcal{O}$, not all in (t), giving us a lifting $\operatorname{Spec}\mathcal{O} \to \mathbb{P}^m_{\mathcal{O}}$. Our characterization of $\mathbb{P}^m_{\mathcal{O}}(\mathcal{O})$ also proves uniqueness of the lifting. □

1.6.7. *Projective and quasi-projective morphisms.*

Definition 1.6.8. Let X be a noetherian k-scheme. A morphism $f : Y \to X$ is called *projective* if f admits a factorization $f = p \circ i$, where $p : \mathbb{P}^n_k \times_k X \to X$ is the projection and $i : Y \to: \mathbb{P}^n_k \times_k X$ is a closed embedding. A morphism $f : Y \to X$ is called *quasi-projective* if f admits a factorization $f = \bar{f} \circ j$, with $j : Y \to \bar{Y}$ an open immersion and $\bar{f} : \bar{Y} \to X$ a projective morphism.

A k-scheme X is called a projective (resp. quasi-projective) k-scheme if the structure morphism $p : X \to \operatorname{Spec} k$ is projective (resp. quasi-projective).

Proposition 1.6.9. *A projective morphism is a proper morphisms of finite type.*

Proof. Since a closed embedding is a proper morphism of finite type, it suffices to prove the case of the projection $\mathbb{P}^n_k \times_k X \to X$, which follows from Proposition 1.6.6 and the fact that the property of a morphism being proper and of finite type is preserved by arbitrary base-change. □

1.6.10. *Globalization.* One can use the operation Proj to define proper morphisms over non-affine schemes as well. One simply replaces graded A-algebras with graded sheaves of $\mathcal{O}_X$-algebras. If $\mathcal{R} = \oplus_{n\geq 0}\mathcal{R}_n$ is a sheaf of graded $\mathcal{O}_X$-algebras for some noetherian scheme X, then $\mathcal{R}(U)$ is a graded $\mathcal{O}_X(U)$-algebra for all open subschemes U of X. We require that X admits an affine open cover $X = \cup_i U_i := \operatorname{Spec} A_i$ such that the graded A_i-algebra $\mathcal{R}(U_i)$ satisfies our conditions (1) and (2) for each i. The A_i-schemes $\operatorname{Proj}_{A_i}\mathcal{R}(U_i)$ then patch together to give the X-scheme $p : \operatorname{Proj}_{\mathcal{O}_X}\mathcal{R} \to X$. p is clearly a proper morphism, as properness is a local property on the base scheme.

One can show that, in case X is a quasi-projective scheme over a field k, then $\operatorname{Proj}_{\mathcal{O}_X}\mathcal{R} \to X$ is actually a projective morphism, i.e., there exists a closed embedding $\operatorname{Proj}_{\mathcal{O}_X}\mathcal{R} \to \mathbb{P}^N_k \times_k X$ for some $N >> 0$. Thus, we are not getting any new schemes over X by this added generality, however, this does make some useful constructions more natural.

Example 1.6.11. Let $\mathcal{E}$ be a rank $n+1$ locally free sheaf of $\mathcal{O}_X$-modules on a noetherian scheme X. Form the sheaf of symmetric algebras $\mathcal{R} := \mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{E})$. Take an affine open cover $X = \cup_i U_i$ trivializing $\mathcal{E}$. If $U_i = \mathrm{Spec}\, A_i$, a choice of an isomorphism $\mathrm{res}_{U_i}\mathcal{E} \cong \mathcal{O}_{U_i}^{n+1}$ gives an isomorphism

$$\mathcal{R}(U_i) \cong A_i[X_0, \ldots, X_n].$$

On $U_i \cap U_j$ the isomorphism $\mathrm{res}_{U_i\cap U_j}\mathrm{res}_{U_i}\mathcal{E} \cong res_{U_i\cap U_j}\mathrm{res}_{U_j}\mathcal{E}$ yields a change of coordinates in the variables X_l, given by an invertible matrix $g_{ij} \in \mathrm{GL}_{n+1}(\mathcal{O}_X(U_i \cap U_j))$. This data gives us a rank $n+1$ vector bundle $E \to X$ with sheaf of sections isomorphic to $\mathcal{E}$, and one has the isomorphism of X-schemes

$$\mathrm{Proj}_{\,\mathcal{O}_X}\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{E}) \cong \mathbb{P}(E^\vee),$$

where $E^\vee$ is the dual bundle and $\mathbb{P}(E^\vee) \to X$ is the fiber bundle with fiber the projective space $\mathbb{P}(E^\vee_x)$ over a point $x \in X$,

$$\mathbb{P}(E^\vee_x) := E^\vee_x \setminus \{0\}/v \sim \lambda v;\ \lambda \in k(x)^*.$$

We write $\mathbb{P}(\mathcal{E})$ for $\mathrm{Proj}_{\,\mathcal{O}_X}\mathrm{Sym}^*_{\mathcal{O}_X}(\mathcal{E})$.

1.6.12. *Blowing up a subscheme.* A very different type of Proj is the blow-up of a subscheme $Z \subset X$ with X in $\mathbf{Sch}_k$. Let $\mathcal{I}_Z$ be the ideal sheaf defining Z. The X-scheme $\pi : \mathrm{Bl}(X, Z) \to X$ is defined as

$$\mathrm{Bl}(X, Z) := \mathrm{Proj}_{\,\mathcal{O}_X}(\oplus_{n\geq 0}\mathcal{I}_Z^n),$$

where we give $\mathcal{I}_Z^* := \oplus_{n\geq 0}\mathcal{I}_Z^n$ the structure of a graded $\mathcal{O}_X$-algebra by using the multiplication maps $\mathcal{I}_Z^n \times \mathcal{I}_Z^m \to \mathcal{I}_Z^{n+m}$. As we have seen, π is a proper morphism, and is projective if for example X is quasi-projective over k.

To analyze the morphism π, let $U = X \setminus Z$. The restriction of $\mathcal{I}_Z$ to U is $\mathcal{O}_U$, so the restriction of $\mathcal{I}_Z^*$ to U is $\oplus_{n\geq 0}\mathcal{O}_U$ with the evident multiplication. This is just the graded $\mathcal{O}_U$-algebra $\mathcal{O}_U[X_0]$ ($\deg X_0 = 1$); using the evident correspondence between graded ideals in $A[X_0]$ and ideals in A, for a commutative ring A, we see that

$$\pi^{-1}(U) = \mathrm{Proj}_{\,\mathcal{O}_U}(\mathcal{O}_U[X_0]) = U,$$

with π the identity map. Over Z, something completely different happens: $Z \times_X \mathrm{Bl}(X, Z)$ is just $\mathrm{Proj}_{\,\mathcal{O}_Z}(\mathcal{I}_Z^* \otimes_{\mathcal{O}_X} \mathcal{O}_Z)$. Since $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}_Z$, we find

$$\mathcal{I}_Z^n \otimes_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}_Z = \mathcal{I}_Z^n/\mathcal{I}_Z^{n+1},$$

hence

$$\pi^{-1}(Z) = \mathrm{Proj}_{\,\mathcal{O}_Z}(\oplus_{n\geq 0}\mathcal{I}_Z^n/\mathcal{I}_Z^{n+1}).$$

The coherent sheaf of $\mathcal{O}_Z$-modules $\mathcal{I}_Z/\mathcal{I}_Z^2$ is called the *conormal sheaf* of Z in X. If Z is locally defined by a regular sequence of length d then

$\mathcal{I}_Z/\mathcal{I}_Z^2$ is a locally free sheaf of $\mathcal{O}_Z$-modules (of rank d); the dual of the corresponding vector bundle on Z is the *normal bundle* of Z in X, $N_{Z/X}$. For example, if both Z and X are smooth over k, and $d = \mathrm{codim}_X Z$, then Z is locally defined by a regular sequence of length d and $N_{Z/X}$ is the usual normal bundle from differential topology.

Assuming that Z is locally defined by a regular sequence of length d, we have the isomorphism of graded $\mathcal{O}_Z$-algebras

$$\oplus_{n\geq 0}\mathcal{I}_Z^n/\mathcal{I}_Z^{n+1} \cong \mathrm{Sym}^*(\mathcal{I}_Z/\mathcal{I}_Z^2).$$

Thus $\pi^{-1}(Z)$ is a $\mathbb{P}^{d-1}$-bundle over Z, in fact

$$\pi^{-1}(Z) = \mathbb{P}(N_{Z/X}).$$

Thus, we have "blown-up" Z in X by replacing Z with the projective space bundle of normal lines to Z.

The simplest example is the blow-up of the origin $0 \in \mathbb{A}^m_k$. This yields

$$\begin{aligned}
&\mathrm{Bl}(X, Z)\\
&\quad = \mathrm{Proj}_{\,k[X_1,\ldots,X_m]}(\oplus_{n\geq 0}(X_1, \ldots, X_m)^n)\\
&\quad = \mathrm{Proj}_{\,k[X_1,\ldots,X_n]}(k[X_1, \ldots, X_m, Y_1, \ldots, Y_m]/(\ldots X_iY_j - X_jY_i \ldots)),
\end{aligned}$$

where we give the Y_i's degree 1. The fiber over 0 is thus

$$\mathrm{Proj}_{\,k}(k[Y_1, \ldots, Y_m]) = \mathbb{P}^{m-1}_k,$$

which we identify with the projective space of lines in $\mathbb{A}^m_k$ through 0.

At this point, we should mention the fundamental result of Hironaka on resolution of singularities:

Theorem 1.6.13 (Hironaka [15])**.** *Let k be an algebraically closed field of characteristic zero, X a reduced finite type k-scheme. Then there is a sequence of morphisms of reduced finite type k-schemes*

$$X_N \to X_{N-1} \to \ldots \to X_1 \to X_0 = X$$

and reduced closed subschemes $Z_n \subset X_{n\,\mathrm{sing}}$, $n = 0, \ldots, N-1$ such that

1. *each Z_n is smooth over k.*
2. *$X_{n+1} \to X_n$ is the morphism* $\mathrm{Bl}(X_n, Z_n) \to X_n$.
3. *X_N is smooth over k.*

The hypothesis that k be algebraically closed was later removed, but the result in characteristic $p > 0$ and in mixed characteristic is still an important open problem.

1.6.14. *The tautological invertible sheaf.* Let $R = \oplus_{n\geq 0} R_n$ be a graded A-algebra satisfying (1) and (2) above. Let $Y = \operatorname{Proj} R$, and fix an integer n. We form the sheaf $\mathcal{O}_Y(n)$ similarly to our construction of $\mathcal{O}_Y$: for $f \in R_m$, the sections of $\mathcal{O}_Y(n)$ over Y_f are the degree n elements in the ring of fractions $S(f)^{-1}R$, i.e., fractions of the form g/f^N with $\deg g = N \deg f + n$. Just as for $\mathcal{O}_Y$, this all patches together to give a well-defined sheaf of abelian groups on Y.

If g/f^a is a section of $\mathcal{O}_Y$ over Y_f and g'/f^b is a section of $\mathcal{O}_Y(n)$ over Y_f, clearly gg'/f^{a+b} is a section of $\mathcal{O}_Y(n)$ over Y_f. Thus, $\mathcal{O}_Y(n)$ has the structure of a sheaf of $\mathcal{O}_Y$-modules.

For $h \in R_a$, multiplication by h gives a morphism of sheaves of $\mathcal{O}_Y$-modules $\mathcal{O}_Y(n) \to \mathcal{O}_Y(a+n)$. Over Y_h, $\times h$ is an isomorphism; as $Y = \cup_{h\in R_n} Y_h$, the sheaves $\mathcal{O}_Y(n)$ are all locally isomorphic (as sheaves of $\mathcal{O}_Y$-modules). In particular, $\mathcal{O}_Y(n)$ is locally isomorphic to $\mathcal{O}_Y(0) = \mathcal{O}_Y$, thus $\mathcal{O}_Y(n)$ is an invertible sheaf.

This construction is canonical, so extends to the setting of $Y = \operatorname{Proj}_{\mathcal{O}_X}(\mathcal{R})$ for a sheaf of graded $\mathcal{O}_X$-algebras $\mathcal{R}$, X some noetherian scheme.

$\mathcal{O}(1)$ is called the *tautological* invertible sheaf on $\operatorname{Proj}_{\mathcal{O}_X}(\mathcal{R})$. The surjection

$$\mathcal{R}_1 \otimes_{\mathcal{O}_X} \mathcal{R} \to \oplus_{n\geq 1} \mathcal{R}_n$$

gives rise to the *tautological quotient map*

$$p^*\mathcal{R}_1 \to \mathcal{O}(1).$$

where $p : \operatorname{Proj}_{\mathcal{O}_X}(\mathcal{R}) \to X$ is the structure morphism.

Example 1.6.15. We examine the case of $\mathcal{R} = \operatorname{Sym}^*_{\mathcal{O}_X}\mathcal{E}$ for a a rank $n+1$ locally free sheaf of $\mathcal{O}_X$-modules on X. Let $E \to X$ be the vector bundle with sheaf of sections $\mathcal{E}$, so we have the isomorphism of $\mathbb{P}^n$-bundles

$$\operatorname{Proj}_{\mathcal{O}_X}(\operatorname{Sym}^*_{\mathcal{O}_X}\mathcal{E}) \cong \mathbb{P}(E^\vee).$$

We have the line bundle $L \to \mathbb{P}(E^\vee)$ with fiber over a point $(x,\ell) \in \mathbb{P}(E_x^\vee)$ being the line through 0 in $E_x^\vee$ corresponding to ℓ. The sheaf of sections of $L^\vee$ is $\mathcal{O}(1)$, and the tautological quotient map becomes

$$p^*\mathcal{E} \to \mathcal{O}(1).$$

If we dualize the map corresponding to the tautological inclusion

$$L \to p^*E^\vee$$

and take sheaves of sections, we arrive at the tautological quotient map.

2. Algebraic cycles, Chow groups and higher Chow groups

The study of algebraic cycles has a rich history, stretching back at least to the study of linear systems on algebraic curves in the late 19th century. In this section, we will sketch some of the main definitions and concepts in this area, with the goal being the description of the Chow groups and higher Chow groups, and their relationship with algebraic K-theory and higher algebraic K-theory.

2.1. Algebraic cycles.

Definition 2.1.1. Let X be in $\mathbf{Sch}_k$. We let $Z_n(X)$ denote the free abelian group on the closed integral subschemes W of X with $\dim_k W = n$. An element $\sum_i n_i W_i$ is called an *algebraic cycle* on X (of dimension n). If X is locally equi-dimensional over k, we let $Z^n(X)$ denote the free abelian group on the codimension n integral closed subschemes of X. Elements of $Z^n(X)$ are *codimension n algebraic cycles on X*.

For $W = \sum_i n_i W_i \in Z_n(X)$ (or in $Z^n(X)$, when defined) with all $n_i \neq 0$, the union $\cup_i |W_i|$ is called the *support* of W, denoted $|W|$.

An element of $Z^1(X)$ is called a *(Weil) divisor* on X; an element of $Z_0(X)$ is a *zero-cycle* on X.

2.1.2. *Push-forward.* Let $f : Y \to X$ be a proper morphism in $\mathbf{Sch}_k$. If $W \subset Y$ is an integral closed subscheme of dimension n, then $f(W)$ (with reduced scheme structure) is an integral closed subscheme of X. Also, since $k(W)$ is a finitely generated field extension of $k(f(W)$ via $f^* : k(f(W)) \to k(W)$, we have $\dim_k W \geq \dim_k f(W)$ and if $\dim_k W = \dim_k f(W)$, then the field extension degree $[k(W) : k(f(W))]$ is finite. We define $f_*(W) \in Z_n(X)$ by

$$f_*(W) := \begin{cases} 0 & \text{if } \dim_k W > \dim_k f(W) \\ [k(W) : k(f(W))] \cdot f(W) & \text{if } \dim_k W = \dim_k f(W). \end{cases}$$

We extend f_* to $f_* : Z_n(Y) \to Z_n(X)$ by linearity.

Since the field extension degree is multiplicative in towers, we have the functoriality

$$f_* \circ g_* = (f \circ g)_*$$

for proper morphisms $g : Z \to Y$, $f : Y \to X$.

If $f : Y \to X$ is not proper, we still have a partially defined push-forward operation. Let $Z_n(Y, f) \subset Z_n(Y)$ be the subgroup generated by the integral closed $W \subset Y$ such that $f_{|W} : W \to X$ is proper. Using the same formula as above yields $f_* : Z_n(Y, f) \to Z_n(X)$, with the same (partially defined) functoriality.

2.1.3. *Pull-back.* Pull-back of cycles is somewhat more complicated. First, consider a smooth morphism $f : Y \to X$ of relative dimension d. For a integral dimension n closed subscheme W, the scheme-theoretic inverse image $f^{-1}(W)$ is reduced and each irreducible component has dimension $d + n$. We therefore define

$$f^*(W) := \sum_Z 1 \cdot Z \in Z_{n+d}(Y),$$

where the sum is over the irreducible components of $f^{-1}(W)$. This pull-back is functorial (for smooth morphisms), $(fg)^* = g^* f^*$.

Essentially the same simple-minded procedure works for *flat* morphisms, except that the coefficient of each component is not necessarily 1. Let $f : Y \to X$ be a flat morphism of relative dimension d, and $W \subset X$ an integral dimension n closed subscheme. Then each reduced irreducible component Z of $f^{-1}(W)$ has dimension $d + n$, but $f^{-1}(W)$ is not necessarily reduced. Letting $\mathcal{O}_{Y,Z}$ be the local ring of the generic point of Z, the closed subscheme $f^{-1}(W) \cap \operatorname{Spec} \mathcal{O}_{Y,Z}$ has underlying space equal to the generic point of Z, so the defining ideal $\mathcal{I}_{f^{-1}(W),Z} \subset \mathcal{O}_{Y,Z}$ has radical equal to the maximal ideal $m_{Y,Z}$ of $\mathcal{O}_{Y,Z}$. Thus $m_{Y,Z}^N \subset \mathcal{I}_{f^{-1}(W),Z} \subset m_{Y,Z}$, hence the $\mathcal{O}_{Y,Z}$-module $\mathcal{O}_{Y,Z}/\mathcal{I}_{f^{-1}(W),Z}$ has *finite length*, that is, there is a finite filtration

$$0 = M_0 \subset \ldots \subset M_\ell = \mathcal{O}_{Y,Z}/\mathcal{I}_{f^{-1}(W),Z} = \mathcal{O}_{f^{-1}(W),Z}$$

with $M_{n+1}/M_n \cong k(Z)$ as an $\mathcal{O}_{Y,Z}$-module. The number ℓ is independent of any choices, and is called the *length* of the $\mathcal{O}_{Y,Z}$-module $\mathcal{O}_{f^{-1}(W),Z}$. We define the multiplicity $m(Z; f^{-1}(W))$ to be this length, and set

$$f^*(W) := \sum_Z m(Z; f^{-1}(W)) \cdot Z \in Z_{n+d}(Y).$$

One can also show fairly easily that this defines a functorial pull-back for flat morphisms. Since the multiplicity $m(Z; f^{-1}(W))$ is 1 if and only if Z is reduced at its generic point, the definitions of flat and smooth pull-back agree.

There are lots of non-flat morphisms, for example, the inclusion of a closed subscheme, so this theory is still not sufficient. To give a unified treatment for a number of different cases, we introduce the notion of a *weak l.c.i.-morphism.*

Definition 2.1.4. (1) A closed embedding $i : Y \to X$ is a *regular embedding* if for each $y \in Y$, the ideal sheaf $\mathcal{I}_{Y,y} \subset \mathcal{O}_{X,y}$ is generated by a regular sequence $t_1, \ldots, t_d$ in $\mathcal{O}_{X,y}$. d is called the *codimension* of the embedding.

(2) A morphism $f : Y \to X$ of finite type k-schemes is an *l.c.i.-morphism* if f admits a factorization in $\mathbf{Sch}_k$ as $f = p \circ i$, where $i : Y \to P$ is a regular embedding and $p : P \to X$ is smooth and quasi-projective. (3) A morphism $f : Y \to X$ of finite type, locally equi-dimensional k-schemes is a *weak l.c.i.-morphism* if f admits a factorization in $\mathbf{Sch}_k$ as $f = p \circ i$, where $i : Y \to P$ is a regular embedding and $p : P \to X$ is flat .

Remark 2.1.5. One can show without difficulty that the composition of two l.c.i.-morphisms is again an l.c.i.-morphism; this property is not clear for weak l.c.i.-morphisms and may in fact be false. In practice, this will not cause any difficulty, as the most common types of weak l.c.i.-morphisms are closed under composition. In particular, the property of being a weak l.c.i.-morphism is stable under base-change by a locally equi-dimensional k-scheme of finite type.

Examples 2.1.6. (1) A *Cartier divisor* D on a scheme X is a closed subscheme such that the inclusion $i : D \to X$ is a regular embedding of codimension one, i.e., $\mathcal{I}_D$ is locally principal, with local generator a non-zero divisor in $\mathcal{O}_X$.

(2) Let $f : Y \to X$ be a morphism of finite type k-schemes with X smooth over k and Y locally equi-dimensional over k. We can factor f as $p \circ i$, where $p : Y \times_k X \to X$ is the projection, and $i : Y \to Y \times_k X$ is the inclusion (id_Y, f), i.e., the graph of f. Since X is smooth over k, each point $x \in |X|$ has local parameters $t_1, \ldots, t_n$, $n = \dim_k(X, x)$ in $\mathcal{O}_{X,x}$; if $x = f(y)$ and we set $f_i := f^*(t_i)$, $i(Y)$ is defined near $(y, f(y))$ by the ideal generated by the elements $f_i \otimes 1 - 1 \otimes t_i$, $i = 1, \ldots, n$. The fact that the t_i form a regular sequence in $\mathcal{O}_{X,x}$ easily implies that these element also form a regular sequence in $\mathcal{O}_{Y \times_k X,(y,f(y))}$, so i is a regular embedding. Since k is a field $Y \to \operatorname{Spec} k$ is flat, hence the projection $Y \times_k X \to X$ is also flat. Thus f is a weak l.c.i.-morphism. If Y is smooth over k, then f is an l.c.i.-morphism.

Since we already know how to define pull-back for flat morphisms, the hard part is to define i^* for a regular embedding $i : Y \to X$. This will be at best a partially defined operation, since not all cycles will intersect Y in the correct dimension. However, we do have the following result, which is a direct consequence of Krull's principal ideal theorem (Theorem 1.5.9).

Lemma 2.1.7. *Let $i : Y \to X$ be a codimension d regular embedding in $\mathbf{Sch}_k$, $W \subset X$ an integral closed subscheme. Suppose that X is locally equidimensional over k. Then Y is also locally equi-dimensional over*

k and each irreducible component Z of $i^{-1}(W)$ has

$$\mathrm{codim}_Y Z \leq \mathrm{codim}_X W.$$

It follows from Lemma 2.1.7 that a regular embedding $i : Y \to X$ in $\mathbf{Sch}_k$, with X locally equi-dimensional over k, is a weak l.c.i.-morphism. Also, for $f : Y \to X$ a weak l.c.i.-morphism, and $W \subset X$ an integral closed subscheme, each irreducible component Z of $f^{-1}(W)$ has $\mathrm{codim}_Y Z \leq \mathrm{codim}_X W$.

Definition 2.1.8. Let $f : Y \to X$ be a weak l.c.i.-morphism in $\mathbf{Sch}_k$ and $W \subset X$ an integral closed subscheme. We say that $f^*(W)$ is defined if each irreducible component Z of $f^{-1}(W)$ has

$$\mathrm{codim}_Y Z = \mathrm{codim}_X W.$$

We let $Z^n(X)_f \subset Z^n(X)$ be the subgroup generated by those codimension n W for which $f^*(W)$ is defined.

If f is a closed embedding, we will also say that Y *intersects* W *properly* if $f^*(W)$ is defined.

Now, suppose we have $f : Y \to X$ a weak l.c.i.-morphism in $\mathbf{Sch}_k$ and $W \subset X$ an integral closed subscheme such that $f^*(W)$ is defined. We set

$$f^*(W) := \sum_Z m(Z; f^*(W)) \cdot Z \in Z^{\mathrm{codim}_W X}(Y),$$

where the sum is over the irreducible components Z of $f^{-1}(W)$, and the multiplicities $m(Z; f^*(W))$ still need to be defined. In general, taking the length of $\mathcal{O}_{f^{-1}(W),Z}$ is *not* the right answer; the resulting pull-back would not in general be functorial even when all terms are defined.

The first correct answer to the problem of defining the multiplicities $m(Z; f^*(W))$ was found by Weil [28], using a technique of specialization from a general transverse intersection (where all multiplicities are 1) to the particular intersection being considered; this gives a good theory in the case of morphisms of smooth k-schemes. The first nice formula in our setting is the homological intersection multiplicity of Serre [23]:

$$m(Z; f^*(W)) = \sum_{i=0}^{\infty} (-1)^i \ell_{\mathcal{O}_{Y,Z}}(\mathrm{Tor}_i^{\mathcal{O}_{X,W'}}(\mathcal{O}_{Y,Z}, \mathcal{O}_{W,W'})).$$

Here W' is the closure in X of $f(Z)$, $\ell_{\mathcal{O}_{Y,Z}}(-)$ means "length as an $\mathcal{O}_{Y,Z}$-module" and the Tor-modules $\mathrm{Tor}_i^{\mathcal{O}_{X,W}}(\mathcal{O}_{Y,Z}, M)$ can be defined as the left-derived functors of the tensor-product functor

$$\begin{aligned} &\mathcal{O}_{Y,Z} \otimes_{\mathcal{O}_{X,W}} - : \mathrm{Mod}_{\mathcal{O}_{X,W}} \to \mathrm{Mod}_{\mathcal{O}_{Y,Z}} \\ &M \mapsto \mathcal{O}_{Y,Z} \otimes_{\mathcal{O}_{X,W}} M. \end{aligned}$$

Although the sum is *a priori* infinite, the condition that f is a weak l.c.i.-morphism implies that $\mathrm{Tor}_i^{\mathcal{O}_{X,W'}}(\mathcal{O}_{Y,Z}, M) = 0$ for $i > \dim_k Y$. The fact that Z is an irreducible component of $f^{-1}(W)$ implies that all the Tor-modules have finite length, so the formula is well-defined.

We extend the definition of f^* to all of $Z^n(X)_f$ by linearity. The work of Serre yields the following partially defined functoriality

Proposition 2.1.9. *Let $f : Y \to X$ and $g : Z \to Y$ be weak l.c.i.-morphisms in $\mathbf{Sch}_k$. Take $W \in Z^n(X)_f$ and suppose that g^*W' is defined for each irreducible component W' of $f^{-1}(W)$. Then W is in $Z^n(X)_{gf}$, $f^*(W)$ is in $Z^n(Y)_g$ and*

$$g^*(f^*(W)) = (fg)^*(W).$$

The identity

$$\mathrm{Tor}_0^{\mathcal{O}_{X,W}}(\mathcal{O}_{Y,Z}, \mathcal{O}_W) = \mathcal{O}_{Y,Z} \otimes_{\mathcal{O}_{X,W}} \mathcal{O}_W = \mathcal{O}_{f^{-1}(W),Z}$$

and the vanishing of $\mathrm{Tor}_i^{\mathcal{O}_{X,W}}(\mathcal{O}_{Y,Z}, M)$ for f flat and $i > 0$ implies that all our intersection multiplicities agree.

Remark 2.1.10. In case f is a closed embedding $i : Y \to X$ (not necessary a regular embedding), with Y irreducible, we often write $Y \cdot W$ for $i^*(W)$ and $m(Z; Y \cdot W)$ for $m(Z, i^*(W))$.

In case X is smooth, each closed embedding $Y \to X$ is a weak l.c.i.-morphism. We extend the partially defined operation $Y \cdot W$ to a partially defined operation on cycles by linearity, where $Y \cdot W$ is defined if for each irreducible component Y' of Y, W' of W and Z of $Y' \cap W'$ we have

$$\mathrm{codim}_X Z = \mathrm{codim}_X Y' + \mathrm{codim}_X W'.$$

This product is called the *intersection product* of cycles.

The commutativity of Tor-modules yields the commutativity of the intersection product:

$$Y \cdot W = W \cdot Y.$$

Similarly, the functoriality of cycle pull-back gives us the associativity of intersection product:

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$

assumimg all products are defined.

Example 2.1.11. Let $i : D \to X$ be the inclusion of a Cartier divisor, $W \subset X$ an integral closed subscheme. Since $\mathrm{codim}_X D = 1$, W intersects D properly if and only if W is not contained in D. As D is locally defined by a non-zero divisor, one sees that this implies

$$\mathrm{Tor}_i^{\mathcal{O}_{X,Z}}(\mathcal{O}_{D,Z}, \mathcal{O}_{W,Z}) = 0$$

for $i > 0$, Z an irreducible component of $D \cap W$ (the closed subscheme of X defined by $\mathcal{I}_D + \mathcal{I}_W$). Since

$$\mathcal{O}_{D\cap W,Z} = \mathcal{O}_{D,Z} \otimes_{\mathcal{O}_{X,Z}} \mathcal{O}_{W,Z} = \mathrm{Tor}_0^{\mathcal{O}_{X,Z}}(\mathcal{O}_{D,Z}, \mathcal{O}_{W,Z})$$

the intersection multiplicity $m(Z; D \cdot W)$ is given by the length of $\mathcal{O}_{D\cap W,Z}$ as an $\mathcal{O}_{X,Z}$-module:

$$D \cdot W = \sum_Z \ell_{\mathcal{O}_{X,Z}}(\mathcal{O}_{D\cap W,Z}) \cdot Z,$$

where the sum is over the irreducible components Z of $D \cap W$.

2.1.12. *Projection formula.* Pull-back and pushforward of cycles are related as follows:

Proposition 2.1.13. *Let*

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"'] & X \end{tikzcd}$$

be a cartesian square in $\mathbf{Sch}_k$ *with* f *projective and* g *weak l.c.i.. Assume that* X, Y, Z *and* W *are all locally equi-dimensional over* k. *Let* A *be in* $Z^n(Y)$ *such that* $g'^*(A)$ *and* $g^*(f_*(A))$ *are defined. Then*

$$f'_*(g'^*(A)) = g^*(f_*(A)).$$

This result follows from a similar behavior of Tor-modules under pull-back and finite extension; we omit the proof.

The classical projection formula is an immediate consequence:

Proposition 2.1.14. *Let* $f : Y \to X$ *be a proper weak l.c.i.-morphism in* $\mathbf{Sch}_k$, *with* X *and* Y *locally equi-dimensional over* k. *Take* $A \in Z^n(Y)$ *and* $B \in Z^m(X)$, *and suppose that* $f^*(B)$ *and the intersection products* $A \cdot f^*(B)$ *and* $f_*(A) \cdot B$ *are defined. Then*

$$f_*(A \cdot f^*(B)) = f_*(A) \cdot B.$$

Proof. Consider the commutative diagram

$$\begin{tikzcd} Y \times_k X \arrow[dd, "f\times \mathrm{id}"'] & Y \times_k Y \arrow[l, "\mathrm{id}\times f"'] & \\ & & Y \arrow[ul, "\delta_Y"'] \arrow[ull, "(\mathrm{id}\times f)\circ\delta_Y"] \arrow[dl, "f"] \\ X \times_k X & X \arrow[l, "\delta_X"] & \end{tikzcd}$$

This identifies Y with the fiber product $(Y \times_k X) \times_{X \times_k X} X$, so we have

$$\begin{aligned} f_*(A \cdot f^*(B)) &= f_*(\delta_Y^*(A \times f^*(B))) \\ &= f_*((\delta_Y \circ (\mathrm{id} \times f))^*(A \times B)) \\ &= \delta_X^*((f \times \mathrm{id})_*(A \times B)) \\ &= f_*(A) \cdot B. \end{aligned}$$

□

2.2. **Chow groups.** The cycle groups are too large to carry useful information, it is necessary to impose an algebraic version of homology to have reasonable groups. This relation is called *rational equivalence*.

2.2.1. *Linear equivalence.* We begin with the classical case of rational equivalence, called *linear equivalence*. Let X be an integral finite type k-scheme, $f \in k(X)^*$ a non-zero rational function on X. As

$$k(X) = \mathcal{O}_{X,x_{\mathrm{gen}}} = \lim_{\emptyset \neq U \subset X} \mathcal{O}_X(U),$$

there is a non-empty open subscheme $U \subset X$ with $f \in \mathcal{O}_X(U) = \mathcal{O}_U(U)$. We note that an element $f \in \mathcal{O}_U(U)$ is the same as a homomorphism of k-algebras $k[T] \to \mathcal{O}_U(U)$, and hence the same as a morphism of k-schemes

$$f : U \to \mathrm{Spec}\,(k[T]) = \mathbb{A}^1_k.$$

We view $\mathbb{A}^1_k$ as the principal affine open subscheme $(\mathbb{P}^1_k)_{X_0}$ of $\mathbb{P}^1_k = \mathrm{Proj}\,_k(k[X_0, X_1])$ by setting $T = X_1/X_0$, so we have $U \times \mathbb{A}^1_k$ as an open subscheme of $X \times \mathbb{P}^1_k$. We let $\Gamma_f \subset X \times_k \mathbb{P}^1$ be the closure of the graph of f.

Let $0, \infty$ be the subschemes of $\mathbb{P}^1_k$ defined by the principal homogeneous ideals (X_1), (X_0), respectively. Clearly 0 and ∞ are Cartier divisors on $\mathbb{P}^1_k$, so $X \times 0$ and $X \times \infty$ are Cartier divisors on $X \times \mathbb{P}^1$. Since f is not identically 0 or infinite, Γ_f is not contained in $X \times 0$ or $X \times \infty$, so the intersection products $\Gamma_f \cdot X \times 0$ and $\Gamma_f \cdot X \times \infty$ are defined. We write

$$\begin{aligned} \mathrm{div}_0(f) &:= p_{1*}(\Gamma_f \cdot X \times 0) \\ \mathrm{div}_\infty(f) &= p_{1*}(\Gamma_f \cdot X \times \infty) \\ \mathrm{div}(f) &= \mathrm{div}_0(f) - \mathrm{div}_\infty(f). \end{aligned}$$

These are all elements in $Z^1(X) = Z_{\dim_k(X)-1}(X)$.

For $D = \sum_i n_i D_i \in Z^1(X)$, we write $D \geq 0$ if $n_i \geq 0$ for all i, and $D \geq D'$ if $D - D' \geq 0$. For $D = \sum_i n_i D_i$, $D' = \sum_i n_i' D_i$, we set $\min(D, D') := \sum_i \min(n_i, n_i') D_i$.

Lemma 2.2.2. *The operation* div *has the following properties:*

(1) $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$
(2) $\mathrm{div}(f+g) \geq \min(\mathrm{div} f, \mathrm{div} g)$
(3) *If f is a regular function on X ($f \in \mathcal{O}_X(X)$), then* $\mathrm{div} f \geq 0$.

Idea of proof. For simplicity, we assume that each codimension one point of X is smooth. These properties follow from another interpretation of $\mathrm{div}_0 f$, $\mathrm{div}_\infty f$ as giving the order of vanishing of f or $1/f$ at the codimension one points of X.

Let W be an integral codimension 1 closed subscheme of X. Then by Proposition 1.5.22, the local ring $\mathcal{O}_{X,W}$ is a DVR; let t be a generator for the maximal ideal. If $f \in k(X)^*$ is a non-zero rational function, then since $k(X)$ is also the quotient field of $\mathcal{O}_{X,W}$, there is a unique integer n and unit $u \in \mathcal{O}_{X,W}^*$ with

$$f = ut^n.$$

We define the map $\mathrm{ord}_W : k(X)^* \to \mathbb{Z}$ by setting $\mathrm{ord}_W(f) = n$. Clearly ord_W is a group homomorphism and

$$\mathrm{ord}_W(f+g) \geq \min(\mathrm{ord}_W(f), \mathrm{ord}_W(g)). \tag{2.2.1}$$

Also $\mathrm{ord}_W(f) > 0$ if and only if W is a component of $\Gamma_f \cap X \times 0$ and $\mathrm{ord}_W(f) < 0$ if and only if W is a component of $\Gamma_f \cap X \times \infty$. In particular, for a given f, $\mathrm{ord}_W(f) = 0$ for all but finitely mny W.

Suppose that $\mathrm{ord}_W(f) > 0$, so f is in $\mathcal{O}_{X,W}$. Since $(t^n)/(t^{n+1}) \cong k(W)$, we have

$$\ell_{\mathcal{O}_{X,W}}(\mathcal{O}_{X,W}/(f)) = \mathrm{ord}_W(f).$$

On the other hand, let U be a neighborhood of the generic point of W over which f is a regular function. The pull-back of the defining equation for $U \times 0$ by the inclusion $U \to U \times \mathbb{P}^1$ defined by (id, f) is just f. Since $\Gamma_f \cap U \times \mathbb{P}^1$ is just the image $(\mathrm{id}, f)(U)$ under this closed embedding, the length $\ell_{\mathcal{O}_{X,W}}(\mathcal{O}_{X,W}/(f))$ computes the intersection multiplicity $m(W \times 0; X \times 0 \cdot \Gamma_f)$, i.e.

$$\mathrm{ord}_W(f) = m(W \times 0; X \times 0 \cdot \Gamma_f).$$

Replacing f with $1/f$, we find similarly

$$\mathrm{ord}_W(f) = -m(W \times \infty; X \times \infty \cdot \Gamma_f)$$

if $\mathrm{ord}_W(f) < 0$. This yields

$$\mathrm{div}(f) = \sum_W \mathrm{ord}_W(f) \cdot W \in Z^1(X).$$

Property (1) thus follows from the fact that $\mathrm{ord}_W : k(X)^* \to \mathbb{Z}$ is a group homomorphism, and property(2) follows from (2.2.1). Finally, if

f is regular, then f is in $\mathcal{O}_{X,W}$ for all W, hence $\mathrm{ord}_W(f) \geq 0$ for all W, whence (3). □

Definition 2.2.3. Let X be an integral k-scheme of finite type. Divisors $D, D' \in Z^1(X)$ are called *linearly equivalent* if there is a rational function $f \in k(X)^*$ with $D - D' = \mathrm{div}(f)$, written $D \sim_l D'$. The quotient group of divisors modulo linear equivalence, $Z^1(X)/\sim_l$, is denoted $CH^1(X)$.

It follows from property (1) of Lemma 2.2.2 that the subgroup $R^1(X)$ of $Z^1(X)$ consisting of $D \sim_l 0$ is exactly the set of $\mathrm{div} f$, $f \in k(X)^*$.

2.2.4. *Invertible sheaves.* $\mathrm{CH}^1(X)$ is closely related to the *Picard group* $\mathrm{Pic}(X)$. For simplicity, we assume that X is smooth over k and that X is irreducible. $\mathrm{Pic}(X)$ is the set of isomorphism classes of locally free $\mathcal{O}_X$-modules of rank one (invertible sheaves). If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves, the tensor product $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ is locally isomorphic to $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{O}_X$, hence also an invertible sheaf. Also the $\mathcal{O}_X$-dual $\mathcal{L}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is locally isomorphic to $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X$, so $\mathcal{L}^\vee$ is an invertible sheaf. Similarly, the canonical "evaluation map" $\mathcal{L} \otimes \mathcal{L}^\vee \to \mathcal{O}_X$ is locally an isomorphism, hence an isomorphism. Thus $\mathrm{Pic}(X)$ is a group under $\otimes_{\mathcal{O}_X}$ with unit $\mathcal{O}_X$ and inverse $\mathcal{L} \mapsto \mathcal{L}^\vee$.

Now suppose we have an integral codimension one closed subscheme W of X. Since X is smooth all the local rings $\mathcal{O}_{X,x}$ are regular, hence are all UFD's. This is the same as saying that the ideal sheaf $\mathcal{I}_W$ has stalk $\mathcal{I}_{W,x}$ which is principal, i.e., W is a Cartier divisor on X. Thus, we can find an affine open cover $\{U_i = \mathrm{Spec}\, A_i\}$ of X such that $W_i := U_i \cap W$ is the subscheme defined by a single element $t_i \in A_i$.

We can use this data to construct an invertible sheaf $\mathcal{O}_X(W)$ on X. The sections $\mathcal{O}_X(W)(U)$ for U open are the rational functions $f \in k(X)$ such that either $f = 0$ or $(\mathrm{div} f + W) \cap U \geq 0$. By Lemma 2.2.2(2), $\mathcal{O}_X(W)(U)$ is a sheaf of abelian groups. $\mathcal{O}_X(W)$ is a a sheaf of $\mathcal{O}_X$-modules, since $f \in \mathcal{O}_X(W)(U)$, $g \in \mathcal{O}_X(U)$ implies $fg \in \mathcal{O}_X(W)$. On U_i, $\mathcal{O}_X(W)$ is just the $\mathcal{O}_X$-module $t_i^{-1}\mathcal{O}_X$, hence $\mathcal{O}_X(W)$ is locally isomorphic to $\mathcal{O}_X$, hence $\mathcal{O}_X(W)$ is an invertible sheaf. We extend the definition of $\mathcal{O}_X(W)$ to arbitrary divisors D by setting

$$\mathcal{O}_X(\sum_i W_i - \sum_j V_j) := (\bigotimes_i \mathcal{O}_X(W_i)) \otimes (\bigotimes_j \mathcal{O}_X(V_j))^\vee.$$

It is easy to see that $\mathcal{O}_X(D)$ is the sheaf with sections

$$\mathcal{O}_X(D)(U) = \{f \in k(X) \mid f = 0 \text{ or } (\mathrm{div} f + D) \cap U \geq 0\}.$$

If $D' = D + \mathrm{div} g$, then we have the isomorphism $\times g : \mathcal{O}_X(D) \to \mathcal{O}_X(D')$. Thus we have defined the group homomorphism

$$\mathcal{O}_X(-) : CH^1(X) \to \mathrm{Pic}(X).$$

If $\mathcal{O}_X(D) \cong \mathcal{O}_X$, then the element $f \in \mathcal{O}_X(D)(X)$ corresponding to $1 \in \mathcal{O}_X(X)$ gives a rational function with $\mathrm{div} f = D$, so $D \sim_l 0$. Thus $\mathcal{O}_X(-)$ is injective.

If $\mathcal{L}$ is an invertible sheaf on X, take an open $U \subset X$ with $\mathcal{L}_{|U} \cong \mathcal{O}_U$. The section 1 in $\mathcal{O}_U$ gives a section s of $\mathcal{L}$ over U. Choosing a local trivialization of $\mathcal{L}$ over another open subscheme V, s transforms to a regular function on $V \cap U$, hence a rational function s_V on V. Since two different local trivializations differ by a nowhere vanishing function, the divisors of the rational functions s_V satisfy

$$\mathrm{div}(s_V) \cap V \cap V' = \mathrm{div}(s_{V'}) \cap V \cap V'$$

and so give a well-defined element $\mathrm{div} s \in Z^1(X)$. The isomorphism $\mathcal{O}_U \cong \mathcal{L}$ extends to an isomorphism $\mathcal{O}_X(\mathrm{div} s) \cong \mathcal{L}$; in particular, the map $\mathcal{O}_X(-)$ is surjective.

We have thus sketched the proof of

Proposition 2.2.5. *Let X be a smooth finite type k-scheme. Sending $D \in Z^1(X)$ to the invertible sheaf $\mathcal{O}_X(D)$ defines an isomorphism*

$$\mathcal{O}_X(-) : \mathrm{CH}^1(X) \to \mathrm{Pic}(X).$$

We denote the inverse to $\mathcal{O}_X(-)$ by

$$c_1 : \mathrm{Pic}(X) \to \mathrm{CH}^1(X).$$

2.2.6. *Rational equivalence.* Linear equivalence extends in a natural way to the relation of rational equivalence for cycles of arbitrary dimension.

Definition 2.2.7. Let X be a finite type k-scheme. $R_n(X)$ is the subgroup of $Z_n(X)$ generated by elements of the form $i_{W*}(\mathrm{div} f)$, where $i_W : W \to X$ is an inclusion of an integral closed subscheme with $\dim_k W = n + 1$, and f is in $k(W)^*$.

The quotient group $Z_n(X)/R_n(X)$ is denoted $\mathrm{CH}_n(X)$; $R_n(X)$ defines the relation of *rational equivalence* of cycles of dimension n on X. $\mathrm{CH}_n(X)$ is called the *Chow group* of dimension n cycles modulo rational equivalence.

There is an alternate description of $R_n(X)$ which is quite useful. We have already used intersections on $W \times \mathbb{P}^1$ to define $\mathrm{div} f$ for $f \in k(W)^*$, giving us the description of $i_{W*}(\mathrm{div} f)$:

$$i_{W*}(\mathrm{div} f) = p_{1*}(\Gamma_f^X \cdot (X \times 0 - X \times \infty))$$

where $\Gamma_f^X \subset X \times \mathbb{P}^1$ is the image of $\Gamma_f \subset W \times \mathbb{P}^1$ under the closed embedding $W \times \mathbb{P}^1 \to X \times \mathbb{P}^1$. Clearly Γ_f has dimension equal to W, i.e., $n+1$ if we are constructing elements of $R_n(X)$. It turns out that a dimension $n+1$ cycle $\Gamma \in Z_{n+1}(X \times \mathbb{P}^1)$ which has no component contained in $X \times \{0, \infty\}$ gives an element of $R_n(X)$ by taking $p_{1*}(\Gamma \cdot (X \times 0 - X \times \infty))$.

To see this, it clearly suffices to take Γ an integral closed subscheme of $X \times \mathbb{P}^1$, not contained in $X \times \{0, \infty\}$. Let $W = p_1(\Gamma)$. If $\dim W < n+1$, then $\Gamma = W' \times \mathbb{P}^1$, so the element we are considering is just 0. If $\dim_k W = n+1$, then $k(W) \subset k(\Gamma)$ is a finite field extension. Via the projection

$$p_2 : \Gamma \to \mathbb{P}^1 = \mathrm{Proj}\,(k[X_0, X_1]),$$

the rational function $T := X_1/X_0$ pulls back to a rational function t on Γ.

Recall that, if $F \subset L$ is a finite extension of field, we have the *norm* homomorphism

$$\mathrm{Nm}_{L/F} : L^* \to F^*.$$

In case L/F is Galois with group g, $\mathrm{Nm}_{L/F}(h) = \prod_{\sigma \in G} h^\sigma$. Let $f = \mathrm{Nm}_{k(\Gamma)/k(W)}(t)$; one can understand f in this geometric setting (assuming k has characteristic 0) by

$$f(w) = \prod_{i=1}^{d} t(s_i)$$

for "general" $w \in W$, where $p_1^{-1}(w) = \{s_1, \dots, s_d\}$. In any case, one shows that

$$i_{W*}(\mathrm{div} f) = p_{1*}(\Gamma \cdot (X \times 0 - X \times \infty)),$$

verifying our claim.

It is also not necessary to use $\mathbb{P}^1$. Identifying $(\mathbb{P}^1 \setminus \{(1:1)\}, 0, \infty)$ with $(\mathbb{A}^1, 1, 0)$, we have the following second definition of rational equivalence:

Definition 2.2.8. $R_n(X)$ is the subgroup of $Z_n(X)$ generated by cycles of the form

$$p_{1*}(\Gamma \cdot (X \times 1 - X \times 0))$$

where $\Gamma \in Z_{n+1}(X \times \mathbb{A}^1)$ has no component contained in $X \times \{0, 1\}$.

Let $Z_{n+1}(X \times \mathbb{A}^1)'$ denote the subgroup of cycles intersecting $X \times 0$ and $X \times 1$ properly, and let $i_0 : X \to X \times \mathbb{A}^1$, $i_1 : X \to X \times \mathbb{A}^1$ be the sections with value 0,1 respectively. We have arrived at the following presentation of $\mathrm{CH}_n(X)$:

$$Z_{n+1}(X \times \mathbb{A}^1)' \xrightarrow{i_1^* - i_0^*} Z_n(X) \to \mathrm{CH}_n(X) \to 0 \tag{2.2.2}$$

Indeed $i_1^*(\Gamma) = p_{1*}(\Gamma \cdot X \times 1)$ and similarly for i_0.

In case X is locally equi-dimensional, we may index with codimension instead of dimension, giving us the subgroup $R^n(X)$ of $Z^n(X)$ and the quotient group $\mathrm{CH}^n(X)$.

We give a list of the main properties of the Chow groups.

(2.2.3)

(1) *Push-foward:* Let $f : Y \to X$ be a proper morphism in $\mathbf{Sch}_k$. The push-forward $f_* : Z_n(Y) \to Z_n(X)$ descends to $f_* : \mathrm{CH}_n(Y) \to \mathrm{CH}_n(X)$. The pushforward $i : X_{\mathrm{red}} \to X$ induces an isomorphism $\mathrm{CH}_n(X_{\mathrm{red}}) \to \mathrm{CH}_n(X)$.

(2) *Pull-back:* Let $f : Y \to X$ be a morphism in $\mathbf{Sch}_k$ with X smooth over k, and Y locally equi-dimensional over k. Then $Z^n(X)_f \to \mathrm{CH}^n(X)$ is surjective and the partially defined pull-back $f^* : Z^n(X)_f \to Z^n(Y)$ descends to a well-defined homomorphism

$$f^* : \mathrm{CH}^n(X) \to \mathrm{CH}^n(Y).$$

If $g : Z \to Y$ is a second morphism in $\mathbf{Sch}_k$ with Z locally-equi-dimensional over k and if Y is smooth over k, then $(fg)^* = g^*f^*$.

The pull-back of cycles $f^* : Z_n(X) \to Z_{n+d}(Y)$ for $f : Y \to X$ flat of relative dimension d descends to $f^* : \mathrm{CH}_n(X) \to \mathrm{CH}_{n+d}(Y)$. In case both pull-backs are defined, they agree and the functoriality $(fg)^* = g^*f^*$ is valid for both types of pull-back, whenever it makes sense.

(3) *Products:* Let X be smooth and of finite type over k, $\delta_X : X \to X \times_k X$ the diagonal. Defining $A \cup B := \delta^*(A \times_k B)$ for A and B integral closed subschemes of X extends (bilinearly) to a well-defined associative and commutative product

$$\cup : \mathrm{CH}^p(X) \otimes \mathrm{CH}^q(X) \to \mathrm{CH}^{p+q}(X).$$

This product makes $\mathrm{CH}^*(X) := \oplus_{n\geq 0}\mathrm{CH}^n(X)$ into a graded ring with unit the *fundamental class* $[X]$. In addition we have

$$f^*(a \cup b) = f^*(a) \cup f^*(b)$$

for $f : Y \to X$ a morphism of smooth k-schemes of finite type, $a, b \in \mathrm{CH}^*(X)$, and

$$f_*(a \cup f^*b) = f_*(a) \cup b$$

for $f : Y \to X$ a proper morphism of smooth k-schemes of finite type, $a \in \mathrm{CH}^*(Y)$, $b \in \mathrm{CH}^*(X)$.

(4) *Localization* Let $i : W \to X$ be a closed embedding in $\mathbf{Sch}_k$ with open complement $j : U \to X$. Then the sequence

$$\mathrm{CH}_n(W) \xrightarrow{i_*} \mathrm{CH}_n(X) \xrightarrow{j^*} \mathrm{CH}_n(U) \to 0$$

is exact.

(5) *Homotopy:* Let $p : V \to X$ be a flat morphism such that $p^{-1}(x) \cong \mathbb{A}^d_{k(x)}$ (as $k(x)$-schemes) for each $x \in X$. Then $p^* : \mathrm{CH}_n(X) \to \mathrm{CH}_{n+d}(V)$ is an isomorphism.

(6) *Projective bundle formula:* Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank $n+1$, $q : \mathbb{P}(\mathcal{E}) \to X$ the associated $\mathbb{P}^n$-bundle with tautological invertible sheaf $\mathcal{O}(1)$. Suppose X is smooth over k. Let $\xi = c_1(\mathcal{O}(1)) \in \mathrm{CH}^1(\mathbb{P}(\mathcal{E}))$. Then the $\mathrm{CH}^*(X)$-module $\mathrm{CH}^*(\mathbb{P}(\mathcal{E}))$ is free with basis $1, \xi, \ldots, \xi^n$.

Remarks 2.2.9. Probably the most difficult aspect is the verification of (2), that the partially defined pull-back maps on cycles lead to well-defined pull-back maps for CH^*. The first proofs used the geometric method of [8], now known as "Chow's moving lemma" (see also [21]). As applied in this setting, this technique works for smooth projective or smooth affine k-schemes, but not does not for smooth quasi-projective varieties in general. One can use Jouanoulou's trick to reduce the problem to the affine case, but I don't know of this approach appearing in the literature. Possibly the first modern and correct proof, due to Grayson [11], uses higher algebraic K theory. Fulton [9] uses an entirely different approach, defining pull-backs of *arbitrary* cycles, but as equivalence classes on the pull-back of the support of the cycle. The theory of the higher Chow groups (see below) has allowed Chow's technique to be extended to prove (2) for arbitrary smooth X.

The fundamental class $[X]$ mentioned in (3) is given by $[X] = \sum_i 1 \cdot X_i$ if X has irreducible components $X_1, \ldots, X_n$. In (6) $\mathrm{CH}^*(\mathbb{P}(\mathcal{E}))$ is a graded module over $\mathrm{CH}^*(X)$ by

$$a \cdot m = q^*(a) \cup m$$

for $a \in \mathrm{CH}^*(X)$, $m \in \mathrm{CH}^*(\mathbb{P}(\mathcal{E}))$.

Some of this theory extends in some form or other to $X \in \mathbf{Sch}_k$, for example Fulton [9] defines for each invertible sheaf $\mathcal{L}$ on $X \in \mathbf{Sch}_k$ an operator $\tilde{c}_1(\mathcal{L}) : \mathrm{CH}_n(X) \to \mathrm{CH}_{n-1}(X)$ which agrees with $\cup c_1(\mathcal{L})$ in case X is smooth. The projective bundle formula, suitably interpreted, then holds for $\mathcal{E}$ a locally free sheaf on $X \in \mathbf{Sch}_k$. The method of Grothendieck [13] for constructing Chern classes (see below) also yields

Chern class operators

$$\tilde{c}_i(\mathcal{E}) : \mathrm{CH}_n(X) \to \mathrm{CH}_{n-i}(X)$$

for $\mathcal{E}$ a locally free sheaf on $X \in \mathbf{Sch}_k$. The operators have been used, e.g., in Roberts' [22] proof of Serre's intersection vanishing conjecture in mixed characteristic.

2.2.10. *Chern classes.* The projective bundle formula leads to a theory of Chern classes for locally free sheaves, following Grothendieck's construction (*loc. cit.*).

Let X be a scheme. We have the category $\mathcal{P}_X$ of locally free sheaves of $\mathcal{O}_X$-modules, of finite rank. A sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

in $\mathcal{P}_X$ is *exact* if it is exact as a sequence of sheaves of abelian groups. We let $[\mathcal{E}]$ denote the isomorphism class of $\mathcal{E}$ in $\mathcal{P}_X$.

Definition 2.2.11. Let X be a scheme. The group $K_0(X)$ is the quotient of the free abelian group on the set of isomorphism classes in $\mathcal{P}_X$, modulo the relations $[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$ if there exists an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

Noting that $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}'$ is in $\mathcal{P}_X$ if both $\mathcal{E}$ and $\mathcal{E}'$ are, $K_0(X)$ becomes a ring. Also, for $f : X \to Y$ a morphism of schemes, we have the pull-back functor

$$f^* : \mathcal{P}_Y \to \mathcal{P}_X,$$

which preserves exact sequences and tensor products and hence induces a ring homomorphism

$$f^* : K_0(X) \to K_0(Y).$$

We also have the functoriality $(fg)^* = g^* f^*$, giving us the functor $K_0 : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Rings}$.

Definition 2.2.12. Let $\mathcal{E}$ be a locally free sheaf of rank n on $X \in \mathbf{Sm}_k$, let $q : \mathbb{P}(\mathcal{E}) \to X$ be the associated $\mathbb{P}^{n-1}$-bundle and let $\xi = c_1(\mathcal{O}(1)) \in \mathrm{CH}^1(\mathbb{P}(\mathcal{E}))$. Define elements $c_i(\mathcal{E}) \in \mathrm{CH}^i(X)$, $i = 1, \ldots, n$, by the formula

$$\xi^n + \sum_{i=1}^{n} (-1)^i \xi^{n-i} q^*(c_i(\mathcal{E})) = 0$$

We set $c_0(\mathcal{E}) := 1 \in \mathrm{CH}^0(X)$. The element $c_i(\mathcal{E})$ is called the *ith Chern class* of $\mathcal{E}$.

The elements $c_i(\mathcal{E})$ are well-defined: it follows from the projective bundle formula (6) that there are unique elements $c_i \in \mathrm{CH}^i(X)$ with $\xi^n + \sum_{i=1}^n (-1)^i \xi^{n-i} q^*(c_i) = 0$. The sum $c(\mathcal{E}) = \sum_{i=0}^{\mathrm{rnk}\mathcal{E}} c_i(\mathcal{E})$ is the *total* Chern class of $\mathcal{E}$.

The Chern classes satisfy the following properties, which in turn characterize the Chern classes:

(1) *naturality:* Let $f : Y \to X$ be a morphism in $\mathbf{Sch}_k$, $\mathcal{E}$ in $\mathcal{P}_X$. Then
$$c_i(f^*(\mathcal{E})) = f^*(c_i(\mathcal{E}))$$
for all i.

(2) *normalization:* The two definitions of $c_1(\mathcal{L})$ for an invertible sheaf $\mathcal{L}$ agree.

(3) *Whitney product formula:* Suppose we have an exact sequence
$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$
in $\mathcal{P}_X$. Then $c(\mathcal{E}) = c(\mathcal{E}')c(\mathcal{E}'')$.

The Chern classes allow one to relate $K_0(X)$ and $\mathrm{CH}^*(X)$. First of all, the Whitney product formula implies that the total Chern class $c(\mathcal{E})$ only depends on the image $[\mathcal{E}]$ in $K_0(X)$. Next, let $\sigma_1(t_1, \ldots), \sigma_2(t_1, \ldots)$, $\ldots$ be the elementary symmetric functions in the variables $t_1, t_2, \ldots$, and let $s_d(T_1, \ldots, T_d)$ be the polynomial (with $\mathbb{Z}$-coefficients) satisfying
$$s_d(\sigma_1, \ldots, \sigma_d) = \sum_i t_i^d.$$
The *Chern character*
$$ch : K_0(X) \to \mathrm{CH}^*(X)_{\mathbb{Q}}$$
is given by
$$ch(\mathcal{E}) := \sum_{i=0}^{\infty} \frac{1}{i!} s_i(c_1(\mathcal{E}), \ldots, c_i(\mathcal{E})).$$
Note that the sum actually stops after $i = \dim_k X$.

It follows from the Whitney product formula that ch is a group homomorphism; using the so-called *splitting principal* one shows that ch is actually a ring homomorphism. A consequence of the Grothendieck-Riemann-Roch theorem [7] is

Theorem 2.2.13. *For $X \in \mathbf{Sm}_k$ the $\mathbb{Q}$-linear extension of ch,*
$$ch_{\mathbb{Q}} : K_0(X)_{\mathbb{Q}} \to \mathrm{CH}^*(X)_{\mathbb{Q}}$$
is an isomorphism.

2.2.14. *The case of zero-cycles.* We look at CH_0 in a little more detail.

Definition 2.2.15. Let X be in $\mathbf{Sch}_k$. For $z = \sum_i n_i z_i \in Z_0(X)$, we set
$$\deg(z) := \sum_i n_i [k(z_i) : k].$$

By Hilbert's Nullstellensatz, the field extension degrees $[k(z_i) : k]$ are all finite, so $\deg : Z_0(X) \to \mathbb{Z}$ is well-defined.

Note that for X irreducible, $Z^0(X) = \mathrm{CH}^0(X) = \mathbb{Z} \cdot [X] \cong \mathbb{Z}$.

Lemma 2.2.16. *Let $p : X \to \operatorname{Spec} k$ be a projective k-scheme, $z \in Z_0(X)$. Let $[z] \in \mathrm{CH}_0(X)$ denote the class of z. Then*
$$p_*([z]) = \deg(z) \cdot [\operatorname{Spec} k].$$
In particular, if $[z] = 0$, then $\deg(z) = 0$.

Proof. By definition of the push-forward homomorphism $p_* : Z_0(X) \to Z_0(\operatorname{Spec} k)$, we have
$$p_*(z) = \deg(z) \cdot (\operatorname{Spec} k).$$
By property (2.2.3)(1), p_* descends to $p_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(\operatorname{Spec} k)$, whence the result. □

Definition 2.2.17. For X projective over k, we let $\deg : \mathrm{CH}_0(X) \to \mathbb{Z}$ be the homomorphism induced by $\deg : Z_0(X) \to \mathbb{Z}$. The kernel of $\deg : \mathrm{CH}_0(X) \to \mathbb{Z}$ is denoted $A_0(X)$.

For the remainder of this section, we take the base field k to be the complex numbers $\mathbb{C}$, and we take $X \subset \mathbb{P}^2_{\mathbb{C}}$ to be a smooth curve defined by a degree three homogeneous polynomial $F(X_0, X_1, X_2) \in \mathbb{C}[X_0, X_1, X_2]$, i.e.
$$X = \operatorname{Proj} \mathbb{C}[X_0, X_1, X_2]/(F).$$
We will assume that $F = X_0 X_1^2 - P(X_0, X_1, X_2)$, where P is a degree-three homogeneous polynomial satisfying $P(0, X_1, X_2) = aX_2^3$ for some $a \in \mathbb{C}^*$. In turns out that every F can be put in this form by a linear change of coordinates,.

We first note that there is a point $* \in X(\mathbb{C})$ and a line $\ell_* \subset \mathbb{P}^2$ such that
$$X \cdot \ell_* = 3 \cdot *.$$
In fact, take $* = (0 : 1 : 0)$ and ℓ_* the line $X_0 = 0$. Now take two points $a, b \in X(\mathbb{C})$. If $a \neq b$ take the line $\ell_{a,b}$ containing a and b; if $a = b$ take $\ell_{a,b}$ to be the tangent line to X at a (this exists since X is smooth). Since F has degree three we have
$$X \cdot \ell_{a,b} = a + b + c$$

for a well-determined point $c \in X(\mathbb{C})$ (c may coincide with a or b). We say that a, b, c *lie on a line*.

Lemma 2.2.18. *If (a, b, c) lie on a line, then*

$$(a - *) + (b - *) + (c - *) = 0$$

in $A_0(X)$.

Proof. The line $\ell_{a,b}$ has defining equation $L(X_0, X_1, X_2) = 0$ for some linear homogeneous polynomial $L \in \mathbb{C}[X_0, X_1, X_2]$. The fraction L/X_0 restricts to a rational function f on X with

$$\mathrm{div}(f) = X \cdot \ell_{a,b} - X \cdot \ell_*$$

which proves the result. □

Now take an arbitrary degree 0 element $\sum_{i=1}^n p_i - \sum_{i=1}^n q_i$ in $Z_0(X)$. We can rewrite this as a difference of degree 0 elements $\sum_{i=1}^n (p_i - *) - \sum_{i=1}^n (q_i - *)$; using the above lemma repeatedly, this zero-cycle is equivalent to a zero-cycle of the form $(p - *) - (q - *)$. If p, $*$ and b lie on a line, and if q, b and a lie on a line, then $p + * \sim a + q$, so $(p - *) - (q - *) \sim a - *$.

Thus, sending $a \in X(\mathbb{C})$ to the class of $(a - *)$ defines a surjection (of sets)

$$\alpha : X(\mathbb{C}) \to A_0(X);$$

if we make $X(\mathbb{C})$ into a group with identity $*$ by defining $a + b = -c$ if a, b, c lie on a line, then α is a surjective group homomorphism.

Lemma 2.2.19. *α is an isomorphism.*

Proof. It suffices to see that α is injective, i.e. that $[a] \neq [*]$ for all $a \in X(\mathbb{C})$, $a \neq *$. For this, first note that by giving $X(\mathbb{C})$ the topology induced from the complex projective plane $\mathbb{CP}^2 = \mathbb{P}^2_{\mathbb{C}}(\mathbb{C})$, $X(\mathbb{C})$ becomes a compact real manifold of dimension 2. We have also defined a group law on $X(\mathbb{C})$, which is commutative and continuous, so $X(\mathbb{C})$ is a 2-dimensional compact commutative Lie group. There is only one, namely $S^1 \times S^1$, so $X(\mathbb{C})$ is a genus 1 surface.

Now suppose that $[a] = [*]$ for some $a \neq *$. Thus, there is a rational function $f \in \mathbb{C}(X)^*$ with $\mathrm{div} f = a - *$. f gives a rational map $f : X \to \mathbb{P}^1_{\mathbb{C}}$, which is actually a morphism because f is a morphism at all codimension one points of X and X is a curve. We already know that $f^*(0) = a$ and $f^*(\infty) - *$; if $(1 : t)$ is an arbitrary point of $\mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$, let $f_t = f - t$. Then $f^*(t) = f_t^*(0)$. But as $\mathrm{div}_\infty(f_t) = \mathrm{div}_\infty(f) = \infty$ and $\mathrm{div}(f_t)$ has degree 0, we must have $f^*(t) = 1 \cdot b$ for some point $b \in X(\mathbb{C})$. From this it follows that $f : X \to \mathbb{P}^1$ is an isomorphism (of $\mathbb{C}$-schemes). But then on $\mathbb{C}$-points we have the homeomorphism of real surfaces $X(\mathbb{C}) = S^1 \times S^1 \to \mathbb{CP}^1 = S^2$, which is impossible. $\square$

Thus we have identified $A_0(X)$ with the points on the genus 1 surface $X(\mathbb{C})$ and we have defined a group law on $X(\mathbb{C})$ so that $X(\mathbb{C}) \cong A_0(X)$ as groups. One can go much further with this, giving $X(\mathbb{C})$ its natural complex structure and using integrals of a holomorphic 1-form on X to give a map $A_0(X) \to X(\mathbb{C})$ inverse to α. For further reading on this topic, we suggest the texts [19, Curves and their Jacobians] and [24].

2.3. **Higher Chow groups.** The right-exactness of the localization sequence (2.2.3)(4) suggests a continuation of CH^* to a *Borel-Moore homology* theory on $\mathbf{Sm}_k$. The analogous localization sequence for algebraic K-theory

$$K_0(W) \xrightarrow{i_*} K_0(X) \xrightarrow{j^*} K_0(U) \to 0$$

for $i : W \to X$ a closed embedding in $\mathbf{Sm}_k$ with complement $j : U \to X$ was extended to the left by Quillen's construction [20] of higher algebraic K-theory $K_*(X)$. The first construction of the CH^*-extension was given by Bloch [5], and was later identified with Voevodsky's motivic cohomology by Voevodsky, Suslin and Friedlander [26]. In this concluding section, we will describe Bloch's construction and list it's main properties.

2.3.1. *Bloch's cycle complex.* The construction relies on the *algebraic n-simplex* Δ^n_k,

$$\Delta^n_k := \operatorname{Spec} k[t_0, \ldots, t_n]/(\sum_i t_i - 1).$$

The assignment $n \mapsto \Delta^n_k$ extends to define a cosimplicial k-scheme, i.e., a functor

$$\Delta_k : \mathbf{Ord} \to \mathbf{Sch}_k,$$

where $\mathbf{Ord}$ is the category with objects the ordered sets $[n] := \{0 < 1 \ldots, < n\}$, $n = 0, 1, \ldots$ and morphisms the order-preserving maps of sets. To define Δ_k, let $g : [n] \to [m]$ be an order-preserving map of sets. Let $\Delta_k(g) : \Delta^n_k \to \Delta^m_k$ be the map induced by the ring homomorphism

$$\Delta_k(g)^* : k[t_0, \ldots, t_m]/(\sum_i t_i - 1) \to k[t_0, \ldots, t_n]/(\sum_i t_i - 1)$$

$$\Delta_k(g)^*(t_i) = \sum_{j \in g^{-1}(i)} t_j.$$

where the empty sum is to be interpreted as 0. We often write $g : \Delta^n_k \to \Delta^m_k$ for $\Delta_k(g)$.

Remark 2.3.2. If we take $k = \mathbb{R}$, then the topological n-simplex Δ^n is the subset of $\Delta^n_{\mathbb{R}}(\mathbb{R})$ consisting of those real points $(r_0, \ldots, r_n)$ with $r_i \geq 0$ for all i. Given $g : [n] \to [m]$, the map $\Delta_{\mathbb{R}}(g) : \Delta^n_{\mathbb{R}} \to \Delta^m_{\mathbb{R}}$ restricts to the map $\Delta^n \to \Delta^m$ used in topology.

A *face* of Δ^n_k is a subscheme defined by equations of the form $t_{i_1} = \ldots = t_{i_s} = 0$; each face of Δ^n is thus isomorphic to Δ^m for some $m \leq n$ and Δ^n is isomorphic to the affine space $\mathbb{A}^n_k$.

Definition 2.3.3. Let X be in $\mathbf{Sch}_k$. The group $z_q(X, p)$ is the subgroup of $z_{p+q}(X \times \Delta^p_k)$ generated by integral closed subschemes W such that, for each face F of Δ^p_k, we have

$$\dim_k[W \cap (X \times F)] = q + \dim_k F$$

or $W \cap (X \times F)] = \emptyset$.

We say for short that a cycle Z in $z_{p+q}(X \times \Delta^p_k)$ is in $z_q(X, p)$ if Z *intersects all faces properly.* If X is equi-dimensional over k of dimension d, we set

$$z^q(X, p) := z_{d-q}(X, p).$$

Indexing by codimension, the proper intersection condition for the generators W of $z^q(X, p)$ becomes

$$\mathrm{codim}_{X \times F}[W \cap (X \times F)] = q$$

for each face F of Δ^p_k.

Let $\delta^{p-1}_i : [p-1] \to [p]$ be the ith face map, i.e., the order-preserving inclusion omitting i from its image. We note the following elementary but crucial fact:

Lemma 2.3.4. *For $Z \in z_q(X, p)$, $(\delta^{p-1}_i)^*(Z)$ is defined and is in the subgroup $z_q(X, p-1)$ of $z_{p+q-1}(X \times \Delta^{p-1}_k)$.*

We may thus define

$$d_p : z_q(X, p) \to z_q(X, p-1)$$

by

$$d_p = \sum_{i=0}^{p} (-1)^i (\delta^{p-1}_i)^*.$$

The usual computation shows that $d_{p-1} d_p = 0$, giving us *Bloch's cycle complex* $(z_q(X, *), d)$. For X locally equi-dimensional over k, we have as well the complex labelled by codimension $(z^q(X, *), d)$.

Definition 2.3.5. The *higher Chow group* $\mathrm{CH}_q(X,p)$ is defined by

$$\mathrm{CH}_q(X,p) := H_p(z_q(X,*)).$$

For X locally equi-dimensional over k, set

$$\mathrm{CH}^q(X,p) := H_p(z^q(X,*)).$$

Remark 2.3.6. Sending (t_0,t_1) to t_1 gives the identification $(\Delta^1_1,\delta^0_0,\delta^0_1)$ with $(\mathbb{A}^1_k,i_1,i_0)$, and thus identifies the presentation of $\mathrm{CH}_q(X,0)$,

$$z_{q+1}(X,1)\xrightarrow{(\delta^0_0)^*-(\delta^0_1)^*} z_q(X,0)\to \mathrm{CH}_q(X,0)\to 0$$

with the presentation (2.2.2) of $\mathrm{CH}_q(X)$. Thus $\mathrm{CH}_q(X,0)=\mathrm{CH}_q(X)$.

The properties (2.2.3) all extend to the higher Chow groups; to avoid any possible confusion as to the indexing, we give the complete list. All the operations defined below on $\mathrm{CH}(X,0)$ agree with those given above for $\mathrm{CH}(X)$.

(2.3.1)

(1) *Push-foward:* Let $f:Y\to X$ be a proper morphism in $\mathbf{Sch}_k$. The push-forward $(f\times\mathrm{id})_*:Z_*(Y\times\Delta^*_k)\to Z_*(X\times\Delta^*_k)$ descends to a map of complexes $f_*:z_q(Y,*)\to z_q(X,*)$, and thus a map on homology $f_*:\mathrm{CH}_q(Y,*)\to\mathrm{CH}_q(X,*)$, satisfying $(fg)_*=f_*g_*$. The pushforward $i:X_{\mathrm{red}}\to X$ induces an isomorphism $\mathrm{CH}_q(X_{\mathrm{red}},p)\to\mathrm{CH}_q(X,p)$.

(2) *Pull-back:* Let $f:Y\to X$ be a morphism in $\mathbf{Sch}_k$ with X smooth over k, and Y locally equi-dimensional over k. There is a map in the derived category $D^-(\mathbf{Ab})$ $f^*:z^q(X,*)\to z^q(Y,*)$. We thus have $f^*:\mathrm{CH}^q(X,p)\to\mathrm{CH}^q(Y,p)$.

The pull-back of cycles $(f\times\mathrm{id})^*:Z_{q+p}(X\times\Delta^p_k)\to Z_{q+p+d}(Y\times\Delta^p_k)$ for $f:Y\to X$ flat of relative dimension d descends to a map of complexes $f^*:z_q(X,*)\to z_{q+d}(Y,*)$ and $f^*:\mathrm{CH}_q(X,p)\to\mathrm{CH}_{q+d}(Y,p)$. In case both pull-backs are defined, they agree and the functoriality $(fg)^*=g^*f^*$ is valid for both types of pull-back, whenever it makes sense.

(3) *Products:* Let X and Y be in $\mathbf{Sch}_k$. Taking products of cycles extends to a map in $D^-(\mathbf{Ab})$

$$\boxtimes_{X,Y}:z_q(X,*)\otimes z_{q'}(Y,*)\to z_{q+q'}(X\times_k Y,*),$$

which is associative, commutative, and compatible with f^* whenever f^* is defined.

Let X be smooth and of finite type over k, $\delta_X : X \to X \times_k X$ the diagonal. Defining

$$\cup := \delta^* \circ \boxtimes_{X,X} : z_q(X,*) \otimes z_{q'}(X,*) \to z_{q+q'}(X,*)$$

gives a well-defined, natural, associative and commutative product in $D^-(\mathbf{Ab})$ and thus an associative product on homology:

$$\cup : \mathrm{CH}^q(X,p) \otimes \mathrm{CH}^{q'}(X,p') \to \mathrm{CH}^{q+q'}(X,p+p').$$

This product makes $\mathrm{CH}^*(X,*) := \oplus_{p,q\geq 0}\mathrm{CH}^q(X,p)$ into a bi-graded ring with unit the *fundamental class* $[X]$, and is commutative with respect to q and graded-commutative with respect to p. In addition we have

$$f^*(a \cup b) = f^*(a) \cup f^*(b)$$

for $f : Y \to X$ a morphism of smooth k-schemes of finite type, $a, b \in \mathrm{CH}^*(X,*)$, and

$$f_*(a \cup f^*b) = f_*(a) \cup b$$

for $f : Y \to X$ a proper morphism of smooth k-schemes of finite type, $a \in \mathrm{CH}^*(Y,*)$, $b \in \mathrm{CH}^*(X,*)$.

(4) *Localization* Let $i : W \to X$ be a closed embedding in $\mathbf{Sch}_k$ with open complement $j : U \to X$. The sequence

$$0 \to z_q(W,*) \xrightarrow{i_*} z_q(X,*) \xrightarrow{j^*} z_q(U,*)$$

is (degree-wise) exact and the quotient complex $z_q(U,*)/\mathrm{im}(j^*)$ is acyclic. Thus, we have the long exact sequence

$$\begin{aligned} \ldots \to \mathrm{CH}_q(W,p) \xrightarrow{i_*} \mathrm{CH}_q(X,p) \xrightarrow{j^*} \mathrm{CH}_q(U,p) \\ \xrightarrow{\delta} \mathrm{CH}_q(W,p-1) \xrightarrow{i_*} \mathrm{CH}_q(X,p-1) \to \ldots \end{aligned}$$

(5) *Homotopy:* Let $p : V \to X$ be a flat morphism such that $p^{-1}(x) \cong \mathbb{A}^d_{k(x)}$ (as $k(x)$-schemes) for each $x \in X$. Then $p^* : \mathrm{CH}_q(X,p) \to \mathrm{CH}_{q+d}(V,p)$ is an isomorphism.

(6) *Projective bundle formula:* Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank $n+1$, $q : \mathbb{P}(\mathcal{E}) \to X$ the associated $\mathbb{P}^n$-bundle with tautological invertible sheaf $\mathcal{O}(1)$. Suppose X is smooth over k. Let $\xi = c_1(\mathcal{O}(1)) \in \mathrm{CH}^1(\mathbb{P}(\mathcal{E})) = \mathrm{CH}^1(\mathbb{P}(\mathcal{E}),0)$. Then the $\mathrm{CH}^*(X,*)$-module $\mathrm{CH}^*(\mathbb{P}(\mathcal{E}),*)$ is free with basis $1, \xi, \ldots, \xi^n$.

Remarks 2.3.7. The most difficult point to prove is the localization property (4). Bloch gives an incorrect proof in the original paper [5] and needs to develop an entirely new method of "moving" cycles, using blow-ups of faces in Δ^p_k, to prove this result in [6].

Once one has the localization property, one immediately has a Mayer-Vietoris sequence for Zariski open covers. This enables one to reduce the existence of a pull-back map in $D^-(\mathbf{Ab})$ to the case of smooth affine X.

For smooth affine X, one considers the subcomplex

$$z^q(X,*)_f \subset z^q(X,*)$$

generated by integral $W \subset X \times \Delta^p_k$ which intersect all faces $X \times F$ properly and for which each irreducible component W' of $(f \times \mathrm{id})^{-1}(W)$ intersects all faces $Y \times F$ properly. One then uses a version of Chow's moving lemma to show that $z^q(X,*)_f \to z^q(X,*)$ is a quasi-isomorphism (see e.g. [17, Theorem 3.5.13]).

2.3.8. *Relations with higher K-theory.* The properties (2.3.1)(1)-(6) allow one to feed $\mathrm{CH}^*(-,*)$ into Gillet's machinery for constructing Chern classes for higher K-theory (see [10]), giving natural maps

$$c_{q,p} : K_p(X) \to \mathrm{CH}^q(X,p)$$

for all $p,q \geq 0$, and for X is smooth and of finite type over k. For $p=0$, we have $c_{q,0} = c_q$. There is also a Chern character

$$ch_* = \oplus_{p\geq 0} ch_p : \oplus_p K_p(X) \to \oplus_{p,q} \mathrm{CH}^q(X,p)_{\mathbb{Q}}$$

which is a ring homomorphism, with $ch_0 = ch$. Using the Grothendieck-Riemann-Roch theorem, together with some facts on the λ-ring structure on K-theory, Bloch shows

Theorem 2.3.9. *Let X be in $\mathbf{Sm}_k$. The $\mathbb{Q}$-linear extension of ch_* is an isomorphism of rings*

$$ch_* : K_*(X)_{\mathbb{Q}} \to \mathrm{CH}^*(X,*)_{\mathbb{Q}}.$$

See [16] for a proof along somewhat different lines; Bloch's first proof in [5] has a gap, fixed in [6].

2.3.10. *Higher Chow groups and motivic cohomology.* Voevodsky's construction of a triangulated category of motives over a perfect field, $DM_{gm}(k)$ (see [26, Chap. 5]) gives rise to a bi-graded cohomology theory on $\mathbf{Sm}^k$, $H^*(X,\mathbb{Z}(*)) := \oplus_{p,q} H^p(X,\mathbb{Z}(q))$. Work of Voevodsky, Suslin and Friedlander (*cf.* [26], [27]) show that, for X in $\mathbf{Sm}_k$ and k perfect, there are natural isomorphisms

$$H^p(X,\mathbb{Z}(q)) \cong \mathrm{CH}^q(X, 2q-p),$$

giving a natural isomorphism of rings

$$H^*(X, \mathbb{Z}(*)) \cong \mathrm{CH}^*(X, *).$$

The identification of the rational higher Chow groups with rational algebraic K-theory via the Chern character thus leads to the same for rational motivic cohomology.

3. Grothendieck topologies and the category of sheaves

We have already seen in §1.2 that one can define presheaves on a topological space T as contravariant functors on the category $\mathrm{Op}(T)$ of open subsets of T, and that sheaves are just presheaves satisfying a condition (*viz.*, the exactness of (1.2.1)) with respect to open covers. Grothendieck showed how to generalize this construction to a small category equipped with an extra structure, called *covering families*. In this section, we recall this theory and discuss its main points.

3.1. **Limits.** Before discussing presheaves and sheaves, we need some basic results on limits.

3.1.1. *Definitions.* Let I be a small category, $\mathcal{A}$ a category and $F : I \to \mathcal{A}$ a functor. Form the category $\underrightarrow{lim}\, F$ with objects

$$(X, \prod_{i \in I} f_i : F(i) \to X)$$

with the condition that, for each morphism $g : i \to i'$ in I, we have $f_{i'} \circ F(g) = f_i$. A morphism

$$\phi : (X, \prod_{i \in I} f_i : F(i) \to X) \to (X', \prod_{i \in I} f'_i : F(i) \to X')$$

is a morphism $\phi : X \to X'$ in $\mathcal{A}$ with $f'_i = \phi \circ f_i$ for all i. Dually, we have the category $\underleftarrow{lim}\, F$ with objects

$$(X, \prod_{i \in I} f_i : X \to F(i))$$

such that for each morphism $g : i \to i'$ in I, we have $f_{i'} = F(g) \circ f_i$; a morphism

$$\phi : (X, \prod_{i \in I} f_i : X \to F(i)) \to (X', \prod_{i \in I} f'_i : X' \to F(i))$$

is a morphism $\phi : X \to X'$ in $\mathcal{A}$ with $f'_i \circ \phi = f_i$ for all i.

Definition 3.1.2. Let I be a small category, $\mathcal{A}$ a category and $F : I \to \mathcal{A}$ a functor. The *inductive limit* $\varinjlim F$ is an initial object in $\mathit{\varinjlim} F$, and the *projective* limit $\varprojlim F$ is a final object in $\mathit{\varprojlim} F$.

Example 3.1.3. For $\mathcal{A} = \mathbf{Sets}$ we have explicit expressions for $\varinjlim F$ and $\varprojlim F$: $\varinjlim F$ is the quotient of the disjoint union $\coprod_{i\in I} F(i)$ by the relation $x_i \in F(i) \sim F(g)(x_i) \in F(i')$ for $g : i \to i'$ in I, and $\varprojlim F$ is the subset of $\prod_{i\in I} F(i)$ consisting of elements $\prod_i x_i$ with $F(g)(x_i) = x_{i'}$ for $g : i \to i'$ in I.

For $\mathcal{A} = \mathbf{Ab}$, replacing disjoint union with direct sum in the above yields $\varinjlim F$; the projective limit $\varprojlim F$ is defined by exactly the same formula as for **Sets**.

One can express the universal property of $\varinjlim F$ and $\varprojlim F$ in terms of limits of sets by the formulas:

$$\mathrm{Hom}_{\mathcal{A}}(\varinjlim F, Z) \cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(F(-), Z)$$
$$\mathrm{Hom}_{\mathcal{A}}(\mathbb{Z}, \varprojlim F) \cong \varprojlim \mathrm{Hom}_{\mathcal{A}}(Z, F(-))$$

Here $\mathrm{Hom}_{\mathcal{A}}(F(-), Z) : I^{\mathrm{op}} \to \mathbf{Sets}$ and $\mathrm{Hom}_{\mathcal{A}}(Z, F(-)) : I \to \mathbf{Sets}$ are the functors $i \mapsto \mathrm{Hom}_{\mathcal{A}}(F(i), Z)$ and $i \mapsto \mathrm{Hom}_{\mathcal{A}}(Z, F(i))$, respectively, and the isomorphisms are induced by the structure morphisms $F(i) \to \varinjlim F$, $\varprojlim F \to F(i)$.

3.1.4. *Functoriality of limits.* Let $F : I \to \mathcal{A}$ and $G : I \to \mathcal{A}$ be functors. A natural transformation of functors $\theta : F \to G$ yields functors $\theta^* : \mathit{\varinjlim} G \to \mathit{\varinjlim} F$ and $\theta_* : \mathit{\varprojlim} F \to \mathit{\varprojlim} G$, thus a morphism on the initial, resp. final, objects

$$\theta_* : \varinjlim F \to \varinjlim G;\ \theta_* : \varprojlim F \to \varprojlim G$$

assuming these exist. These satisfy $(\theta \circ \theta')_* = \theta_* \circ \theta'_*$.

Similarly, a functor $f : J \to I$ induces $f_* : \varinjlim F \circ f \to \varinjlim F$ and $f^* : \varprojlim F \to \varprojlim F \circ f$, with $(ff')_* = f_* \circ f'_*$ and $(ff')^* = f'^* \circ f^*$.

3.1.5. *Representability and exactness.* A functor $G : \mathcal{C} \to \mathcal{D}$ is called *left-exact* if G commutes with projective limits, *right exact* if G commutes with inductive limits and *exact* if both left and right exact. From our formulas for $\varinjlim F$ and $\varprojlim F$, we have:

Proposition 3.1.6. *For* $\mathcal{A} = \mathbf{Sets}, \mathbf{Ab}$*:*

(1) $\varinjlim F$ *and* $\varprojlim F$ *both exist for arbitrary functors* $F : I \to \mathcal{A}$*.*

(2) $F \mapsto \varinjlim_I F$ *is right exact*
(3) $F \mapsto \varprojlim_I F$ *is left exact.*

In general, $F \mapsto \varinjlim_I F$ is not left-exact, however, under special conditions on I, this is the case. The usual conditions are:

L1. Given $i \to j$, $i \to j'$ in I, there exists a commutative square

$$\begin{tikzcd} & j \arrow[dr] & \\ i \arrow[ur] \arrow[dr] & & k \\ & j' \arrow[ur] & \end{tikzcd}$$

L2. Given morphisms $i \overset{f}{\underset{g}{\rightrightarrows}} j$ there is a morphism $h : j \to k$ with $hf = hg$.

L3. Give i, i', there are morphisms $i \to j$, $i' \to j$.

A category I satisfying $L1 - L3$ is called a *filtering* category. The main result on filtered inductive limits is

Proposition 3.1.7. *Let I be a small filtering category, $\mathcal{A} =$ **Sets** or $\mathcal{A} =$ **Ab**. Then $F \mapsto \varinjlim_I F$ is exact.*

For a proof, see [3].

3.1.8. *Cofinality.* It is often useful to replace an index category I with a subcategory $\epsilon : J \to I$. In general, this changes inductive and projective limits, but it is useful to have a criterion which ensures that $\varinjlim_I F = \varinjlim_J F \circ \epsilon$.

Definition 3.1.9. A subcategory J of a small category I is called *cofinal* if

(1) Given $i \in I$, there is a morphism $i \to j$ with $j \in J$.
(2) Given a morphism $i \to j$ with $j \in J$, there are morphisms $i \to j'$ and $j \to j'$ making

$$\begin{tikzcd} i \arrow[r] \arrow[dr] & j \arrow[d] \\ & j' \end{tikzcd}$$

commute.

Here the main result is:

Lemma 3.1.10. *Suppose that $\epsilon : J \to I$ is a cofinal subcategory of a small category I. Then for $F : I \to$ **Sets** or $F : I \to$ **Ab**, the map*

$$\epsilon_* : \varinjlim_J F \circ \epsilon \to \varinjlim_I F$$

is an isomorphism.

3.2. **Presheaves.** Fix a small category $\mathcal{C}$. A *presheaf* P on $\mathcal{C}$ with values in a category $\mathcal{A}$ is simply a functor

$$P : \mathcal{C}^{\mathrm{op}} \to \mathcal{A}.$$

Morphisms of presheaves are natural transformations of functors. This defines the category of $\mathcal{A}$-valued presheaves on $\mathcal{C}$, $PreSh_{\mathcal{A}}(\mathcal{C})$.

Remark 3.2.1. We require $\mathcal{C}$ to be small so that the collection of natural transformations $\vartheta : F \to G$, for presheaves F, G, form a set. It would suffice that $\mathcal{C}$ be essentially small (the collection of isomorphism classes of objects form a set). In practice, one often ignores the smallness condition on $\mathcal{C}$.

3.2.2. *Limits and exactness.* One easily sees that the existence, exactness and cofinality of limits in **Sets** or **Ab** (Proposition 3.1.6, Proposition 3.1.7, Lemma 3.1.10) is inherited by the presheaf category:

Proposition 3.2.3. *For $PreSh_{\mathbf{Sets}}(\mathcal{C})$ or $PreSh_{\mathbf{Ab}}(\mathcal{C})$:*

(1) *$\varinjlim F$ and $\varprojlim F$ both exist for arbitrary functors $F : I \to \mathcal{A}$.*
(2) *$F \mapsto \varinjlim F$ is right exact*
(3) *$F \mapsto \varprojlim F$ is left exact.*
(4) *If I is filtering, then $F \mapsto \varinjlim F$ is exact*
(5) *If $\epsilon : J \to I$ is cofinal, then $\epsilon_* : \varinjlim_J F \circ \epsilon \to \varinjlim_I F$ is an isomorphism.*

Indeed, since inductive and projective limits are functors, we have the formulas (for $F : I \to PreSh_{\mathcal{A}}(\mathcal{C})$)

$$(\varinjlim F)(X) = \varinjlim_{i \in I} F(i)(X)$$
$$(\varprojlim F)(X) = \varprojlim_{i \in I} F(i)(X).$$

3.2.4. *Functoriality and generators for presheaves.* Let $f : \mathcal{C} \to \mathcal{C}'$ be a functor of small categories. Composition with f defines the presheaf pull-back

$$f^p : PreSh_{\mathcal{A}}(\mathcal{C}') \to PreSh_{\mathcal{A}}(\mathcal{C}).$$

Lemma 3.2.5. *For $\mathcal{A} = \mathbf{Sets}, \mathbf{Ab}$, f^p has a left adjoint*

$$f_p : PreSh_{\mathcal{A}}(\mathcal{C}) \to PreSh_{\mathcal{A}}(\mathcal{C}').$$

f_p is right-exact and f^p is exact.

Proof. Recall that a functor $L : \mathcal{B} \to \mathcal{B}'$ is left-adjoint to a functor $R : \mathcal{B}' \to \mathcal{B}$ (equivalently, R is right adjoint to L) if there is a natural isomorphism

$$\theta_{X,Y} : \mathrm{Hom}_{\mathcal{B}}(X, R(Y)) \to \mathrm{Hom}_{\mathcal{B}'}(L(X), Y). \tag{3.2.1}$$

Taking $Y = L(X)$ and $X = R(Y)$ and applying the isomorphism to the respective identity elements, we find morphisms $\psi_X : X \to R \circ L(X)$ and $\phi_Y : L \circ R(Y) \to Y$ which yield the isomorphism $\theta_{X,Y}$ by $(f : X \to R(Y)) \mapsto \phi_Y \circ L(f)$ and its inverse by $(g : L(X) \to Y) \mapsto R(g) \circ \psi_X$. Conversely, given natural transformations $\psi : \mathrm{id} \to R \circ L$ and $\phi : L \circ R \to \mathrm{id}$, L and R form an adjoint pair if the compositions

$$R \xrightarrow{\psi \circ R} R \circ L \circ R \xrightarrow{R \circ \phi} R$$

$$L \xrightarrow{L \circ \psi} L \circ R \circ L \xrightarrow{\phi \circ L} L$$

are the identity natural transformations. It follows immediately from the existence of the natural isomorphism (3.2.1) that L is right-exact and R is left-exact.

To define the functor f_p, take $Y \in \mathcal{C}'$ and let I_Y be the category with objects pairs (X, ϕ), where $\phi : Y \to f(X)$ is a morphism in $\mathcal{C}'$. A morphism $g : (X, \phi) \to (X', \phi')$ is a morphism $g : X \to X'$ in $\mathcal{C}$ with $\phi' = f(g) \circ \phi$. I_Y is a small category.

For $F \in PreSh_{\mathcal{A}}(\mathcal{C})$, let $F_Y : I_Y^{\mathrm{op}} \to \mathcal{A}$ be the functor $F_Y(X, \phi) = F(X)$ and set

$$(f_p F)(Y) := \lim_{\to} F_Y.$$

A morphism $g : Y' \to Y$ in $\mathcal{C}'$ gives the functor $g^* : I_Y \to I_{Y'}$ by $g^*(X, \phi) = (X, \phi \circ g)$, and we have the identity $F_Y = F_{Y'} \circ g^*$. Thus, the functor

$$(g^*)_* : \lim_{\to} F_Y \to \lim_{\to} F_{Y'}$$

gives the morphism

$$(f_p F)(g) : (f_p F)(Y) \to (f_p F)(Y')$$

satisfying $(f_pF)(gg') = (f_pF)(g') \circ (f_pF)(g)$. We have therefore constructed a presheaf f_pF on $\mathcal{C}'$.

To show that f_p is left-adjoint to f^p, we need to construct natural maps $\psi_F : F \to f^p f_p F$ and $\phi_G : f_p f^p G \to G$ for $F \in PreSh_{\mathcal{A}}(\mathcal{C})$ and $G \in PreSh_{\mathcal{A}}(\mathcal{C}')$. For this, note that $f^p f_p F(X) = (f_pF)(f(X))$. We have the object $(X, \mathrm{id}_{f(X)})$ in $I_{f(X)}$, giving the canonical map

$$F(X) = F_{f(X)}(X, \mathrm{id}_{f(X)}) \to \lim_{\to} F_{f(X)} = (f_pF)(f(X)),$$

which defines ψ_F. For $Y \in \mathcal{C}'$, $f_p f^p G(Y)$ is the limit over I_Y of the functor $(X, \phi) \mapsto G(f(X))$. For each $(X, \phi) \in I_Y$, we have the map $G(\phi) : G(f(X)) \to G(Y)$; the universal property of $\lim\limits_{\to}$ thus yields the desired map $\phi_G(Y) : f_p f^p G(Y) \to G(Y)$. One checks the necessary compatibility $(R \circ \phi) \circ (\psi \circ R) = \mathrm{id}_R$, $(\phi \circ L) \circ (L \circ \psi) = \mathrm{id}_L$ without trouble.

The right-exactness of f_p follows from the adjoint property; the exactness of f^p follows from the fact that limits in the presheaf category are taken pointwise: $(\lim F)(X) = \lim F(X)$. $\square$

3.2.6. *Generators for presheaves.* Recall that a *set of generators* for a category $\mathcal{C}$ is a set of objects $\mathcal{S}$ of $\mathcal{C}$ with the property that, if $f, g : A \to B$ are morphisms in $\mathcal{C}$ with $f \neq g$, then there exists an $X \in \mathcal{S}$ and a morphism $h : X \to A$ with $fh \neq gh$. If $\mathcal{S} = \{X\}$, X is called a generator for $\mathcal{C}$.

Example 3.2.7. **Sets** has as generator the one-point set 0, and **Ab** has the generator $\mathbb{Z}$. If inductive limits exist in $\mathcal{C}$ and $\mathcal{C}$ has a set of generators $\mathcal{S}$, then $\coprod_{X \in \mathcal{S}} X$ is a generator for $\mathcal{C}$.

Let $\mathcal{C}$ be a small category. We will use the functor f_p to construct generators for the presheaf categories $PreSh_{\mathcal{A}}(\mathcal{C})$, $\mathcal{A} = \mathbf{Sets}, \mathbf{Ab}$.

Take X in $\mathcal{C}$, let $*$ denote the one-point category (with only the identity morphism) and $i_X : * \to \mathcal{C}$ the functor with $i_X(*) = X$. Since $PreSh_{\mathcal{A}}(*) = \mathcal{A}$, we have the functor

$$i_{Xp} : \mathbf{Sets} \to PreSh_{\mathbf{Sets}}(\mathcal{C}).$$

Let $\Xi_X = i_{Xp}(0)$. Similarly, we have $i_{Xp} : \mathbf{Ab} \to PreSh_{\mathbf{Ab}}(\mathcal{C})$; we set $\mathcal{Z}_X := i_{Xp}(\mathbb{Z})$. As $i_X^p(P) = P(X)$, the adjoint property of i_{Xp} and i_X^p gives

$$P(X) = \mathrm{Hom}_{\mathbf{Sets}}(0, i_X^p(P)) = \mathrm{Hom}_{PreSh_{\mathbf{Sets}}(\mathcal{C})}(\Xi_X, P)$$
$$P(X) = \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}, i_X^p(P)) = \mathrm{Hom}_{PreSh_{\mathbf{Ab}}(\mathcal{C})}(\mathcal{Z}_X, P)$$

for P in $PreSh_{\mathbf{Sets}}(\mathcal{C})$, resp. P in $PreSh_{\mathbf{Ab}}(\mathcal{C})$. Clearly this shows

Proposition 3.2.8. *The set $\{\Xi_X \mid X \in \mathcal{C}\}$ is a set of generators for $PreSh_{\mathbf{Sets}}(\mathcal{C})$ and $\{\mathcal{Z}_X \mid X \in \mathcal{C}\}$ is a set of generators for $PreSh_{\mathbf{Ab}}(\mathcal{C})$.*

3.2.9. *$PreSh_{\mathbf{Ab}}(\mathcal{C})$ as an abelian category.* We begin this section by recalling some of the basic facts on abelian categories.

Let $\mathcal{A}$ be an additive category, i.e., the Hom-sets are given the structure of abelian groups such that, for each morphism $f : X \to Y$, and object Z of $\mathcal{A}$, the maps $f_* : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$ and $f^* : \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ are group homomorphisms. In addition, one requires that finite coproducts (direct sums) exist. One shows that this implies that finite products also exist, and products and coproducts agree. In particular, $\mathcal{A}$ has an initial and final object 0.

If $\mathcal{A}$ is an additive category, and $f : X \to Y$ a morphism, $i : \ker f \to X$ is a morphism which is universal for maps $g : Z \to X$ such that $fg = 0$, i.e. there exists a unique morphism $\phi : Z \to \ker f$ such that $i\phi = g$. Dually, $j : Y \to \mathrm{coker} f$ is universal for morphisms $h : Y \to Z$ such that $hf = 0$, in that there is a unique morphism $\psi : \mathrm{coker} f \to Z$ with $h = \psi j$. These morphisms, if they exist, are called the *categorical kernel* and *categorical cokernel*, respectively.

An *abelian category* is an additive category $\mathcal{A}$ such that each morphism $f : X \to Y$ admits a categorical kernel and cokernel, and the canonical map

$$\mathrm{coker}(\ker f) \to \ker(\mathrm{coker} f)$$

is an isomorphism. The object $\ker(\mathrm{coker} f)$ is the *image* of f, denoted $\mathrm{im} f$. The primary example of an abelian category is the category Mod_R of (left) modules over a ring R, for example $\mathbf{Ab} := \mathrm{Mod}_{\mathbb{Z}}$.

A *complex* in an abelian category $\mathcal{A}$ is a sequence of morphisms

$$\ldots \xrightarrow{d^{n-2}} M^{n-1} \xrightarrow{d^{n-1}} M^n \xrightarrow{d^n} M^{n+1} \xrightarrow{d^{n+1}} \ldots$$

with $d^n \circ d^{n-1} = 0$ for all n. Under this condition, the map d^{n-1} factors as

$$M^{n-1} \to \mathrm{im}\, d^{n-1} \to \ker d^n \to M^n.$$

The *cohomology* of a complex (M^*, d^*) is defined by

$$H^n(M^*, d^*) := \mathrm{coker}(\mathrm{im}\, d^{n-1} \to \ker d^n),$$

i.e., the familiar quotient object $\ker d^n / \mathrm{im}\, d^{n-1}$. We call the complex *exact* if all cohomology objects H^n vanish.

An *injective* object in an abelian category $\mathcal{A}$ is an object I such that, for each monomorphism $i : N' \to N$, the map $i^* : \mathrm{Hom}_{\mathcal{A}}(N, I) \to \mathrm{Hom}_{\mathcal{A}}(N', I)$ is surjective. Dually a *projective* object in $\mathcal{A}$ is an object P such that, for each epimorphism $j : N \to N'$ the map $j_* :$

$\mathrm{Hom}_{\mathcal{A}}(P, N) \to \mathrm{Hom}_{\mathcal{A}}(P, N')$ is surjective. We say that $\mathcal{A}$ *has enough injectives* if each object M of $\mathcal{A}$ admits a monomorphism $M \to I$ with I injective; $\mathcal{A}$ *has enough projectives* if each M admits an epimorphism $P \to M$ with P projective.

These conditions are useful, as we shall see later, for defining right and left derived functors of left- and right-exact functors. For now, we will only recall that a right-resolution of an object M in an abelian category $\mathcal{A}$ is an exact complex of the form

$$0 \to M \to I^0 \to \ldots \to I^n \to \ldots$$

and dually a left-resolution is an exact complex of the form

$$\ldots \to P^n \to \ldots \to P^0 \to \; M \to 0.$$

A right-resolution with all I^n injective is an *injective* resolution of M; a left-resolution with all P^n projective is a *projective* resolution of M. Clearly, if $\mathcal{A}$ has enough injectives, then each M in $\mathcal{A}$ admits an injective resolution, dually if $\mathcal{A}$ has enough projectives.

We now return to our discussion of the presheaf category. Let $\mathcal{C}$ be a small category, $\mathcal{A}$ an abelian category and $f : F \to G$ a map in $PreSh_{\mathcal{A}}(\mathcal{C})$. Since the categorical ker and coker are defined by universal properties, it is clear that $X \mapsto \ker(f(X) : F(X) \to G(X))$ and $X \mapsto \mathrm{coker}(f(X) : F(X) \to G(X))$ define $\mathcal{A}$-valued presheaves $\ker f$, $\mathrm{coker} f$ on $\mathcal{C}$ and that these are the respective categorical kernel and cokernel of f. Thus

Proposition 3.2.10. *Let $\mathcal{A}$ be an abelian category. Then $PreSh_{\mathcal{A}}(\mathcal{C})$ is an abelian category where, for $f : F \to G$ a morphism,* $\ker f$*, resp.* $\mathrm{coker} f$ *are the presheaves*

$$(\ker f)(X) = \ker f(X); \quad (\mathrm{coker} f)(X) = \mathrm{coker} f(X).$$

We recall the basic result of Grothendieck on the existence of enough injective objects:

Theorem 3.2.11 (Grothendieck [12])**.** *Let $\mathcal{A}$ be an abelian category. Suppose that*

(1) *(small) inductive limits exist in $\mathcal{A}$.*
(2) *if I is a small filtering category, then $(F : I \to \mathcal{A}) \mapsto \lim\limits_{\to} F$ is exact.*
(3) *$\mathcal{A}$ has a set of generators.*

Then $\mathcal{A}$ has enough injectives.

Remark 3.2.12. The condition (1) is equivalent to the condition (AB3) in [12]; the conditions (1) and (2) together is equivalent to the condition (AB5) in [12].

Proposition 3.2.13. *Let $\mathcal{C}$ be a small category. The abelian category $PreSh_{\mathbf{Ab}}(\mathcal{C})$ has enough injectives.*

Proof. The conditions (1) and (2) of Theorem 3.2.11 follow from Proposition 3.2.3. Condition (3) is Proposition 3.2.8. □

3.3. **Sheaves.** We recall the definition of a Grothendieck pre-topology on a category $\mathcal{C}$, and the resulting category of sheaves. Unless explicitly mentioned to the contrary, we will assume that the value category $\mathcal{A}$ is either **Sets** or **Ab**; we leave it to the reader to make the necessary changes for more general value categories, such as G-**Sets** for a group G or Mod_R for a ring R.

Definition 3.3.1. Let $\mathcal{C}$ be a category. A *Grothendieck pre-topology* τ on $\mathcal{C}$ is given by the following data: for each object $X \in \mathcal{C}$ there is a class $\mathrm{Cov}_\tau(X)$ of *covering families* of X, where a covering family of X is a set of morphisms $\{f_\alpha : U_\alpha \to X \mid \alpha \in A\}$ in $\mathcal{C}$. The $\mathrm{Cov}_\tau(X)$ should satisfy the following axioms:

A1. $\{\mathrm{id}_X\}$ is in $\mathrm{Cov}_\tau(X)$ for each $X \in \mathcal{C}$.
A2. For $\{f_\alpha : U_\alpha \to X\} \in \mathrm{Cov}_\tau(X)$ and $g : Y \to X$ a morphism in $\mathcal{C}$, the fiber products $U_\alpha \times_X Y$ all exist and $\{p_2 : U_\alpha \times_X Y \to Y\}$ is in $\mathrm{Cov}_\tau(Y)$.
A3. If $\{f_\alpha : U_\alpha \to X\}$ is in $\mathrm{Cov}_\tau(X)$ and if $\{g_{\alpha\beta} : V_{\alpha\beta} \to U_\alpha\}$ is in $\mathrm{Cov}_\tau(U_\alpha)$ for each α, then $\{f_\alpha \circ g_{\alpha\beta} : V_{\alpha\beta} \to X\}$ is in $\mathrm{Cov}_\tau(X)$.

We will not need the notion of a Grothendieck topology in order to make the construction of primary interest for us, namely sheaves. Roughly speaking giving a pre-topology is analogous to giving a basis of open subsets for a topology on a set; in particular each Grothendieck pre-topology generates a Grothendieck topology. For this reason, we will often omit the distinction between a pre-topology and a topology on a category.

Examples 3.3.2. (1) The "classical" example is the topology T on $\mathcal{C} = \mathrm{Op}(T)$, for T a topological space, where for $U \subset T$ open, $\{f_\alpha : U_\alpha \to U\}$ is in $\mathrm{Cov}_T(U)$ if each f_α is the inclusion of an open subset $U_\alpha \subset U$ and $U = \cup_\alpha U_\alpha$. A somewhat less classical example is to let $\mathcal{C}$ be the category of topological spaces **Top**. Setting $\mathrm{Cov}_{\mathbf{Top}}(U) = \mathrm{Cov}_U(U)$ gives a topology on **Top**. Note that for $U_\alpha \subset U$ open and $f : V \to U$ a continuous map, the fiber product $V \times_U U_\alpha$ is just the open subset $f^{-1}(U_\alpha)$ of V. In particular, for $\{f_\alpha : U_\alpha \to U\}$ in $\mathrm{Cov}(U)$, the fiber product $U_\alpha \times_U U_\beta$ is just the intersection $U_\alpha \cap U_\beta$.

(2) We will be most interested in Grothendieck topologies on subcategories of $\mathbf{Sch}_k$, k a field. The first example is the category $\mathrm{Op}(|X|)$, $X \in \mathbf{Sch}_k$, with the covering families as in (1). This gives the *Zariski* topology, denoted X_{Zar}. We can extend this topology to all of $\mathbf{Sch}_k$, as in example (1), giving the topology $\mathbf{Sch}_{k\mathrm{Zar}}$.

(3) Take $X \in \mathbf{Sch}_k$. The *étale* topology $X_{\text{ét}}$ has as underlying category the schemes over X, $f : U \to X$ such that f is an étale morphism of finite type. For such a U, a covering family $\{f_\alpha : U_\alpha \to U\}$ is a set of étale morphisms (of finite type) such that $\coprod_\alpha |U_\alpha| \to |U|$ is surjective. As in (2) one can use the same definition of covering families to define the étale topology on $\mathbf{Sch}_k$.

(4) The *Nisnevic* topology is between the étale and Zariski topologies: an étale covering family $\{f_\alpha : U_\alpha \to U\}$ is a Nisnevic covering family if for each field extension $L \supset k$, the induced map on the L-valued points

$$\coprod_\alpha U_\alpha(L) \to U(L)$$

is surjective; this condition clearly implies that $\{f_\alpha : U_\alpha \to U\}$ is in $\mathrm{Cov}_{\text{ét}}(U)$. This defines the Nisnevic topology on X, X_{Nis} and on $\mathbf{Sch}_k$, $\mathbf{Sch}_{k\mathrm{Nis}}$.

One can phrase the condition on the L-valued points somewhat differently: For each point $w \in U$, there is an α and a $w_\alpha \in U_\alpha$ with $f_\alpha(w_\alpha) = w$ and with $f_\alpha^* : k(w) \to k(w_\alpha)$ an isomorphism. This condition for a particular $w \in |U|$ with closure $W \subset U$ is equivalent to saying that for some α, the map $f_\alpha : U_\alpha \to U$ admits a section over some open subscheme W^0 of W.

(5) The h-topology and variants (see [26, Chap. 2, §4]). We recall that a map of schemes $f : Y \to X$ is a *topological epimorphism* if the map on the underlying spaces $|f| : |Y| \to |X|$ identifies $|X|$ with a quotient of $|Y|$ ($|f|$ is surjective and $U \subset |X|$ is open if and only if $|f|^{-1}(U)$ is open in $|Y|$). f is a *universal* topological epimorphism if $p_2 : Y \times_X Z \to Z$ is a topological epimorphism for all $Z \to X$.

The covering families in the h-topology are finite sets $\{f_i : U_i \to X\}$ such that each f_i is of finite type and $\coprod_i U_i \to X$ is a universal topological epimorphism.

The covering families in the qfh-topology are the h-coverings $\{f_i : U_i \to X\}$ such that each f_i is quasi-finite.

The covering families in the cdh-topology are those generated (i.e., by iteratively applying the axioms A2 and A3) by

a) Nisnevic covering families

b) families of the form $\{X' \amalg F \xrightarrow{p\amalg i} X\}$, with $p : X' \to X$ proper, $i : F \to X$ a closed embedding and $p : p^{-1}(X \setminus i(F)) \to X \setminus i(F)$ an isomorphism.

(6) The *indiscrete* topology on a category $\mathcal{C}$ is the topology *ind* with $\mathrm{Cov}_{ind}(X) = \{\{\mathrm{id}_X\}\}$.

To define $\mathcal{A}$-valued sheaves for some value-category $\mathcal{A}$, we need to be able to state the sheaf axiom, so we require that $\mathcal{A}$ admits arbitrary products (indexed by sets). Recall that the *equalizer* of two morphisms $f, f' : A \to B$ in $\mathcal{A}$, is a morphism $i : A_0 \to A$ which is universal for morphisms $g : Z \to A$ such that $fg = f'g$, i.e., there exists a unique morphism $\phi : Z \to A_0$ with $g = i\phi$. If $i : A_0 \to A$ is the equalizer of f and f', we say the sequence $A_0 \xrightarrow{i} A \overset{f}{\underset{f'}{\rightrightarrows}} B$ is *exact.*

Remarks 3.3.3. (1) The equalizer of f and f' is the same is the projective limit over the category $I := A \overset{f}{\underset{f'}{\rightrightarrows}} B$ of the evident inclusion functor $I \to \mathcal{A}$.

(2) If $\mathcal{A}$ is an abelian category, then $A_0 \xrightarrow{i} A \overset{f}{\underset{f'}{\rightrightarrows}} B$ is exact if and only if $0 \to A_0 \xrightarrow{i} A \xrightarrow{f-f'} B$ is exact in the usual sense.

If $\mathcal{A}$ is as above, S an $\mathcal{A}$-valued presheaf on $\mathcal{C}$ and $\{f_\alpha : U_\alpha \to X \mid \alpha \in A\} \in \mathrm{Cov}_\tau(X)$ for some $X \in \mathcal{C}$, we have the "restriction" morphisms

$$\begin{aligned} &f_\alpha^* : S(X) \to S(U_\alpha) \\ &p_{1,\alpha,\beta}^* : S(U_\alpha) \to S(U_\alpha \times_X U_\beta) \\ &p_{2,\alpha,\beta}^* : S(U_\beta) \to S(U_\alpha \times_X U_\beta). \end{aligned}$$

Taking products, we have the diagram in $\mathcal{A}$

$$S(X) \xrightarrow{\prod f_\alpha^*} \prod_\alpha S(U_\alpha) \overset{\prod p_{1,\alpha,\beta}^*}{\underset{\prod p_{2,\alpha,\beta}^*}{\rightrightarrows}} \prod_{\alpha.\beta} S(U_\alpha \times_X U_\beta). \tag{3.3.1}$$

Definition 3.3.4. Let $\mathcal{A}$ be a category having arbitrary (small) products, and let τ be a Grothendieck pre-topology on a small category $\mathcal{C}$. An $\mathcal{A}$-valued presheaf S on $\mathcal{C}$ is a *sheaf* for τ if for each covering family $\{f_\alpha : U_\alpha \to X\} \in \mathrm{Cov}_\tau$, the sequence (3.3.1) is exact. The category $Sh^\tau_{\mathcal{A}}(\mathcal{C})$ of $\mathcal{A}$-valued sheaves on $\mathcal{C}$ for τ is the full subcategory of $PreSh_{\mathcal{A}}(\mathcal{C})$ with objects the sheaves.

Remark 3.3.5. For the examples discussed in 3.3.2, we use the following notation: If X is a topological space, we write $Sh_{\mathcal{A}}(X)$ for $Sh^X_{\mathcal{A}}(\mathrm{Op}(X))$. For X a finite type k-scheme, we write $Sh^\tau_{\mathcal{A}}(X)$ for $Sh^\tau_{\mathcal{A}}(X_\tau)$, where $\tau = \mathrm{Zar}, \text{ét}$, etc. We use a similar notation, $PreSh_{\mathcal{A}}(X)$ (X a topological space) or $PreSh_{\mathcal{A}}(X_\tau)$ (X a finite type k-scheme, $\tau = \mathrm{Zar}, \text{ét}$, etc.) for the respective presheaf categories.

For the indiscrete topology on $\mathcal{C}$, we have

$$Sh^{ind}_{\mathcal{A}}(\mathcal{C}) = PreSh_{\mathcal{A}}(\mathcal{C})$$

so the presheaf category is also a category of sheaves.

3.3.6. *Projective limits.* The left-exactness of projective limits allows one to construct projective limits of sheaves easily:

Lemma 3.3.7. *Let $F : I \to PreSh_{\mathcal{A}}(\mathcal{C})$ be a functor such that $F(i)$ is a τ-sheaf for all $i \in I$, $\mathcal{A} = \mathbf{Sets}$ or $\mathcal{A} = \mathbf{Ab}$. Then the presheaf $\varprojlim F$ is a τ-sheaf.*

Proof. Take $\{f_\alpha : U_\alpha \to X\}$ in Cov_τ. For each i, the sequence

$$F(i)(X) \xrightarrow{\prod f^*_\alpha} \prod_\alpha F(i)(U_\alpha) \underset{\prod p^*_{2,\alpha,\beta}}{\overset{\prod p^*_{1,\alpha,\beta}}{\rightrightarrows}} \prod_{\alpha.\beta} F(i)(U_\alpha \times_X U_\beta).$$

is exact. The left-exactness of $\varprojlim$ in $\mathcal{A}$ and the definition of the presheaf limit implies (see Remark 3.3.3) that

$$(\varprojlim F)(X) \xrightarrow{\prod f^*_\alpha} \prod_\alpha (\varprojlim F)(U_\alpha) \underset{\prod p^*_{2,\alpha,\beta}}{\overset{\prod p^*_{1,\alpha,\beta}}{\rightrightarrows}} \prod_{\alpha.\beta} (\varprojlim F)(U_\alpha \times_X U_\beta).$$

is exact, whence the result. □

This yields

Proposition 3.3.8. *Let $\mathcal{C}$ be a small category, $\mathcal{A} = \mathbf{Sets}, \mathbf{Ab}$, τ a Grothendieck pre-topology on $\mathcal{C}$. Then*
1. *small projective limits exist in $Sh^\tau_{\mathcal{A}}(\mathcal{C})$.*
2. *$(F : I \to Sh^\tau_{\mathcal{A}}(\mathcal{C})) \mapsto \varprojlim F$ is left-exact.*

3.3.9. *Sheafification.* Let $i : Sh^\tau_{\mathcal{A}}(\mathcal{C}) \to PreSh_{\mathcal{A}}(\mathcal{C})$ be the inclusion functor. A modification of the sheafification construction for sheaves on a topological space yields:

Theorem 3.3.10. *For $\mathcal{A} = \mathbf{Sets}, \mathbf{Ab}$, the inclusion i admits a left-adjoint $a_\tau : PreSh_{\mathcal{A}}(\mathcal{C}) \to Sh^\tau_{\mathcal{A}}(\mathcal{C})$.*

The proof proceeds in a number of steps. To begin, a presheaf P is called *separated* (for the pre-topology τ) if for each $\{f_\alpha : U_\alpha \to X\}$ in Cov_τ, the map

$$P(X) \to \prod_\alpha P(U_\alpha)$$

is a monomorphism. We first construct a functor $P \mapsto P^+$ on presheaves with the property that P^+ is separated for each presheaf P.

Definition 3.3.11. Let $\mathcal{U} := \{f_\alpha : U_\alpha \to X \mid \alpha \in A\}$ and $\mathcal{V} := \{g_\beta : V_\beta \to X \mid \beta \in B\}$ be in $\mathrm{Cov}_\tau(X)$. We say that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if there exists a map of sets $r : B \to A$ and morphisms $\phi_\beta : V_\beta \to U_{r(\beta)}$ in $\mathcal{C}$ for each $\beta \in B$, such that $g_\beta = f_{r(\beta)} \circ \phi_\beta$ for all β. The pair $\rho := (r, \{\phi_\beta, \beta \in B\})$ is called a *refinement mapping*, written

$$\rho : \mathcal{V} \to \mathcal{U}.$$

Let $\mathcal{U} = \{f_\alpha : U_\alpha \to X \mid \alpha \in A\}$ be in $\mathrm{Cov}_\tau(X)$. For a presheaf P, define $P_X(\mathcal{U})$ by the exactness of

$$\begin{tikzcd} P_X(\mathcal{U}) \arrow[r, "\prod f_\alpha^*"] & \prod_\alpha P(U_\alpha) \arrow[r, shift left, "\prod p^*_{1,\alpha,\alpha'}"] \arrow[r, shift right, "\prod p^*_{2,\alpha,\alpha'}"'] & \prod_{\alpha.\alpha'} P(U_\alpha \times_X U_{\alpha'}). \end{tikzcd}$$

$P_X(\mathcal{U})$ exists since projective limits exist in **Sets** and in **Ab**. Since $f_\alpha \circ p_{1,\alpha,\alpha'} = f'_\alpha \circ p_{2,\alpha,\alpha'}$, the universal property of the equalizer gives us a canonical map

$$\epsilon_{\mathcal{U}} : P(X) \to P_X(\mathcal{U}).$$

Similarly, each refinement mapping $\rho = (r, \{\phi_\beta\}) : \mathcal{V} \to \mathcal{U}$ gives a commutative diagram

$$\begin{tikzcd} P(X) \arrow[r, "\epsilon_{\mathcal{U}}"] \arrow[dr, "\epsilon_{\mathcal{V}}"'] & P_X(\mathcal{U}) \arrow[d, "\rho^*"] \\ & P_X(\mathcal{V}) \end{tikzcd}$$

where ρ^* is induced by the commutative diagram

$$\begin{tikzcd} P_X(\mathcal{U}) \arrow[r, "\prod f_\alpha^*"] \arrow[dd, "\rho^*"'] & \prod_\alpha P(U_\alpha) \arrow[r, shift left, "\prod p^*_{1,\alpha,\alpha'}"] \arrow[r, shift right, "\prod p^*_{2,\alpha,\alpha'}"'] \arrow[d, "\pi"'] & \prod_{\alpha.\alpha'} P(U_\alpha \times_X U_{\alpha'}) \arrow[d, "\pi"] \\ & \prod_\beta P(U_{r(\beta)}) \arrow[r, shift left, "\prod p^*_{1,r(\beta),r(\beta')}"] \arrow[r, shift right, "\prod p^*_{2,r(\beta),r(\beta')}"'] \arrow[d, "\prod_\beta \phi^*_\beta"'] & \prod_{\beta,\beta'} P(U_{r(\beta)} \times_X U_{r(\beta')}) \arrow[d, "\prod_{\beta,\beta'} \phi^*_\beta \times \phi^*_{\beta'}"] \\ P_X(\mathcal{V}) \arrow[r, "\prod f_\beta^*"'] & \prod_\beta P(V_\beta) \arrow[r, shift left, "\prod p^*_{1,\beta,\beta'}"] \arrow[r, shift right, "\prod p^*_{2,\beta,\beta'}"'] & \prod_{\beta.\beta'} P(V_\beta \times_X V_{\beta'}), \end{tikzcd}$$

and where the maps π are the respective projections.

Lemma 3.3.12. *Suppose we have two refinement mappings $\rho, \rho' : \mathcal{V} \to \mathcal{U}$. Then $\rho^* = \rho'^*$.*

Proof. Write $\rho = (r, \{\phi_\beta\})$, $\rho' = (r', \{\phi'_\beta\})$. For each β. let $\psi_\beta : V_\beta \to U_{r(\beta)} \times_X U_{r'(\beta)}$ be the map $(\phi_\beta, \phi'_\beta)$. If $\prod_\alpha x_\alpha$ is in $P_X(\mathcal{U}) \subset \prod_\alpha P(U_\alpha)$ then

$$\begin{aligned}\phi^*_\beta(x_{r(\beta)}) &= \psi^*_\beta \circ p^*_{1,r(\beta),r'(\beta)}(x_{r(\beta)})\\ &= \psi^*_\beta \circ p^*_{2,r(\beta),r'(\beta)}(x_{r'(\beta)})\\ &= \phi'^*_\beta(x_{r'(\beta)}).\end{aligned}$$

□

Now let $Cov_\tau(X)$ be the category with objects $\mathrm{Cov}_\tau(X)$ and a unique morphism $\mathcal{V} \to \mathcal{U}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$. We have defined the functor $P_X : Cov_\tau(X) \to \mathcal{A}$ sending $\mathcal{U}$ to $P_X(\mathcal{U})$ and $\mathcal{V} \to \mathcal{U}$ to ρ^* for any choice of a refinement mapping $\rho : \mathcal{V} \to \mathcal{U}$. For later use, we record the structure of the category $Cov_\tau(X)$:

Lemma 3.3.13. *$Cov_\tau(X)$ is a small filtering category.*

Proof. Since $\mathrm{Hom}_{Cov_\tau(X)}(\mathcal{V}, \mathcal{U})$ is either empty or has a single element, properties L1 and L2 follow from L3. If $\mathcal{U} = \{f_\alpha : U_\alpha \to X\}$ and $\mathcal{V} = \{g_\beta : V_\beta \to X\}$ are in $\mathrm{Cov}_\tau(X)$, then $\{U_\alpha \times_X V_\beta \to X\}$ is a common refinement, verifying L3. □

We set

$$P^+(X) := \varinjlim_{Cov_\tau(X)} P_X.$$

The maps $\epsilon_{\mathcal{U}} : P(X) \to P_X(\mathcal{U})$ define the map $\epsilon_X : P(X) \to P^+(X)$.

Let $g : Y \to X$ be a morphism in $\mathcal{C}$. If $\mathcal{U} = \{f_\alpha : U_\alpha \to X\}$ is in $\mathrm{Cov}_\tau(X)$, we have the covering family $g^*\mathcal{U} := \{p_2 : U_\alpha \times_X Y \to Y\}$ in $\mathrm{Cov}_\tau(Y)$. The operation $\mathcal{U} \mapsto g^*\mathcal{U}$ respects refinement, giving the functor $\hat{g}^* : Cov_\tau(X) \to Cov_\tau(Y)$, with $\widehat{(gh)}^*$ canonically isomorphic to $\hat{h}^*\hat{g}^*$. If $\mathcal{U} = \{f_\alpha : U_\alpha \to X\}$ is in $\mathrm{Cov}_\tau(Y)$, pull-back by the projections $U_\alpha \times_X Y \to U_\alpha$ defines the map $g^*_{\mathcal{U}} : P_X(\mathcal{U}) \to P_Y(\hat{g}^*\mathcal{U}))$. The maps $g^*_{\mathcal{U}}$ define a natural transformation $g^*_? : P_X \to P_Y \circ \hat{g}^*$. Thus, we have the map on the limits

$$g^* := (\hat{g}^*)_* \circ (g^*_?)_* : P^+(X) \to P^+(Y).$$

With these pull-back maps, we have defined a presheaf P^+ on $\mathcal{C}$; the maps ϵ_X define the map of presheaves $\epsilon_P : P \to P^+$.

If $f : P \to Q$ is a morphism of presheaves, f induces in the evident manner a natural transformation $f_X : P_X \to Q_X$ and hence a map on the limits $f^+(X) : P^+(X) \to Q^+(X)$. The maps $f^+(X)$ define a morphism of presheaves $f^+ : P^+ \to Q^+$, compatible with the maps ϵ_P, ϵ_Q. Thus we have define a functor $^+ : PreSh_{\mathcal{A}}(\mathcal{C}) \to PreSh_{\mathcal{A}}(\mathcal{C})$ and a natural transformation $\epsilon : \mathrm{id} \to {}^+$.

Lemma 3.3.14. *Let P be a presheaf on $\mathcal{C}$ with values in* **Sets**, **Ab**.

(1) *P^+ is a separated presheaf.*
(2) *if P is separated, then P^+ is a sheaf*
(3) *if P is a sheaf, then $\epsilon_P : P \to P^+$ is an isomorphism.*

Proof. For (1), take $\{f_\alpha : U_\alpha \to X\}$ in $\mathrm{Cov}_\tau(X)$. Since $Cov_\tau(X)$ is filtering, each element $x \in P^+(X)$ is represented by an $x_{\mathcal{V}} \in P_X(\mathcal{V})$ for some $\mathcal{V} \in \mathrm{Cov}_\tau(X)$, and $x_{\mathcal{V}}$, $x_{\mathcal{V}'}$ represent the same element in $P^+(X)$ if and only if there is a common refinement $\rho : \mathcal{W} \to \mathcal{V}$, $\rho' : \mathcal{W} \to \mathcal{V}'$ with $\rho^*(x_{\mathcal{V}}) = \rho'^*(x_{\mathcal{V}'})$ in $P(\mathcal{W})$. Similar remarks describe $P^+(U_\alpha)$.

Now, if for $x, x' \in P^+(X)$ have $f_\alpha^*(x) = f_\alpha^*(x')$ for all α, choose a covering family $\mathcal{V}$ and elements $x_{\mathcal{V}}, x'_{\mathcal{V}} \in P_X(\mathcal{V})$ representing x, x'. Replacing $\mathcal{V}$ and $\mathcal{U}$ with a common refinement, we may assume that $\mathcal{V} = \mathcal{U}$. Write

$$x_{\mathcal{U}} = \prod_\alpha x_\alpha \in P(U_\alpha),$$
$$x'_{\mathcal{U}} = \prod_\alpha x'_\alpha \in P(U_\alpha).$$

The element $f_\alpha^*(x_{\mathcal{U}})$ is represented by the collection $\prod_{\alpha'} p_{1\alpha'}^*(x_{\alpha'})$ in $P_{U_\alpha}(\mathcal{U} \times_X U_\alpha)$, where $\mathcal{U} \times_X U_\alpha$ is the covering family $\{U_{\alpha'} \times_X U_\alpha \to U_\alpha\}$ of U_α. The diagonal $U_\alpha \to U_\alpha \times_X U_\alpha$ gives the refinement $\{\mathrm{id}\} \to \mathcal{U} \times_X U_\alpha$, so $f_\alpha^*(x_{\mathcal{U}})$ is also represented by $x_\alpha \in P(U_\alpha)$.

The identity $f_\alpha^*(x_{\mathcal{U}}) = f_\alpha^*(x'_{\mathcal{U}})$ thus is equivalent to the assertion that there is a covering family $\mathcal{V}_\alpha = \{g_{\alpha\beta} : V_{\alpha\beta} \to U_\alpha\}$ in $\mathrm{Cov}_\tau(U_\alpha)$ such that $g_{\alpha\beta}^*(x_\alpha) = g_{\alpha\beta}^*(x'_\alpha)$ for all β. Replacing $\mathcal{U}$ with the covering family $\{f_\alpha \circ g_{\alpha\beta} : V_{\alpha\beta} \to X\}$, we may assume that $\mathcal{V}_\alpha$ is the identity covering family, i.e. that $x_\alpha = x'_\alpha$ in $P(U_\alpha)$. But then $x = x'$, as desired.

For (2), suppose that P is separated, take $\{f_\alpha : U_\alpha \to X\}$ in $\mathrm{Cov}_\tau(X)$, and suppose we have elements $x_\alpha \in P^+(U_\alpha)$ with

$$p_{1,\alpha,\alpha'}^*(x_\alpha) = p_{2,\alpha,\alpha'}^*(x_{\alpha'}) \text{ in } P^+(U_\alpha \times_X U_{\alpha'})$$

for all α, α'. We need to show that there exists an element $x \in P^+(X)$ with $f_\alpha^*(x) = x_\alpha$, as the injectivity of $P^+(X) \to \prod_\alpha P^+(U_\alpha)$ follows from (1).

The element x_α is represented by a collection

$$\prod_\beta x_{\alpha\beta} \in P_{U_\alpha}(\mathcal{V}_\alpha) \subset \prod_\beta P(V_{\alpha\beta})$$

for some $\mathcal{V}_\alpha := \{V_{\alpha\beta} \to U_\alpha\} \in \mathrm{Cov}_\tau(U_\alpha)$. Since P is separated, the relation $p^*_{1,\alpha,\alpha'}(x_\alpha) = p^*_{2,\alpha,\alpha'}(x_{\alpha'})$ in $P^+(U_\alpha \times_X U_{\alpha'})$ implies the relation

$$p^*_{1,\alpha\beta,\alpha'\beta'}(x_{\alpha\beta}) = p^*_{2,\alpha\beta,\alpha'\beta'}(x_{\alpha'\beta'})$$

in $P(V_{\alpha\beta} \times_X V_{\alpha'\beta'})$. This in turn implies that the collection $\prod_{\alpha,\beta} x_{\alpha\beta} \in \prod_{\alpha,\beta} P(V_{\alpha\beta})$ is in the subset $P_X(\mathcal{V})$, where $\mathcal{V}$ is the covering family $\{V_{\alpha\beta} \to X\}$. This yields the desired element of $P^+(X)$.

For (3), let $\mathcal{U}$ be in $\mathrm{Cov}_\tau(X)$ and suppose P is a τ-sheaf. Then $P(X) \to P(\mathcal{U})$ is an isomorphism, hence we have an isomorphism on the limit $\epsilon_P(X) : P(X) \to P^+(X)$. □

We define the functor $a_\tau : PreSh_{\mathcal{A}}(\mathcal{C}) \to Sh^\tau_{\mathcal{A}}(\mathcal{C})$ by $a_\tau = {}^+ \circ {}^+$. By the lemma just proved, a_τ is well-defined and $a_\tau \circ i$ is naturally isomorphic to the identity on $Sh^\tau_{\mathcal{A}}(\mathcal{C})$. We call a_τ the *sheafification* functor.

Lemma 3.3.15. *The sheafification functor $a_\tau : PreSh_{\mathcal{A}}(\mathcal{C}) \to Sh^\tau_{\mathcal{A}}(\mathcal{C})$ is left-adjoint to the inclusion functor i.*

Proof. Let P be a presheaf, S a sheaf and $f : P \to i(S)$ a morphism of presheaves. Applying a_τ gives the commutative diagram

$$\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, "\epsilon_P"'] & i(S) \arrow[d, "\epsilon_{i(S)}"] \\
a_\tau(P) \arrow[r, "a_\tau(f)"'] & a_\tau(i(S))
\end{tikzcd}$$

As $\epsilon_{i(S)}$ is an isomorphism, sending f to $\epsilon^{-1}_{i(S)} \circ a_\tau(f)$ gives a natural transformation

$$\theta_{P,S} : \mathrm{Hom}_{PreSh^\tau_{\mathcal{A}}(\mathcal{C})}(P, i(S)) \to \mathrm{Hom}_{Sh^\tau_{\mathcal{A}}(\mathcal{C})}(a_\tau(P), S).$$

Similarly sending $g : a_\tau(P) \to S$ to $\epsilon^{-1}_{i(S)} \circ g \circ \epsilon_P$ gives the inverse to $\theta_{P,S}$. □

This completes the proof of Theorem 3.3.10. We record a useful property of sheafification.

Proposition 3.3.16. *The sheafification functor $a_\tau : PreSh_{\mathcal{A}}(\mathcal{C}) \to Sh^\tau_{\mathcal{A}}(\mathcal{C})$ is exact.*

Proof. Since a_τ is a left-adjoint, a_τ is right-exact. Similarly, the inclusion functor i is left-exact. In particular, a projective limit of sheaves is given point-wise:

$$(\lim_{\leftarrow} F : I \to Sh^\tau_{\mathcal{A}}(\mathcal{C}))(X) = \lim_{\leftarrow} F(X) : I \to \mathcal{A}.$$

Since the category $Cov_\tau(X)$ is filtering, the functor $^+$ is exact, so the functor $i \circ a_\tau$ is left-exact, hence a_τ is left-exact. $\square$

3.3.17. *Inductive limits of sheaves.* We have seen that the theory of projective limits of sheaves is essentially trivial, in that it coincides with the theory of projective limits of presheaves. To construct inductive limits of sheaves, we use the inductive limit of presheaves plus the sheafification functor.

Proposition 3.3.18. *Let I be a small category, $F : I \to Sh^\tau_{\mathcal{A}}(\mathcal{C})$ a functor. Then $\varinjlim F$ exists and is given by the formula*

$$\varinjlim F = a_\tau(\varinjlim i \circ F).$$

The functor $F \mapsto \varinjlim F$ is right-exact; if I is a filtering category, then $F \mapsto \varinjlim F$ is exact.

Proof. We verify the universal property: Take a sheaf S. Then

$$\begin{aligned}
\mathrm{Hom}_{Sh^\tau_{\mathcal{A}}(\mathcal{C})}(a_\tau(\varinjlim i \circ F), S) &= \mathrm{Hom}_{PreSh^\tau_{\mathcal{A}}(\mathcal{C})}(\varinjlim i \circ F, i(S)) \\
&= \varprojlim \mathrm{Hom}_{PreSh^\tau_{\mathcal{A}}(\mathcal{C})}(i \circ F, i(S)) \\
&= \varprojlim \mathrm{Hom}_{Sh^\tau_{\mathcal{A}}(\mathcal{C})}(F, S).
\end{aligned}$$

The exactness assertions follow from the exactness properties of the presheaf inductive limit and the exactness of a_τ. $\square$

3.3.19. *Epimorphisms of sheaves.* Since projective limits of sheaves and preheaves agree, a map of sheaves $S' \to S$ is a monomorphism if and only if $f(X) : S'(X) \to S(X)$ is a monomorphism for each X. We now make explicit the condition that a map of sheaves $f : S \to S'$ be an epimorphism. Let 0 be the final object in $\mathcal{A}$ and $*$ the sheafification of the constant presheaf 0 with value 0. It is easy to see that f is an epimorphism if and only if the canonical map $S'/S \to *$ is an isomorphism, where the quotient sheaf S'/S is defined as the inductive limit over the category I

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "\pi"'] & S' \\
*
\end{tikzcd}$$

of the evident inclusion functor $I \to Sh^\tau_{\mathcal{A}}(\mathcal{C})$. Noting that the sheaf inductive limit is the sheafification of the presheaf inductive limit we see that S'/S is $*$ if and only if the presheaf $i(S')/i(S)$ has $*$ as sheafification. Since P^+ is separated for every presheaf P, we see that

$a_\tau(i(S')/i(S)) = *$ if and only if $(i(S')/i(S))^+ \to 0^+ = *$ is an isomorphism. This in turn yields the criterion:

Proposition 3.3.20. *A map of sheaves $f : S \to S'$ is an epimorphism if and only if, for each $X \in \mathcal{C}$, and each $x \in S'(X)$, there exists a $\{g_\alpha : U_\alpha \to X\} \in \mathrm{Cov}_\tau(X)$ and $y_\alpha \in S(U_\alpha)$ with $f(y_\alpha) = g_\alpha^*(x)$.*

3.3.21. *Functoriality.* Let $f : \mathcal{C} \to \mathcal{C}'$ be a functor of small categories, where $\mathcal{C}$ and $\mathcal{C}'$ are endowed with pre-topologies τ, τ', respectively. f is called *continuous* if for each $\{g_\alpha : U_\alpha \to X\} \in \mathrm{Cov}_\tau$, $\{f(g_\alpha) : f(U_\alpha) \to f(X)\}$ is in $\mathrm{Cov}_{\tau'}$. Thus, if f is continuous, the presheaf pull-back $f^p : PreSh_{\mathcal{A}}(\mathcal{C}') \to PreSh_{\mathcal{A}}(\mathcal{C})$ restricts to give the sheaf pull-back $f^s : Sh^{\tau'}_{\mathcal{A}}(\mathcal{C}') \to Sh^{\tau}_{\mathcal{A}}(\mathcal{C})$, i.e.

$$i \circ f^s = f^p \circ i'$$

where i and i' are the respective inclusions.

We define the sheaf push-forward by $f_s := a_{\tau'} \circ f_p \circ i$.

Proposition 3.3.22. *Let $f : (\mathcal{C}, \tau) \to (\mathcal{C}', \tau')$ be a continuous functor of small categories with pre-topologies. Then*

(1) *f_s is left-adjoint to f^s*
(2) *f^s is left-exact and f_s is right-exact.*

Proof. The proof of adjointness is formal:

$$\begin{aligned}
\mathrm{Hom}_{Sh^{\tau'}_{\mathcal{A}}(\mathcal{C}')}(f_s(S), T) &= \mathrm{Hom}_{Sh^{\tau'}_{\mathcal{A}}(\mathcal{C}')}(a_{\tau'} \circ f_p \circ i(S), T) \\
&= \mathrm{Hom}_{PreSh_{\mathcal{A}}(\mathcal{C}')}(f_p \circ i(S), i'(T)) \\
&= \mathrm{Hom}_{PreSh_{\mathcal{A}}(\mathcal{C})}(i(S), f^p \circ i'(T)) \\
&= \mathrm{Hom}_{PreSh_{\mathcal{A}}(\mathcal{C})}(i(S), i \circ f^s(T)) \\
&= \mathrm{Hom}_{Sh^{\tau}_{\mathcal{A}}(\mathcal{C})}(S, f^s(T)).
\end{aligned}$$

The exactness assertions follow from the adjoint property. $\square$

Examples 3.3.23. We consider the case of a continuous map $f : X \to Y$ of topological spaces. Let $F : \mathrm{Op}(Y) \to \mathrm{Op}(X)$ be the inverse image functor $F(V) = f^{-1}(V)$, which is clearly continuous with respect to the topologies X, Y on $\mathrm{Op}(X)$, $\mathrm{Op}(Y)$. The functor $F^s : Sh(X) \to Sh(Y)$ is usually denoted f_* and $F_s : Sh(Y) \to Sh(X)$ is denoted f^*.

More generally, if τ is a pre-topology on $\mathcal{C}$ and X is an object in $\mathcal{C}$, let X_τ be the full subcategory of the category of morphisms to X, $\mathcal{C}/X$, with objects the morphisms $f : U \to X$ which occur as a member of a covering family of X. Clearly τ restricts to a pre-topology on X_τ; we write X_τ for the category with this pre-topology. If $f : X \to Y$ is a morphism in $\mathcal{C}$ then sending $f : U \to X$ to $p_2 : U \times_X Y \to Y$ is a well-defined continuous functor

$$f^{-1} : X_\tau \to Y_\tau.$$

We denote $(f^{-1})^s$ by f_* and f_s^{-1} by f^*.

For example, if X is a k-scheme of finite type, and $\tau =$ Zar, ét, etc., on $\mathbf{Sch}_k$, then X_τ agrees with our definition of X_τ given in Example 3.3.2.

3.3.24. *Generators and abelian structure.* Just as for presheaves, we can use the inclusion functor $i_X : * \to \mathcal{C}$ corresponding to an object $X \in \mathcal{C}$ to construct generators for the sheaf category. Let $\chi_X = i_{Xs}(0) \in Sh^\tau_{\mathbf{Sets}}(\mathcal{C})$ and $\mathbb{Z}_X = i_{Xs}(\mathbb{Z}) \in Sh^\tau_{\mathbf{Ab}}(\mathcal{C})$.

Proposition 3.3.25. *Let τ be a pre-topology on a small category $\mathcal{C}$. Then $\{\chi_X \mid X \in \mathcal{C}\}$ forms a set of generators for $Sh^\tau_{\mathbf{Sets}}(\mathcal{C})$ and $\{\mathbb{Z}_X \mid X \in \mathcal{C}\}$ forms a set of generators for $Sh^\tau_{\mathbf{Ab}}(\mathcal{C})$.*

The category $Sh^\tau_{\mathbf{Ab}}(\mathcal{C})$ is an abelian category: for a morphism $f : S \to T$, $\ker f$ is just the presheaf kernel (since i is left-exact, this is automatically a sheaf). The sheaf cokernel is given by

$$\mathrm{coker}^s f = a_\tau(\mathrm{coker}^p f).$$

One easily checks that this formula does indeed give the categorical cokernel and that the natural map $\ker(\mathrm{coker} f) \to \mathrm{coker}(\ker f)$ is an isomorphism. Just as for presheaves, the existence of a set of generators, and the exactness of filtered inductive limits gives us

Proposition 3.3.26. *The abelian category $Sh^\tau_{\mathbf{Ab}}(\mathcal{C})$ has enough injectives.*

3.3.27. *Cohomology.* Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $f : \mathcal{A} \to \mathcal{B}$ an additive functor. If $\mathcal{A}$ has enough injectives, then each object M in $\mathcal{A}$ admits an injective resolution

$$0 \to M \to I^0 \to \ldots \to I^n \to \ldots .$$

The *right-derived functors* $R^q f$ of f are then defined (see e.g. [25]) and are given by the formula

$$R^q f(M) = H^q(f(I^0) \to f^(I^1) \to \ldots \to f(I^n) \to \ldots).$$

The $R^q f$ form a *cohomological functor* in that each short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{A}$ yields a long exact sequence

$$\begin{aligned} 0 \to R^0 f(M') \to R^0 f(M) \to R^0 f(M'') \to R^1 f(M') \to \ldots \\ \to R^n f(M') \to R^n f(M) \to R^n f(M'') \to R^{n+1} f(M') \to \ldots \end{aligned}$$

natural in the exact sequence. If f is left-exact, then $R^0f = f$. For example, the nth right-derived functor of the left-exact functor

$$N \mapsto \mathrm{Hom}_{\mathcal{A}}(M, N)$$

is denoted $\mathrm{Ext}^n_{\mathcal{A}}(M, -)$, and $\mathrm{Ext}^0_{\mathcal{A}}(M, N) = \mathrm{Hom}_{\mathcal{A}}(M, N)$.

We apply this to sheaves of abelian groups on $(\mathcal{C}, \tau)$. Take $X \in \mathcal{C}$ and consider the left-exact functor

$$i^s_X : Sh^\tau_{\mathbf{Ab}}(X) \to Sh_{\mathbf{Ab}}(*) = \mathbf{Ab}$$

This is just the functor $S \mapsto S(X)$. We define the *cohomology* $H^n(X, S)$ by

$$H^n(X, S) := R^n i^s_X(S).$$

As $i^s_X(S) = \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Z}, i^s_X(X)) = \mathrm{Hom}_{Sh^\tau_{\mathbf{Ab}}(\mathcal{C})}(\mathbb{Z}_X, S)$, we have another interpretation of $H^n(X, S)$:

$$H^n(X, S) = \mathrm{Ext}^n_{Sh^\tau_{\mathbf{Ab}}(\mathcal{C})}(\mathbb{Z}_X, S).$$

In the same way, one can take the right-derived functors of the left-exact functors $F^s : Sh^{\tau'}_{\mathbf{Ab}}(\mathcal{C}') \to Sh^\tau_{\mathbf{Ab}}(\mathcal{C})$ for any continuous functor $F : (\mathcal{C}, \tau) \to (\mathcal{C}'\tau')$. Taking $F = f^{-1}$ for $f : Y \to X$ a continuous map of topological spaces, or the fiber product functor for $f : Y \to X$ a morphism in $\mathbf{Sch}_k$, we have $F^s = f_*$, giving us the right-derived functors $R^n f_*$ of f_*.

3.3.28. *Cofinality.* It is often convenient to consider sheaves on a full subcategory $\mathcal{C}_0$ of a given small category $\mathcal{C}$; under certain circumstances, the two categories of sheaves are equivalent.

Definition 3.3.29. Let $i : \mathcal{C}_0 \to \mathcal{C}$ be a full subcategory of a small category $\mathcal{C}$, τ a pre-topology on $\mathcal{C}$. If

(1) each $X \in \mathcal{C}$ admits a covering family $\{f_\alpha : U_\alpha \to X\}$ with the U_α in $\mathcal{C}_0$,
(2) if Y and Z are in $\mathcal{C}_0$, X is in $\mathcal{C}$ and $Y \to X$, $Z \to X$ are morphisms such that $Y \times_X Z$ exists in $\mathcal{C}$, then $Y \times_X Z$ is in $\mathcal{C}_0$,

we say that $\mathcal{C}_0$ is a *cofinal* subcategory of $\mathcal{C}$, for the pre-topology τ.

If $i : \mathcal{C}_0 \to \mathcal{C}$ is cofinal for τ, we define the *induced* pre-topology τ_0 on $\mathcal{C}_0$, where a covering family $\{f_\alpha : U_\alpha \to X\}$, $X \in \mathcal{C}_0$, is an element of $\mathrm{Cov}_\tau(X)$ with the U_α in $\mathcal{C}_0$. Then $i : (\mathcal{C}_0, \tau_0) \to (\mathcal{C}, \tau)$ is continuous.

Proposition 3.3.30. *Let $i : (\mathcal{C}_0, \tau_0) \to (\mathcal{C}, \tau)$ be a cofinal subcategory of $(\mathcal{C}, \tau)$ with induced topology τ_0. Then*

$$i^s : Sh^\tau_{\mathcal{A}}(\mathcal{C}) \to Sh^{\tau_0}_{\mathcal{A}}(\mathcal{C}_0)$$

is an equivalence of categories (for $\mathcal{A} = \mathbf{Sets}, \mathbf{Ab}$).

Proof. Let S be a sheaf on $\mathcal{C}$. The adjoint property of i_s and i^s yields the canonical morphism $\theta_S : i_s i^s S \to S$, with $i^s\theta_S : i^s i_s i^s S \to i^s S$ an isomorphism. In addition, for $X \in \mathcal{C}$, we can take a covering family $\{f_\alpha : U_\alpha \to X\}$ with the U_α in $\mathcal{C}_0$. Since $U_\alpha \times_X U_\beta$ is also in $\mathcal{C}_0$, the exactness of

$$S(X) \longrightarrow \prod_\alpha S(U_\alpha) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \prod_{\alpha,\beta} S(U_\alpha \times_X U_\beta)$$

implies that i^s is a faithful embedding and θ_S is an isomorphism. By the adjoint property again, i^s is fully faithful. As $i^s i_s$ is naturally isomorphic to the identity, the proof is complete. ☐

Example 3.3.31. The category of affines k-schemes of finite type is cofinal in $\mathbf{Sch}_k$ for the topologies Zar, ét and Nis considered in Example 3.3.2.

3.3.32. *Sub-canonical topologies.* Let $\mathcal{C}$ be a small category. The representable functors $\mathrm{Hom}_\mathcal{C}(-, X) = \Xi_X \in PreSh_{\mathbf{Sets}}(\mathcal{C})$ give a ready supply of presheaves of sets. For many interesting topologies these presheaves are already sheaves; such topologies are called *sub-canonical*. For example, all the topologies of Example 3.3.2 are sub-canonical.

The *canonical* topology on $\mathcal{C}$ is the finest sub-canonical topology. One can define the canonical topology explicitly as follows:

A set of morphisms in $\mathcal{C}$, $\mathcal{U} := \{f_\alpha : U_\alpha \to X\}$, is an *effective epimorphism* if for each Y in $\mathcal{C}$, the sequence of Hom-sets

$$\mathrm{Hom}_\mathcal{C}(X, Y) \xrightarrow{\prod f_\alpha^*} \prod_\alpha \mathrm{Hom}_\mathcal{C}(U_\alpha, Y) \underset{\prod p_{2,\alpha,\beta}^*}{\overset{\prod p_{1,\alpha,\beta}^*}{\rightrightarrows}} \prod_{\alpha.\beta} \mathrm{Hom}_\mathcal{C}(U_\alpha \times_X U_\beta, Y)$$

is exact, i.e., if for each $Y \in \mathcal{C}$, the representable presheaf Ξ_Y on $\mathcal{C}$ satisfies the sheaf axiom for the set of morphisms $\mathcal{U}$. $\mathcal{U}$ is called a *universal* effective epimorphism if for each morphism $Z \to X$ in $\mathcal{C}$, the fiber products $U_\alpha \times_X Z$ exist and the family $\{U_\alpha \times_X Z \to Z\}$ is an effective epimorphism. Clearly the universal effective epimorphisms define a pre-topology on $\mathcal{C}$ which agrees with the canonical pre-topology.

If τ is a sub-canonical topology, then the representable presheaves Ξ_Y are already sheaves, hence $\chi_Y := a_\tau(\Xi_Y) = \Xi_Y$. We thus have the functor

$$\begin{aligned} \chi : \mathcal{C} &\to Sh^\tau_{\mathbf{Sets}}(\mathcal{C}) \\ Y &\mapsto \mathrm{Hom}_\mathcal{C}(-, Y) = \chi_Y \end{aligned}$$

which by the Yoneda lemma is a fully faithful embedding.

In addition, each sheaf S is a quotient of a coproduct of such sheaves. Indeed, let S be a sheaf, and take $X \in \mathcal{C}$ and $x \in S(X)$. Since $S(X) = \mathrm{Hom}_{Sh^\tau_{\mathbf{Sets}}(\mathcal{C})}(\chi_X, S)$, x uniquely corresponds to a morphism $\phi_x : \chi_X \to S$. Taking the coproduct over all pairs (x, X), $x \in S(X)$, we have the morphism

$$L_0(S) := \coprod_{\substack{(x,X)\\ x\in S(X)}} \chi_X \xrightarrow{\phi:=\coprod \phi_x} S$$

which is easily seen to be an epimorphism (even of presheaves). If we let $\mathcal{R}$ be the presheaf with

$$\mathcal{R}(Y) = \{(y, y') \in L_0(S)(Y) \mid \phi(Y)(y) = \phi(Y)(y')\}$$

then $\mathcal{R}$ is also a sheaf; composing the projection $p_i : \mathcal{R} \to L_0(S)$ with the epimorphism $L_0(\mathcal{R}) \to \mathcal{R}$ yields the presentation of S as

$$L_1(S) := L_0(\mathcal{R}) \underset{\phi_2}{\overset{\phi_1}{\rightrightarrows}} L_0(S) \xrightarrow{\phi} S$$

Thus we have presented S as an inductive limit of the representable sheaves χ_X, $X \in \mathcal{C}$.

From this point of view, the sheaf category $Sh^\tau_{\mathbf{Sets}}(\mathcal{C})$ can be viewed as an extension of the original category $\mathcal{C}$, in which arbitrary (small) projective and inductive limits exist, containing $\mathcal{C}$ as a full subcategory, and with the closure of $\mathcal{C}$ under inductive limits being the entire category $Sh^\tau_{\mathbf{Sets}}(\mathcal{C})$.

As these properties are valid for *all* sub-canonical topologies on $\mathcal{C}$, one is entitled to ask why one particular sub-canonical topology is chosen over another. The answer here seems to be more art than science, depending strongly on the properties one wishes to control. For the category of motives, the Nisnevic topology has played a central role, but in questions of arithmetic, such as values of L-functions, the étale topology seems to be a more natural choice, and for issues involving purely geometric properties of schemes, the Zariski topology is often more applicable. In fact, some of the most fundamental questions in the theory of motives and its relation to algebraic geometry and arithmetic can be phrased in terms of the behavior of sheaves under a change of topology.

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Department of Mathematics, Northeastern University, Boston, MA 02115, USA

E-mail address: `marc@neu.edu`