

Research Seminar WS 26/27: Perverse Sheaves on Abelian Varieties

Introduction

Abelian varieties

An abelian variety over $\mathbb{C}$ is a complete connected algebraic group. Completeness implies that the group law is commutative and that the variety is projective. Analytically, a complex abelian variety of dimension g is a complex torus $X(\mathbb{C}) \simeq V/\Lambda$ satisfying a projectivity condition. In particular, $\pi_1(X, 0) \simeq \Lambda \simeq \mathbb{Z}^{2g}$, and every character $\rho: \pi_1(X, 0) \rightarrow \mathbb{C}^\times$ defines a rank-one local system L_ρ on X .

The group law $a: X \times X \rightarrow X$ gives rise to the convolution product

$$K * L = Ra_*(K \boxtimes L)$$

on constructible complexes. This additional monoidal structure makes the theory of perverse sheaves on abelian varieties particularly rich and eventually connects it with the representation theory of algebraic groups.

Perverse sheaves

Perverse sheaves are constructible complexes satisfying support and cosupport conditions. They form the heart $\text{Perv}(X, \mathbb{C}) \subset D_c^b(X, \mathbb{C})$ of the middle perverse t -structure and hence constitute an abelian category. Natural examples include shifted local systems on smooth varieties and intersection complexes of singular varieties. Perverse sheaves are stable under Verdier duality and play a central role in the decomposition theorem, making them useful for studying singular spaces and proper morphisms.

Convolution does not preserve all perverse sheaves. Nevertheless, after passing to a suitable quotient by negligible objects, convolution produces Tannakian categories. The corresponding Tannaka groups encode the decomposition of convolution powers and therefore contain geometric information about subvarieties of abelian varieties.

Goal of the seminar

The main goal of the seminar is to study the generic vanishing theorem of Krämer–Weissauer [11, Theorem 1.1]. Let X be a complex abelian variety and $P \in \text{Perv}(X, \mathbb{C})$. Then

$$H^i(X, P \otimes_{\mathbb{C}} L_\rho) = 0 \quad \text{for all } i \neq 0$$

for all characters $\rho: \pi_1(X, 0) \rightarrow \mathbb{C}^\times$ outside a finite union of translates of proper algebraic subtori of the character variety. The proof combines character twists and convolution, an axiomatic Tannakian construction, the theory of perverse multipliers, and the classification of perverse sheaves with Euler characteristic zero. Along the way, the seminar develops the required background on abelian varieties, constructible and perverse sheaves, the Riemann–Hilbert correspondence and Tannakian categories. The final talks discuss consequences of generic vanishing and an application to theta divisors.

Talks

For Talks 3-5 the lecture notes on intersection cohomology of Jochen Heinloth [8] are available upon request.

The *'s after the talk title describe its difficulty, ranging from * (easier) to *** (harder)

Talk 1: Introduction and generic vanishing (**)

Give an overview of the seminar and explain why character twists of perverse sheaves on abelian varieties are natural objects to study. Introduce the character variety $\Pi(X) = \text{Hom}(\pi_1(X, 0), \mathbb{C}^\times)$, associate to $\rho \in \Pi(X)$ the local system L_ρ , and state the main theorem [11, Theorem 1.1]. Two examples should illustrate the statement. For a skyscraper sheaf δ_x , twisted cohomology is concentrated in degree zero for every character, see [11, §4 and the beginning of §11]. If $g = \dim X$ and $P = L_\varphi[g]$, then P is perverse and its twisted cohomology vanishes completely unless $\varphi\rho$ is trivial, see [11, Remark 5.3]. State the constructible-sheaf and relative versions [11, Theorems 1.2 and 2.1], and conclude with a roadmap of the proof in [11, §4–11].

Talk 2: Abelian varieties and character varieties (*)

Introduce abelian varieties over $\mathbb{C}$ and the analytic description $X(\mathbb{C}) \simeq V/\Lambda$, following [13, Chapter I, §1–2]. Explain the resulting description of the fundamental group and character variety. Discuss homomorphisms, isogenies, abelian subvarieties and quotient abelian varieties, focusing on an inclusion $i: A \hookrightarrow X$ with quotient $q: X \rightarrow X/A$. The associated exact sequence

$$0 \longrightarrow \Pi(X/A) \xrightarrow{\Pi(q)} \Pi(X) \xrightarrow{\Pi(i)} \Pi(A) \longrightarrow 0$$

and the subtorus $K(A) = \text{im}(\Pi(q))$ should be explained carefully, since these subtori describe the exceptional sets in generic vanishing, see [11, Remark 11.1]. Briefly introduce the dual abelian variety and $\text{Pic}^0(X)$, using [13, Chapter I, §7–8].

Talk 3: Constructible sheaves and local systems (*)

Character twists are rank-one local systems, whereas perverse sheaves are constructible complexes. Recall Whitney stratifications and explain how they decompose singular varieties into smooth pieces along which the topology is locally constant, see [16, §2]. Define local systems and their description through representations of the fundamental group, in particular the local system L_ρ attached to a character $\rho: \pi_1(X, 0) \rightarrow \mathbb{C}^\times$. Introduce constructible sheaves, using skyscraper sheaves and the extension of a local system from a locally closed stratum as examples, see [16, §5]. Finally, introduce the constructible derived category $D_c^b(X, \mathbb{C})$ and hypercohomology as the setting for the seminar, following [16, §6.1]. Mention briefly that analogous constructions exist for constructible $\mathbb{Q}_\ell$ -complexes in the étale setting.

Talk 4: Verdier duality and t -structures (*)

Motivate Verdier duality as an extension of Poincaré duality to constructible complexes, and introduce the duality functor D_X together with its basic functorial properties, following [16, §6.2–6.3]. Then define a t -structure on a triangulated category, its truncation functors and its heart. Starting from the standard t -structure, explain the idea of gluing t -structures along an open–closed decomposition and use the examples in [16, §7.1–7.4] to motivate the perverse shift. This prepares the support and dual support conditions used to define perverse sheaves, see also [2, §1.3–1.4].

Talk 5: Perverse sheaves and intersection complexes (**)

Define the middle perverse t -structure by its support and cosupport conditions, and explain the equivalent semiperversity formulation used in [11, §1]. Show that $L[d]$ is perverse when L is a local system on a smooth variety of pure dimension d , and discuss perverse sheaves on curves as accessible examples, see [16, §7.5 and 8]. Introduce the intermediate extension $j_{!*}$ and the intersection complex $\mathrm{IC}_Y(L) = j_{!*}(L[\dim Y])$, following [16, §11]. State the classification of simple perverse sheaves as intermediate extensions of irreducible local systems, and conclude with the statement and geometric meaning of the decomposition theorem, see [16, §12] and [2, Théorème 6.2.5].

Talk 6: Riemann–Hilbert correspondence and characteristic cycles (***)

This talk introduces the complex-analytic input used in [11, §10]. Begin with the relation between local systems and vector bundles with integrable connection, and state the Riemann–Hilbert equivalence between regular holonomic $\mathcal{D}_X$ -modules and perverse sheaves, using [9, Chapters 4, 6 and 7]. For an overview you may also consult [5]. Introduce the characteristic cycle

$$CC(P) = \sum_Z n_Z \Lambda_Z$$

only to the extent needed later: its components are conormal varieties and they recover the support of P , cf. [10, §IX] and [11, Proof of Prop. 10.1]. For an abelian variety, explain the Gauss map $\Lambda_Z \rightarrow H^0(X, \Omega_X^1)$ and the formula expressing $\chi(P)$ as a sum of its generic degrees, see [6, Proposition 2.2 and Corollary 1.4] and [11, Lemmas 10.3–10.4]. The purpose is to prepare the key ideas in the classification of perverse sheaves with Euler characteristic zero.

Talk 7: Character twists and convolution (*)

Define convolution by $K * L = Ra_*(K \boxtimes L)$ and explain its rigid symmetric monoidal structure, with unit δ_0 and adjoint dual $K^\vee = (-\mathrm{id}_X)^* D_X K$, following the beginning of [11, §4]. Prove explicitly that $\delta_x^\vee * \delta_x \simeq \delta_0$. For a character ρ , show that $a^* L_\rho \simeq L_\rho \boxtimes L_\rho$ and explain the resulting compatibility $(K * L)_\rho \simeq K_\rho * L_\rho$. The main result is the tensor functoriality of character twists [11, Proposition 4.1], although the longest coherence verifications may be abbreviated. Finally, use Künneth to explain why hypercohomology is a tensor functor to super vector spaces and prove the invariance of the Euler characteristic under character twists [11, Corollary 4.2].

Talk 8: Tannakian categories (**)

Motivate Tannakian formalism through the category $\mathrm{Rep}_\Lambda(G)$ of finite-dimensional representations of an algebraic group. Define rigid symmetric monoidal abelian categories and fibre functors $\omega: \mathcal{T} \rightarrow \mathrm{Vect}_\Lambda$, and state the reconstruction equivalence $\mathcal{T} \simeq \mathrm{Rep}_\Lambda(\mathrm{Aut}^\otimes(\omega))$, following [4, Theorem 2.11]. Explain the tensor subcategory generated by one object and how it corresponds to a faithful representation of an algebraic quotient, see [4, Propositions 2.20–2.21]. The relation between semisimplicity and reductivity [4, Proposition 2.23] should also be discussed. The goal is to make the later definition of the Tannaka group generated by a perverse sheaf transparent.

Talk 9: The axiomatic framework and the André–Kahn quotient (**)

Explain the axioms (D1)–(D3) from [11, §5], emphasizing the roles of perverse truncations, semisimplicity and hard Lefschetz. Define the categorical trace and the André–Kahn radical, and construct the quotient $\overline{D} = D/N_D$ following [11, §6]. The main goal is to prove [11, Proposition 6.2 and Corollaries 6.3–6.4], especially that an object becomes zero in $\overline{D}$ precisely

when all simple constituents of its perverse cohomology have Euler characteristic zero. Conclude by explaining why cohomology takes values in super vector spaces and how Deligne’s criterion yields a reductive super-Tannaka group for every finitely generated tensor subcategory, see [11, Theorem 7.1]. Only the super-Tannakian notions required for this statement should be introduced.

Talk 10: Perverse multipliers ()**

Introduce the negligible category through axioms (N1)–(N4) in [11, §5], and define N -multipliers and zero types following [11, Definition 8.1]. Prove [11, Lemma 8.2], which explains why the multiplier property forces the cohomology of a generic character twist to be concentrated in degree zero. Then explain [11, Lemma 8.3]: if a perverse sheaf is not a multiplier, a convolution power of $P * P^\vee$ contains a shifted Tate object $\mathbf{1}[2i](i)$ with $i \neq 0$. The talk should culminate in the proof of the multiplier theorem [11, Theorem 9.1], where the shifted Tate object produces a quotient torus of the Tannaka supergroup and the central-torus argument from [11, Proposition 15.3] gives a contradiction.

Talk 11: Euler characteristic zero and the proof of generic vanishing ()**

Use the characteristic-cycle description from Talk 6 to explain the key ideas of [11, Proposition 10.1]. First treat a simple abelian variety: the Gauss-map interpretation of the coefficients in the index formula gives [11, Lemma 10.3], and $\chi(P) = 0$ forces the characteristic cycle to be supported on the zero section. The argument of [11, Lemma 10.4] then identifies P with a shifted rank-one local system. Explain how the general classification leads to

$$P \simeq L_\varphi \otimes q^*Q[\dim A]$$

for a quotient by a positive-dimensional abelian subvariety. Prove [11, Corollary 10.2] by computing $Rq_*(L_{\varphi\rho})$, verify axiom (N2), and combine this with [11, Theorem 9.1 and Lemma 8.2] to complete the proof of Theorem 1.1.

Talk 12: The exceptional spectrum ()**

Define the spectrum

$$S(P) = \{\rho \in \Pi(X) \mid H^i(X, P \otimes L_\rho) \neq 0 \text{ for some } i \neq 0\}$$

as in [11, §11]. For a simple perverse sheaf of Euler characteristic zero, use Proposition 10.1 to prove $S(P) = \varphi^{-1}K(A)$, where $A = \text{Stab}(P)^0$, see [11, Lemma 11.2(a)]. Then explain how the multiplier theorem and convolution reduce the spectrum of an arbitrary semisimple perverse sheaf to the spectra of negligible constituents. This proves that $S(P)$ is a finite union of translates of subtori $K(A_i)$, see [11, Lemma 11.2(b)]. Finally, discuss the torsion statement for perverse sheaves of geometric origin in [11, Lemma 11.2(c)] and restate the main theorem in its precise form.

Talk 13: Relative and Kodaira–Nakano-type generic vanishing ()**

This talk studies the consequences developed in [11, §2–3]. For a quotient $f: X \rightarrow B = X/A$, prove the relative theorem that $Rf_*(P \otimes L_\rho)$ is perverse for most characters, see [11, Theorem 2.1]. The proof should focus on the perverse-cohomology spectral sequence on a fibre and the resulting support-dimension estimate. Then explain the passage from constructible to coherent generic vanishing for the Albanese morphism $Y \rightarrow \text{Alb}(Y)$. Introduce the invariant $w(Y)$ measuring the failure of the Albanese morphism to be semismall, and prove the Kodaira–Nakano-type

vanishing theorem [11, Theorem 3.1] using unitary local systems, Hodge decomposition and Serre duality. State the more refined bounds of [11, Theorem 3.2] and explain how they recover the theorem of Green–Lazarsfeld [7].

Talk 14: Theta divisors and Tannaka groups (***)

The final talk illustrates how the Tannakian formalism gives geometric invariants. Introduce principally polarized abelian varieties and symmetric theta divisors, using [13, Chapter I, §11] and [12, §1.1]. For an irreducible theta divisor Θ , explain the category generated by its intersection complex δ_Θ and the associated group $G(\delta_\Theta)$, focusing on the properties in [12, §1.2 and Remark 1.1]. State the known groups for Jacobian theta divisors [12, Theorem 1.2] and the generic result [12, Theorem 1.4]. The main proof to be discussed is the upper bound in [12, Lemma 2.1]: for a smooth symmetric theta divisor, compute $\chi(\delta_\Theta) = g!$, use self-duality, and determine from the symmetry constraint whether the corresponding representation is orthogonal or symplectic. The later degeneration arguments of the article are not part of the talk.

References

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