SEMINAR ON ALGEBRAIC SURFACES

DR. A. PIEPER, WS 2026/27

We will study algebraic surfaces, more precisely smooth projective varieties of dimension 2 over an algebraically closed field. While the restriction to surfaces simplifies many things in comparison to higher dimension, the theory is already considerably more vast than for algebraic curves, i.e., 1-dimensional varieties.

Therefore, we will focus primarily on understanding a few interesting classes of algebraic surfaces and (some of) the most beautiful examples.

Mostly we will follow Chapter V of Hartshorne's Algebraic Geometry. In the first few talks we will discuss some general results from algebraic geometry which we will use later on. See the bibliography for further references.

Preliminaries Knowledge of algebraic geometry, for instance in the scope of Prof. Görtz' lecture series AG I (schemes, morphisms of schemes, $\mathbb{P}^n$) and AG II (algebraic curves, cohomology of quasi-coherent sheaves, Riemann–Roch for curves). Moreover, talk 7 uses a theorem which is discussed in AGIII. Therefore, participating in that course could be helpful (but is not mandatory).

Venue: Wed., 4pm–6pm, WSC-S-U-3.01. We will begin with an introductory session on Oct 14th.

If you are interested, feel free to contact `andreas.pieper@uni-due.de` to register.

Program

Note to the speakers: You should plan a blackboard talk not longer than 80min so that there is enough time for questions. In most talks you will not be able to present all details, so you should make sure to plan ahead what exactly you want to present, and what to omit. This is true in particular for talks 1-3 which partly will have a flavor of an overview talk. Usually, it is best to *not omit examples*!

Regardless of this remark, as a first step you should make sure that you yourself do understand all the details (including *exercises* and *details left to the reader*). It is recommended to meet the organizer at least once in order to discuss the preparation of the talk.

1. **Divisors and line bundles, ampleness.** An important tool to study a surface will be looking at the curves contained in it, or more precisely at *divisors* on the surface.

Content of the talk:

- Give a brief "reminder" on *Weil divisors* versus *Cartier divisors* versus *line bundles* on a noetherian locally factorial integral scheme. ([H], II.6, but obviously you will have to shorten this quite a lot; cf. [GW] Ch. 11)
- Ample and very ample line bundles, [H] II.7 up to Lemma 7.8 (cf. [GW] Ch. 13)
- Ampleness and cohomology, [H] III.5, Thm. 5.2 and Prop. 5.3
- Mention also Bertini's Theorem ([H] II.8.18)

2. **The theorem on formal functions.** We will need the theorem on formal functions [H] III.11.1, and its corollaries: (a version of) Zariski's main theorem [H] III.11.4 and Stein factorization [H] III.11.5. (For other variants and other approaches to Zariski's main theorem, see e.g., [GW] Ch. 12 and Mumford's Red Book.)

Content of the talk: A quick "reminder" on higher direct image sheaves ([H] III.8). [H] III.11. Make sure you have enough time left to explain how the corollaries (11.2–5) follow from the theorem; if required, you may be sketchier on the proof of the theorem itself.

3. **Serre duality.** Although duality was already stated in Prof. Görtz' lectures, we want to revisit the theory here.

Content of the talk: [H] III.7 up to 7.12.3. In the seminar, we will need the theory only for smooth varieties, so you may and should make this assumption, either throughout or whenever it simplifies the presentation. In particular, make sure to state the main result in this case (i.e., the combination of loc. cit., Cor. 7.7 and Cor. 7.12).

Further References: Compare [L] 6.4 and the references given there.

4. **The intersection pairing on an algebraic surface.** Given two curves on a surface, typically the intersection will consist of finitely many points, so it is a natural idea to think about the "intersection number" of two curves, or more generally of two divisors on a surface. Obviously, to get a reasonable notion, one has to be a little careful (e.g., count points with appropriate multiplicity). In this talk we will define the intersection pairing on the divisor class group of a surface.

Content of the talk: [H] V.1 until (and including) Prop. 1.4.

Further References: [Ba] Ch. 1.

5. **The theorem of Riemann–Roch for surfaces.** Similarly as in the theory of curves, there is a Riemann–Roch theorem for surfaces (albeit with a slightly more complicated statement), and it is of similar importance for the development of the theory.

Content of the talk: Then prove the Riemann–Roch theorem for surfaces [H] V.1, 1.4.1 until (and including) Cor. 1.8.

6. **The Hodge index theorem and the Nakai–Moishezon criterion.** In this talk, we look at two important applications of the theorem of Riemann–Roch.

Content of the talk: [H] V.1, Definition after Cor. 1.8 until end of chapter.

7. **Ruled surfaces I.**

Er redet nur von Regelflächen,
ich werd' mich an dem Flegel rächen!

(He always talks about ruled surfaces
I'm going to take revenge on that rascal!)

Dedicated to the memory of Karl Stein (1913-2000) [FP].

We now start the study of the class of *ruled surfaces*, i.e., of surfaces X with a surjective morphism $\pi\colon X \to C$ to a smooth projective curve, such that all fibers are projective lines.

Content of the talk: Explain what we need about projective bundles $\mathbb{P}(\mathscr{E})$, see [H] II.7, including the relevant exercises. Also say something about loc.cit., Ex. V.1.7. Then cover [H] V.2 until Remark 2.7.1.

Remark: This talk uses a result discussed in AGIII, namely Grauert's theorem on cohomology and base change.

8. **Ruled surfaces II.** We continue the study of ruled surfaces, in particular the relation with locally free sheaves of rank 2 on the base curve.

Content of the talk: [H] V.2, Prop. 2.8 until (and including) Cor. 2.14.

9. **Ruled surfaces III.** We continue the study of ruled surfaces and use our results to prove a theorem of Atiyah on vector bundles of rank 2 on an elliptic curve.

Content of the talk: Recall some results on elliptic curves, in particular the classification of line bundles (see [H] IV.1.3.7, IV.4). [H] V.2, Theorem 2.15 until (and including) Remark 2.16.1.

10. **Ruled surfaces IV.** To conclude our study of ruled surfaces, we will determine all ample divisors on rational ruled surfaces, and also consider this question in the general case.

Content of the talk: [H] V.2, 2.17 until the end of the chapter. You may be more sketchy on the final part (Prop. 2.20 ...), or omit it, if required.

11. **Monoidal transformations.** A *monoidal transformation* of a surface is simply the blow-up of a single closed point. This is an important tool to produce new surfaces from a given one. We need to understand its effect on the divisors of the surface.

Content of the talk: Start with a short reminder on blow-ups ([H] I.4, II.7; [GW] Ch. 13). [H] V.3 until (and including) Prop. 3.8. If there is time, you could say a few words (and draw a picture or two) about the final part of V.3.

12. **Castelnuovo's contractibility criterion.** When we blow-up a surface X at a closed point x, we get a map $\pi : \tilde{X} \to X$ so that $E := \pi^{-1}(x)$ is a curve, called the *exceptional curve*. The *Castelnuovo contractibility criterion* is concerned with the converse question: Given a surface $\tilde{X}$ and a curve $E \subset \tilde{X}$, under which condition is E the exceptional curve of some monoidal transformation?

Content of the talk: Prove [H] Theorem 5.7 (alternatively see [Be] Théorème II.17) and illustrate it with some example(s), for instance loc. cit. 5.7.1.

13. **Cubic Surfaces I.** Now we will consider *cubic surfaces*, i.e., surfaces defined as the zero set in $\mathbb{P}^3_k$ of a homogeneous polynomial of degree 3. We start by studying curves in the projective plane containing a prescribed set of points.

Content of the talk: [H] V.4 until (and including) 4.5.1.

14. **Cubic Surfaces II: The 27 Lines.** With the results of the previous talk at our disposal, we can analyze what happens if we blow up the projective plane in a number of points. For us, the most important case will be the blow up of $\mathbb{P}^2_k$ in 6 points, no three of which are collinear, and not all six lying on a conic. In this case we obtain a cubic surface. On the other hand, it can be shown that every cubic surface arises in this way, up to isomorphism. We will show that every such surface contains exactly 27 lines, a famous classical result in algebraic geometry.

Content of the talk: [H] V.4, 4.6 until the end of the chapter.

Further References: For an approach to "the 27 lines" on a cubic surface not using cohomology, see [GW] Ch. 16.

Literatur

[Ba] L. Badescu, *Algebraic Surfaces*, Springer Universitext.
[Be] A. Beauville, *Surfaces Algébriques Complexes*, Astérisque 54.
[FP] G. Fischer, J. Piontkowski, *Ruled Varieties*, Vieweg+Teubner Verlag.
[GW] U. Görtz, T. Wedhorn, *Algebraic Geometry I*, Vieweg.
[H] R. Hartshorne, *Algebraic Geometry*, Springer GTM 52.
[L] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford Graduate Texts in Mathematics 6.
[R] M. Reid, *Chapters on Algebraic Surfaces*, in: Complex algebraic geometry, IAS/Park City Mathematics Series; `arxiv:alg-geom/9602006`