Quot Schemes, Picard Schemes and Néron Models

This seminar is separated into two parts.

The first part would serve as an introduction to the construction of Quot schemes and Picard schemes.

Quot schemes arise as moduli spaces that parametrize quotients of sheaves with fixed Hilbert polynomial. These arise naturally as generalizations of projective spaces, though their construction is already non-trivial, requiring the notions of flattening stratification and Castelnuovo-Mumford regularity.

Picard schemes parametrize line bundles (up to appropriate equivalence) on a scheme; the proof uses as input the representability of the Quot scheme. Picard schemes arise naturally in various topics, such as the construction of the Jacobian variety, Brill-Noether theory, deformation theory or the classification of surfaces. For instance, the representability of Picard schemes is crucial to the proof that smooth minimal surfaces with non-nef canonical divisor admit a ruling (except $\mathbb{P}^2$).

In the second part, we would study the construction of Néron models. Let R be a DVR with fraction field K and let G be a commutative group scheme over K. A Néron model for G is a group scheme $\mathcal{G}/\mathrm{Spec}(R)$ with the Néron mapping property: for every smooth R-scheme X with a map $X_K \to G$, there is an extension to a map $X \to \mathcal{G}$.

Néron models have applications in arithmetic geometry, such as the proof of the Néron-Ogg-Shafarevich theorem or Tate's algorithm for elliptic curves.

In good situations, the Jacobian of a family of curves can be compared with its Néron model. These results in turn are useful for constructing compactifications of the moduli of curves. Time permitting we would study this at the end of the seminar.

Here is a rough outline of the talks. The main references are Bosch's book on Néron Models and a set of lecture notes on Picard Schemes which I can share.

Talk 1: Representability of Hom Sheaves, definition of Quot schemes with fixed Hilbert polynomial (chapters 3–4 in lecture notes).

Talk 2: CM regularity, flattening stratification. Hopefully skip some proofs in both chapters and explain CM regularity in more detail (chapters 5–6, skip chapter 7 in lecture notes).

Talk 3: Introduce Picard schemes, give the examples where Pic is not a sheaf, not separated, not universally closed. Maybe finish with discussion of relative effective Cartier divisors (chapter 8, beginning of 9 in lecture notes).

Talk 4: Picard representability I: reduction steps and setup (chapter 9 in lecture notes).

Talk 5: Picard representability II: core argument (chapters 10–11 in lecture notes).

Talk 6: Jacobians of curves.

Talk 7: Weak Néron models (maybe blackbox smoothening, chapter 3 in Bosch).

Talk 8: Birational group laws — Weil theorem (chapter 4 in Bosch).

Talk 9: Construction of Néron models (chapter 5 in Bosch).

Talk 10: Comparison of Jacobians with Néron models and moduli compactification (time permitting).