Babyseminar Proposal: Shimura Varieties

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1 Introduction

Shimura varieties lie at the intersection of algebraic geometry, Hodge theory, Lie theory, and arithmetic geometry. In this seminar, we will build toward them as parameter spaces for Hodge structures with prescribed symmetries and Hodge tensors. Along the way, we will develop the necessary language of reductive algebraic groups, period domains, hermitian symmetric domains, and arithmetic quotients. The main goal is to understand the definition of a Shimura datum, the construction of the associated Shimura varieties, and how natural variations of Hodge structures live on them.

2 Preliminary plan

The seminar will be divided into three parts:

I.
- Begin with basic theory of algebraic groups over $\mathbb{Q}$ and $\mathbb{R}$, and their representations. The structural notions needed later include derived and adjoint groups, reductive and semisimple groups, parabolic subgroups, real forms, and Cartan involutions.
- Recall classical Hodge theory for compact Kähler manifolds, then pass to the abstract language of Hodge structures and explain the equivalent description in terms of the Deligne torus $\mathbb{S}$. Next, study polarized variation of Hodge structures, period domains and period maps.
- Define the Mumford–Tate group of a rational Hodge structure, both as the smallest $\mathbb{Q}$-algebraic group containing the image of Hodge homomorphism and as the stabilizer of its Hodge tensors. Discuss generic Mumford–Tate groups in families and Mumford–Tate domains as real group orbits inside period domains.

II.
- Introduce hermitian symmetric domains, then explain their equivalent definitions and classifications. Study the quotients $\Gamma\backslash D$ of hermitian symmetric domains by discrete groups, and then specialize to arithmetic locally symmetric varieties. Explain the statement and consequences of the Baily–Borel theorem. Explain how to classify variations of Hodge structures on arithmetic locally symmetric varieties in terms of certain auxiliary reductive groups.
- Introduce Shimura data (G, X) consisting of a reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class of maps in $\mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ that satisfy some properties. For a compact open subgroup $K \subset G(\mathbb{A}_f)$, define the Shimura variety as the double quotient $\mathrm{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash\big(X \times G(\mathbb{A}_f)/K\big)$, and show its decomposition into arithmetic locally symmetric varieties.

III. In the final part, we have some choices of additional topics:
   - Explain the Baily–Borel Theorem in more details.
   - Study examples, including Siegel modular varieties (arithmetic quotients of type III domains) and orthogonal Shimura varieties of type IV, and explain their respective relations to moduli spaces of principally polarized abelian varieties and polarized K3 surfaces.
   - Other possible topics are CM points, Hodge loci and special subvarieties.

We will mainly use two sets of notes by Milne [Mil05, Mil13], which can provide the general contents of the seminar. Carlson–Müller-Stach–Peters [CMSP17] is a useful source for Hodge structures, variations, and period domains, while Kerr [Ker14] directly develops the Hodge-theoretic perspective on Shimura varieties. The precise balance between algebraic groups, Hodge theory, and examples can be adjusted according to the background and interests of the participants.

References

[CMSP17] James Carlson, Stefan Müller-Stach, and Chris Peters. *Period Mappings and Period Domains*. Second edition, Cambridge Studies in Advanced Mathematics 168. Cambridge University Press, 2017. `https://doi.org/10.1017/9781316995846`.

[Ker14] Matt Kerr. “Shimura Varieties: A Hodge-Theoretic Perspective.” In *Hodge Theory*, edited by Eduardo Cattani, Fouad El Zein, Phillip A. Griffiths, and Lê Dũng Tráng, Mathematical Notes 49, pp. 525–566. Princeton University Press, 2014. `https://www.math.wustl.edu/~matkerr/SV.pdf`.

[Mil05] J. S. Milne. “Introduction to Shimura Varieties.” In *Harmonic Analysis, the Trace Formula, and Shimura Varieties*, Clay Mathematics Proceedings 4, pp. 265–378. American Mathematical Society, 2005. Updated online notes: `https://www.jmilne.org/math/xnotes/svi.pdf`.

[Mil13] J. S. Milne. “Shimura Varieties and Moduli.” In *Handbook of Moduli*, Vol. II, Advanced Lectures in Mathematics 25, pp. 467–548. International Press, 2013. Online notes: `https://www.jmilne.org/math/xnotes/svh.pdf`.