# Hodge theory and mixed Hodge structures

Anna Borri

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## 1 Introduction

Hodge theory establishes a decomposition of the cohomology groups of a smooth projective complex variety (or a compact Kähler manifold) $X$

$$H^k(X) = \bigoplus_{p+q=k} H^{p,q}(X), \tag{1.1}$$

where the classes in $H^{p,q}$ are represented by forms of type $(p,q)$, i.e. forms that can locally be written as

$$\sum_{|I|=p,|J|=q} f_{I,J}(z)\, dz_I \wedge d\overline{z_J}.$$

The spaces $H^{p,q}(X)$ have the property that

$$\overline{H^{p,q}(X)} = H^{q,p}(X). \tag{1.2}$$

A decomposition as in (1.1) respecting the property (1.2) is called a *pure Hodge structure of weight $k$* on the cohomology group $H^k(X)$. Giving such a decomposition is equivalent to giving the decreasing filtration

$$F^p H^k := \bigoplus_{r\geq p} H^{r,k-r}.$$

The conditions (1.1) and (1.2) become

$$F^p H^k \cap \overline{F^{k-p+1}H^k} = \{0\}, \quad H^k = F^p H^k \oplus \overline{F^{k-p+1}H^k}.$$

The datum of the filtration is equivalent to the Hodge decomposition, because the spaces $H^{p,q}$ can be recovered from the filtration via

$$H^{p,q} = F^p H^k \cap \overline{F^q H^k}.$$

The Hodge structure gives numerical bounds on the Betti numbers of smooth projective varieties and may help to distinguish complex manifolds whose underlying topological spaces are homeomorphic.

A natural question is whether varieties which are not projective or smooth also admit a similar decomposition. Deligne showed in [Del71; Del74] that complex varieties admit a generalized notion, called a *Mixed Hodge Structure* (MHS). A mixed Hodge structure consists of

(a) a finite $\mathbb{Z}$-module (or $\mathbb{Q}$-module) $H_{\mathbb{Z}}$;

(b) a finite increasing filtration $W$ defined at the rational level (i.e. on $H_{\mathbb{Q}}$), called the *weight filtration*;

(c) a finite decreasing filtration $F$ defined at the complex level (i.e. on $H_{\mathbb{C}}$), called the *Hodge filtration*.

Moreover, it is required that for all $k$ the Hodge filtration $F$ induces a pure Hodge structure of weight $k$ on the $k$-th graded piece induced by the weight filtration $W$.

Then the main results of [Del71; Del74] can be phrased as

**Theorem.** *The cohomology groups of complex varieties admit natural mixed Hodge structures, functorial with respect to algebraic morphisms.*

The main purpose of the seminar is to introduce the machinery involved in the proof of this theorem and then to understand Deligne's argument.

# 2 Program of the seminar

The core of the seminar will be Deligne's construction of mixed Hodge structures. If enough people take part in the seminar, at the end we may have 2 or 3 extra talks on additional topics.

## 2.1 Mixed Hodge structures

The content is based on Deligne's classical papers [Del71; Del74]. An (unofficial) english translation can be found at `https://translations.thosgood.net/hodge-theory/`.

During the seminar we will mainly follow the approach illustrated in [ET13]. Beyond Deligne's original papers, a thorough reference you can use to fill in the details missing from [ET13] or to find additional topics is [PS08].

**Notation.** To keep referencing lighter, in the rest of the program I have written directly in black the references coming from [ET13] and in cyan the references coming from [PS08]. Any other reference is explicitly cited.

### 2.1.1 Talk 0 (15/04/26): Introduction

In this talk I will introduce the topic of the seminar and present the program.

### 2.1.2 Talk 1 (22/04/26): Classical Hodge theory

This is going to be mainly a survey to recall the results from classical Hodge theory for smooth projective complex varieties (or more generally for compact Kähler manifolds). The proofs involve quite a lot of differential machinery and terminology, which will not be needed in the rest of the seminar (in particular the theory of harmonic forms). For this reason, feel free to skip any of this or include as little as you want. The main things to include are:

- Hodge decomposition theorem (Theorem 1.1). Explain what the spaces $H^{p,q}$ are and what it means to act by conjugation (see §1.2, i.e. 1.2.2, 1.2.3, 1.12, 1.13 or 1.8, §1.1.3)
- Comparison between De Rham and Dolbeault cohomology(1.2.5, 1.16, 1.17, 1.5.2). Holomorphic Poincaré lemma (1.2.6).
- The category of Hodge structures: properties and Hodge filtration (§2.2, 2.8, 2.11, §2.3, 2.12, 2.13. See also 3.10-11-12-13). Tate twist (2.14, 2.15).
- If you have time, explain the numerical bounds that follow from the Hodge decomposition and include some examples and counterexamples to these numerical bounds. For instance, $S^1 \times S^3$ doesn't admit the structure of a complex Kähler manifold (2.26).

### 2.1.3 Talk 2 (29/04/26): The category of mixed Hodge structures

Explain why in the noncompact setting we expect to see different weights appearing in a cohomology group. The key points are

- the Gysin map preserves HS up to Tate twist (2.15, 1.19);

- the weights appearing in the Gysin sequence obtained by a closed immersion $i\colon Z \hookrightarrow X$ don't match.

Then, the main purpose of this talk is to introduce the general definition of $A$-Mixed Hodge Structure for $A = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. The main results are

- the category of mixed Hodge structures is abelian;
- morphisms of MHS are strict with respect to both filtrations.

This is the content of §3 (exclude: 3.2.2, 3.4) or §3.1.

#### 2.1.4 Talk 3 (06/05/26): Filtered derived category and spectral sequences

In this talk we will introduce the technical notions needed to study mixed Hodge structures on smooth varieties. This corresponds to §4 and the first part of §5. Recall that a filtered complex in an abelian category defines a spectral sequence (this is introduced in 4.0.2 and explained in detail in 4.3). In the case we will need, the spectral sequence degenerates on the first page, so the general formulas explained in 4.3 are not really needed. Skip the construction of the derived category in the classical setting, and just explain what happens in the filtered setting (from 4.1.7. to 4.14). Recall the hypercohomology spectral sequence in the filtered case, with focus on the *canonical* and on the *trivial* (or *stupid*) filtration. (4.3.2, 4.3.3).

Finally, explain Deligne's theorem about the comparison of the spectral sequence in the bifiltered case (the "two filtrations lemma"). This is 5.12 to 5.16, or §3.2.

All of the content explained is contained (without the language of derived categories) in [Del71, §1].

#### 2.1.5 Talk 4 (13/05/26): The mixed Hodge complex

This talk will include the content of §5 not covered in the previous one. Start by explaining the notions of Hodge Complex (HC) and Cohomologycal Hodge complex (CHC). Explain how the Hodge decomposition of the cohomology of a smooth compact variety can be expressed in terms of a CHC. Prove the decomposition by reducing to the projective case. (5.1-5.5, §5.1). This is the kind of argument that we want to generalize to the non compact setting. Give the mixed equivalent of the previous notions: MHC and CMHC. The main result of this talk is Theorem 5.9:

**Theorem.** *The cohomology of a Mixed Hodge Complex carries a Mixed Hodge Structure.*

Go through the proof of this theorem (§5.2), using the results about the comparison of the filtrations from the previous talk. The discussion of MHC is the content of §3.3, which may have a more clean approach to the proof. In Deligne's papers, this is [Del74, §8].

#### 2.1.6 Talk 5 (20/05/26): Mixed Hodge structure on the cohomology of a smooth variety

We finally come to one of the main results of the seminar. Let $U$ be a smooth variety and $X$ a compactification with $U = X - D$, where $D$ is a simple normal crossing divisor. The key idea is that the cohomology of $U$ coincides with the hypercohomology of the complex

$$\Omega^\bullet_X(\log D)$$

of differential forms on $X$ with logarithmic poles along $D$. Then, one needs to show that this is a CMHC to get the desired mixed Hodge structure. The theorem is explained in detail in §4.1, 4.2, 4.3, or more briefly in 6.1.1, 6.1.2, 6.1.3, 6.1.4. If there is time, the talk may be concluded with one or more examples, see 4.19, 6.8, 6.9.

#### 2.1.7 Talk 6 (27/05/26): Hypercoverings and cohomological descent

Show that the mixed Hodge structure found in the previous talk doesn't depend on the compactification and is functorial (4.12, 4.18 or 6.1.5). Explain the numerical bounds on Hodge numbers (4.20, 4.22 or 6.7).

Start explaining the idea behind how to get a MHS in the general case. The key point is to give a "simplicial approximation" of the variety, that in each degree is the complement of a SNCD in a proper variety and then to use the *cohomological descent* property to descend the Hodge structure. This is Theorem 7.7. Explain the terms that appear in this statement, starting from the description of sheaves and derived category on a simplicial space. Explain the cohomological descent property (7.1.2 to 7.1.7). The proof of Theorem 7.7. is left to the following talk.

For more about hypercoverings or cohomological descent you can also look at [Con]. In particular, a more general version of the first part of 7.7 is [Con, Theorem 4.7].

#### 2.1.8 Talk 7 (03/06/26): Mixed Hodge structure on general varieties

Start again from the statement of Theorem 7.7. Explain the main steps that appear in the proof, see section 6.2 of [Del74] (you can also look at [Con, §4]). Once we have the hypercovering of $S$ given by Theorem 7.7, the construction of the MHS on proceeds similarly to the smooth case. Using the cohomological descent property, it suffices to find a Mixed Hodge Structure on the cohomology of the simplicial space $U_*$. This amounts to finding a CMHC quasi isomorphic to the constant sheaf. This can be again done by taking at each level the De Rham complex with logarithmic singularities at infinity. Explain in detail this argument, corresponding to 7.19 to 7.14 (this is [Del74, §8], in particular Theorem 8.1.15 and 8.1.19).

#### 2.1.9 Talk 8 (10/06/26): Examples and complementary results

Include some examples of weights computations. A nice reference with some examples is [Dur83], but there are many more and you may choose the approach you prefer. Some possible examples include

- the union of two projective smooth varieties intersecting transversally;
- a variety with an isolated singular point;
- the complement of a smooth divisor, for instance a punctured curve.

Notice that having a functorial Hodge theory that includes open varieties is useful even if one is just interested to the smooth projective case, if the space admits a stratification by subvarieties with a simple enough cohomology. Indeed, the Gysin sequence helps to compare the cohomology of the open and closed strata, building recursively the cohomology of the total space. Explain this concept with one or more examples, for instance looking at the cohomology of projective spaces or Grassmanians.

Conclude with some complementary results regarding MHS (to choose depending on time constraints):

- numerical restrictions on the Hodge numbers (§5.3.3);
- interaction of Hodge structures with Künneth decomposition and cup product (§5.4);
- Hodge structure on relative cohomology and compactly supported cohomology (§5.5).

### 2.2 Additional topics

The seminar may be concluded with two or three talks about additional topics in Hodge theory. These talks are to be considered complementary, i.e. only to be filled when the previous talks are filled, if there are additional volunteers. The speakers should feel free to propose any topic in Hodge Theory that they would like to talk about!

As a temporary tentative topic, I will suggest an introduction to Variations of Hodge Structures (VHS), which is related to how Hodge Structures change in families of complex varieties. However if there is something you would like to talk about, feel free to discuss another topic in Hodge Theory.

There are many references explaining VHS's, one that may be useful for these talks is [Cat+14, §7]

#### 2.2.1 Talk 9 (17/06/26): Variations of Hodge structures I

This could include [Cat+14, §7.1, 7.2].

#### 2.2.2 Talk 10 (24/06/26): Variations of Hodge structures II

This could include [Cat+14, §7.3, 7.4].

#### 2.2.3 Talk 11 (08/07/26): Variations of mixed Hodge structures

This could include [Cat+14, §7.5].

## References


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