Research Seminar (WS 2024-2025)

Compact subvarieties of $\mathcal{A}_g(\mathbb{C})$

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1. Introduction

The aim of the seminar is to discuss the results in [11] by Grushevsky, Mondello, Salvati Manni and Tsimerman, where it is determined the maximal dimension of a compact subvariety of the moduli space of complex principally polarized abelian varieties of dimension g. The seminar is structured as follows.

We will start by presenting the setting and the problem: we introduce the moduli space of principally polarized abelian varieties (Talk 1), we observe it is not compact, but it admits a compactification that can be embedded into projective space using modular forms (Talk 2). In Talk 3 we see an overview of some of the main results on the dimension of a compact subvariety of the moduli space over any field.

After introducing the setting, we prepare the necessary background to approach the proof in [11], which makes use of the theory of Shimura varieties. Talk 4,5,6 are devoted to discussing some Hodge theory and the definition of special (or Shimura) subvariety of the moduli space. After this, we discuss a fundamental ingredient for the proof: the Ax-Schanuel theorem for Shimura varieties. Talk 7 is dedicated to introducing and motivating the statement of this theorem and to mentioning its relation with o-minimality, which is a branch of model theory, on which the proof of Ax-Schanuel and many interesting recent results in Hodge theory rely. In Talk 8 we discuss some consequences of this theorem.

At this point we are ready to address the results in [11]. The proof consists of two parts: first, the determination of the maximal dimension of a Hodge-generic compact subvariety of $\mathcal{A}_g$ (Talk 9) and, secondly, the determination of the maximal dimension of a compact Shimura subvariety $S \subsetneq \mathcal{A}_g$ (Talks 10, 11). Talk 12 is dedicated to discussing some consequences for the indecomposable locus and the locus of Jacobians. Talk 13 is left free for discussions on further topics.

Here are two videos in which two of the autors present the results in the paper:

`Talk Grushevsky`,

`Talk Tsimerman`.

2. Talks

Talk 0 - Introduction - 10/10. Overview of the seminar.

Talk 1 - Abelian varities and their moduli space - 17/10. The goal of this talk is to introduce the moduli space of principally polarized abelian varieties. Start by recalling

the definition of abelian variety, of polarization and principal polarization over any field. Recall the characterization, over the complex numbers, of an abelian variety as a complex torus together with a polarization, you can find this e.g in [1, Section 1.4] or in [13], or [3, Chapter 4]. Hence present the construction of the moduli space of complex principally polarized abelian varieties, say as in [1]. The main goal is to present the content of [1, Section 4.5], see also [3, Chapter 8]. Time permitting, it would be nice to present also (part of) the considerations explained in Sections 4.1, ..., 4.4. State that the moduli space is quasi-projective (this will be the object of the following talk) and that it exists over any field.
See also [25] as a reference for the talk.

Talk 2 - Modular forms and Baily-Borel compactification - 24/10. The goal of this talk is to survey basic facts about the Satake-Baily-Borel compactification of the moduli space, in particular the fact that it can be embedded in projective space using modular forms. You can follow e.g. [24] and give the definition of Siegel Modular form as in Section 3, hence move to Section 5. Define the Hodge bundle $\mathbb{E}_g$ over the moduli space, and present the relation between Siegel modular forms and sections of (powers) of $\det(\mathbb{E}_g)$. You can present with more details the special case of weight $2k$ modular forms in $g = 1$ following [15, Section 4] in the subsection *Modular forms as sections of line bundles*. Hence present the Baily-Borel theorem as in [24, Section 11]: $\det(\mathbb{E}_g)$ is an ample line bundle on the Satake-Baily-Borel compactification. If there is time you can also present the discussion about toroidal compactifications present in the section. In [25] you can find some history of the construction of the compactification, first over $\mathbb{C}$, and then over $\mathbb{Z}$.
Other references are [1], [10] and [14].

Talk 3 - Compact subvarieties of $\mathcal{A}_g \otimes k$ - 31/10. The goal of this talk is to explain the main results about the dimension of a compact subvariety of $\mathcal{A}_g$ over any field. Start explaining the consequences of what we have seen in the previous talk about the Satake compactification for compact subvarieties of the moduli space as in [11, Section 1.1], getting a bound from below on the dimension. Hence discuss how to get a bound on the dimension of a complete subvariety from above, following [23] (in particular, discuss [23, Corollary 2.5]). This result uses that $\det(\mathbb{E}_g)$ is ample as we have seen in the previous talk, together with the properties of the Chern classes of the Hodge bundle presented in [23, Section 2]. Then explain that over $\mathbb{F}_p$ the upper bound is reached (e.g. this is stated in [23, Lemma 2.3]), whereas over $\mathbb{C}$ it was first conjectured by Oort and then proved by Keel and Sadun [12] that is is not the case. For comparison, you can also state the main results that we are going to prove in the seminar [11]. Depending on the taste of the speaker, any of the points below can be explored in more detail.

(1) You can explain more about the results in [23] on the subring of the Chow ring of $\mathcal{A}_g$ generated by the Chern classes of the Hodge bundle, called tautological subring (hence giving more detail on the proof of [23, Corollary 2.5]).
(2) Or you can decide to explain more about the bound reached over $\mathbb{F}_p$ and discuss the results [19], where it is proved that the locus of abelian varieties of p-rank 0 is complete and of maximal dimension.
(3) Or you can decide to give more details of why the bound fails to be reached over the complex numbers from Keel and Sadun [12].
(4) As a final option, you can explain the consequences of the bounds we got for $\mathcal{A}_g$ for the moduli space of curves $\mathcal{M}_g$. See [7], [26]

Talk 4 - Hodge structures and Mumford-Tate groups - 7/11. With this talk, we start the topic of Hodge theory and Shimura varieties. The material for this talk is contained in [17, Sections 1,3,4]: explain the definition of Hodge structure, some properties (in particular, the link between complex abelian varieties and Hodge structures of type (-1,0) (0,-1)), the equivalent definitions of Hodge structure via Hodge decomposition, Hodge filtration and as a representation of the Deligne torus, and then define the Mumford-Tate group associated with a Hodge structure. If there is time, you can present some of the examples in Section 5 on explicit computations of Mumford-Tate groups related to Abelian varieties and/or discuss Section 2 about the Hodge conjecture. Other references are [6], [18], [21], [22].

Talk 5 - Variations of Hodge structures and special subvarieties - 14/11. The goal of this talk is to discuss the behavior in families of the objects introduced in the previous talk. Present the content of [17, Section 6]. Hence, focus on $\mathcal{A}_g$: starting by recalling the equivalence between complex abelian varieties and polarized HS of type (-1,0) (0,-1), describe the Siegel space as period domain, from here, explain also the description of the Siegel space as space of complex structures with certain properties as in [21, Lemma 1.2.7] and describe the tautological VHS on the Siegel space, see [9]. Finally, describe the *special subvarieties* of $\mathcal{A}_g$ as orbits of their generic Mumford-Tate group, as in [9, Section 3]. If there is time, you could also mention the definition of PEL-type Shimura subvariety. Other references are [6], [14], [18], [22].

Talk 6 - Connected Shimura data - 21/11. In this talk we first translate what we have seen in terms of Hodge theory about special subvarieties of $\mathcal{A}_g$ into the more algebraic language of (connected) Shimura data and, secondly, present some more properties of these subvarieties. In other words, the talk focuses on the Mumford-Tate group G, presents some of the properties of this algebraic group, of the group $G_{\mathbb{R}}$ of its real points, and consequences for the associated Shimura subvariety of $\mathcal{A}_g$. The material to be presented is the one in [11, Section 2]. This contains recollections about the basics of algebraic groups, the definition of connected Shimura datum together with main properties (in particular a criterion for compactness), the definition of Shimura sub-datum, of special subvariety, and of weakly special subvariety. It concludes by discussing the structure of symplectic representations and, as a consequence, with a result ([11, Theorem 2.9]) about products of Shimura subvarieties in $\mathcal{A}_g$. Other references are [14] and [16].

Talk 7 - Ax-Schanuel and functional transcendence - 28/11. This talk is aimed to introduce and motivate the statement of the Ax-Schanuel Theorem for Shimura varieties. Start by presenting the material in [2, Sections 1.1, 1.2] about classical transcendence of the exponential function, concluding with Theorem 1.2.11 Corollary 1.2.12, and Corollary 1.2.13 that are the Ax-Schanuel, weak Ax-Schanuel, and Ax-Lindemann-Weierstrass for tori. Hence state the Ax-Schanuel Theorem for period maps as in [2, Section 6.1] and the weak Ax-Schanuel version we will need [11, Theorem 3.1]. When presenting these statements, try to stress the analogy with the already-seen statements for tori. Following [20, Section 4] give the definition of o-minimal structure and the notion of *definability*, hence move to [20, Section 6] and present the content of the subsections *The setting, Definability, Characterising the algebraic part*, that motivate and partially explain the link between o-minimality, $\mathcal{A}_g$ and Ax-Schanuel.

Talk 8 - Some consequences of Ax-Schanuel for Shimura varieties - 5/12. The goal of this talk is to present some interesting consequences of Ax-Schanuel for Shimura varieties. The talk is going to be divided into two parts: the first one regards the content of [11, Section 3.1]. The two results in this section give sufficient conditions for certain intersections with a Shimura subvariety of $\mathcal{A}_g$ to have the expected dimension. They are fundamental ingredients for getting the bound on the maximal dimension of a Hodge-generic compact subvariety in $\mathcal{A}_g$, which is going to be the subject of the following talk. In the remaining time explain more about the relation between o-minimality, Ax-Schanuel type of theorems, and Diophantine geometry, as in [20]. This is a survey about o-minimality and "special point" problems.

Talk 9 - Maximal dimension of a Hodge-generic compact subvariety - 12/12. The goal of this talk is to present the proof of Theorem A in [11]. This gives the maximal dimension of a Hodge-generic compact subvariety of the moduli space and uses the consequences of Ax-Schanuel discussed in the previous talk. The proof is the content of [11, Section 3.2], but you could start by explaining the strategy of the proof as described in [11, Section 1.4]. The videos linked at the end of the introduction could also be useful.

Talk 10 - Compact Shimura subvarieties from decoupled representations - 19/12. The goal of Talks 10 and 11 is to obtain the maximal dimension of a compact Shimura subvariety $S \subsetneq \mathcal{A}_g$. The proof consists of analyzing compact Shimura subvarieties by going through the list of real, simply connected almost-simple groups and their irreducible linear representations that occur as constituents of a complex symplectic representation. In this talk, you should explain the content of [11, Section 4], which deals with a class of complex symplectic representations, called *decoupled*. Note that the section contains a case-by-case analysis, which we don't need to see in detail. You can focus on the statements (especially Proposition 4.14) and on explaining the strategy of the proof.

Talk 11 - Maximal dimension of a compact Shimura subvariety - 09/01. Talk 11 concludes the proof of Theorem B in [11]. The material for this talk is [11, Section 5], where it is shown that compact Shimura subvarieties arising from non-decoupled representations never have higher dimensions than the already considered Shimura subvarieties (arising from decoupled representations), and hence that what explained in the previous talk actually gives the dimension of the largest compact Shimura in $\mathcal{A}_g$. The same warning as in Talk 10 holds: the section contains a case-by-case analysis, which we don't need to see in detail. You can focus on the statements and on explaining the strategy of the proof.

Talk 12 - Indecomposable locus and locus of Jacobians - 16/01. Depending on whether or not this was already treated in Talk 3, you can discuss or simply recall the known bounds on the dimension of a compact subvariety of $\mathcal{M}_g$ over any field. Hence, explain the consequences of the results presented in the seminar for the indecomposable locus and the locus of Jacobians over the complex numbers. This is the content of [11, Section 6], which comes together with some open questions and considerations in their directions. The discussion in [12, Section 1] is also interesting to present. Depending on the taste of the speaker, any of the points below can then be investigated in more detail.

(1) More about the link between compact subvarities and product loci I: present the characterization in [8, Section 2] of the geometry of a maximal compact subvariety of the moduli space of curves of compact type $\mathcal{M}_g^{ct}$ (over any field). In particular, Lemma 2.3, says that a maximal compact in $\mathcal{M}_g^{ct}(k)$ contains points corresponding to chains of g elliptic curves.
(2) More about the link between compact subvarieties and product loci II: discuss the results in [4, 5] and the reason why they are mentioned in [11, Section 6] in relation to our problem. These works study the Chow ring of the moduli space and contain interesting results concerning the classes of product loci.
(3) Discuss what is known about explicit examples of compact subvarieties in $\mathcal{M}_g$. See [7, 26] and references therein.

Talk 13 - Further topics - 23/01. To be discussed. Possible topics include any of the remaining optional topics from Talk 3 or Talk 12. Alternatively, we can see additional aspects of o-minimality, or of Hodge-theory, or about Chow and cohomology rings of moduli spaces. Let me know if you have further ideas!

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