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Variations of Hodge structures and density of the Hodge locus

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Contents

Introduction

The central aim of this thesis is to introduce the notion of variation of Hodge structures and to discuss a particular aspect of the geometry of the Hodge locus for an integral polarized variation of Hodge structures on a smooth quasi-projective variety, that is the question of its density in the complex analytic topology.

The motivation for the concept of Hodge structure comes from the structure of the cohomology of a compact Kähler manifold or, more in particular, of a smooth projective variety X over $\mathbb{C}$. In this case, the singular cohomology $H^k(X^{\mathrm{an}}, \mathbb{Z})$ carries a Hodge structure of weight k, namely we have a decomposition

$$H^k(X^{\mathrm{an}}, \mathbb{Z}) \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X^{\mathrm{an}})$$

where $H^{p,q}(X^{\mathrm{an}}) = \overline{H^{q,p}(X^{\mathrm{an}})}$ and $H^{p,q}(X^{\mathrm{an}}) \cong H^q(X^{\mathrm{an}}, \Omega^p_{X^{\mathrm{an}}})$. Furthermore, this structure is polarized by the intersection form given as

$$Q(\alpha, \beta) = \int_{X^{\mathrm{an}}} \omega^{\dim X - k} \wedge \alpha \wedge \beta$$

where ω is an integral Kähler form on X^{an}, $\alpha, \beta \in H^k(X^{\mathrm{an}}, \mathbb{Z})$ and we are using de Rham Theorem to identify $H^k(X^{\mathrm{an}}, \mathbb{Z}) \otimes \mathbb{C}$ with the complexified de Rham cohomology of X^{an}.

Chapter 1 of the thesis is essentially dedicated to a brief review of Hodge theory for compact Kähler manifolds and of the general notion of Hodge structure.

Studying how this Hodge structure varies for a family of projective varieties varying holomorphically over a base S motivates the definition of a variation of Hodge structures, and is our central aim in **Chapter 2**.

Let $f : X \to S$ be a smooth projective morphism of smooth connected algebraic varieties over $\mathbb{C}$, which gives, passing to analytification, a projective holomorphic submersion f^{an} of complex manifolds. Then, thanks to a Theorem of Ehresmann, the complex analytic fibers X_s^{an} of f^{an} are diffeomorphic, so their cohomologies $H^k(X_s^{\mathrm{an}}, \mathbb{Z})$ are isomorphic and glue together into the locally constant sheaf $R^k f_*^{\mathrm{an}} \mathbb{Z}$. Hence, the holomorphic vector bundle $R^k f_*^{\mathrm{an}} \mathbb{Z} \otimes \mathscr{O}_{S^{\mathrm{an}}}$ is naturally endowed with a flat connection ∇, called Gauss-Manin connection. Moreover, the Hodge numbers $h^{p,q}(s) = \dim H^{p,q}(X_s^{\mathrm{an}})$ are constant and the Hodge filtration on the cohomology of each fiber induces a filtration $\mathscr{F}^\bullet$ of $R^k f_*^{\mathrm{an}} \mathbb{Z} \otimes \mathscr{O}_{S^{\mathrm{an}}}$ by holomorphic subbundles satisfying Hodge symmetry and the additional constraint

$$\nabla(\mathscr{F}^p) \subseteq \Omega^1_{S^{\mathrm{an}}} \otimes \mathscr{F}^{p-1},$$

known as Griffiths transversality. This motivates the abstract definition of an integral variation of Hodge structures as a locally constant abelian sheaf $\mathbb{V}$ together with a filtration $\mathscr{F}^\bullet$ of the associated holomorphic vector bundle satisfying Hodge symmetry and Griffiths transversality. Furthermore, having in mind the previous geometric situation, we can put the additional structure of a polarization.

A crucial tool to study a polarized variation of Hodge structures $(\mathbb{V}, \mathscr{F}^\bullet)$ on S is the period map $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$, sending a point s to its corresponding Hodge structure, seen, up to the action of the monodromy group, as a filtration on the vector space $\mathbb{V}_{s_0} \otimes \mathbb{C}$, for a fixed $s_0 \in S$. This tool will also be introduced in Chapter 2.

The Chapter ends with the definition of the Hodge locus of an integral (or rational) variation of Hodge structures on S. In particular, we show that the Mumford-Tate group associated with the Hodge structure on each stalk is locally constant outside a subset of S, called Hodge locus, where it shrinks, as *exceptional* Hodge classes (and tensors) appear.

One of the aim of this thesis is to keep an eye both on the *complex-analytic* and on the *algebraic* point of view. Indeed, if X a smooth projective variety, one can prove, essentially from GAGA correspondence and the holomorphic Poincaré Lemma, that

$$H^k(X^{\mathrm{an}}, \mathbb{C}) \cong \mathbf{H}^k(X, \Omega_X^\bullet)$$

where the right hand side is the hypercohomology, in the Zariski topology, of the algebraic de Rham complex. Moreover, the Hodge filtration on $H^k(X^{\mathrm{an}}, \mathbb{C})$ comes, under this isomorphism, from the naive filtration of the de Rham complex. Similarly, in our *relative* case, with $f : X \to S$ smooth projective morphism of algebraic varieties, a relative version of the holomorphic Poincaré Lemma gives

$$(\mathbf{R}^k f_* \Omega^\bullet_{X/S})^{\mathrm{an}} \cong R^k f_*^{\mathrm{an}} \mathbb{Z} \otimes \mathscr{O}_{S^{\mathrm{an}}}$$

and also the Gauss-Manin connection has a purely algebraic definition. While in the first two Chapters we adopt a more complex-analytic perspective, this algebraic point of view is discussed in two interludes at the end these Chapters.

In **Chapter 3** we finally adress the question of the density of the Hodge locus for an integral polarized variation of Hodge structures on a smooth quasi-projective variety S.

The crucial point of view that inspires this criterion consists in seeing Hodge loci as intersection loci: indeed the period map $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$ sends irreducible components of the Hodge locus to irreducible components of the intersection, inside $\Gamma\backslash D$, of the image of the period map with *special* subvarieties of $\Gamma\backslash D$ which arise as quotients of period sub-domains of D corresponding to Mumford-Tate groups that are smaller than the generic one. This perspective and its implications will be discussed in details in the beginning of Chapter 3.

Then, we focus on the proof of a density criterion, due to Khelifa-Urbanik [33], which takes, under a couple of additional assumptions, the following simple form:

Theorem. *Let $(\mathbb{V}, \mathscr{F}^\bullet)$ be an integral polarized variation of Hodge structures on a smooth connected quasi-projective variety S with generic Hodge datum $(\mathbf{G}, D)$ and period map Φ. Furthermore, assume that the algebraic monodromy group is $\mathbf{H} = \mathbf{G}^{\mathrm{der}}$ and is $\mathbb{Q}$-simple. Then, the Hodge locus is analytically dense in S^{an} as soon as there exists a strict Hodge sub-datum $(\mathbf{M}, D_M)$ of $(\mathbf{G}, D)$ such that*

$$\dim \Phi(S^{\mathrm{an}}) + \dim D_M - \dim D \geq 0.$$

As we will see, the essential ingredient for the proof is the Ax-Schanuel Theorem for variations of Hodge structures, which allows to control atypical intersections of $\Phi(S^{\mathrm{an}})$ with (real) translates of special subvarieties of $\Gamma\backslash D$ passing through a Hodge generic point in terms of the so-called *weakly special* subvarieties of S. A section of the Chapter is dedicated to a discussion about the heuristic behind this Theorem, relying on the fact that the weakly special subvarieties of S are exactly the *bi-algebraic* subvarieties for the *bi-algebraic* structure induced on S be the given variation of Hodge structures.

In the end of Chapter 3, we discuss some applications of the density criterion of Khelifa-Urbanik. In particular, we show how to use it to deduce density results for the Hodge locus of some universal families of projective hypersurfaces and complete intersections and for families of curves in $\mathcal{M}_g$ with non-simple Jacobian.

The thesis ends with two **Appendices**. In the first one, we discuss the relationship between polarized Hodge structures of type $(1,0),(0,1)$ and complex abelian varieties and the construction of their moduli space $\mathcal{A}_g$, while the other one collects some basic facts about reductive groups that are used at some points of the thesis.

Chapter 1

Complex geometry and Hodge theory

In this first Chapter we recollect some basic knowledge of complex geometry, in particular we introduce all terms involved in Hodge decomposition, which is the starting point of Hodge theory, and we fix the notations we will use throughout this work. The reader is supposed to be already familiar with complex geometry, so only few details and few proofs are provided in this Chapter, whose aim is just to fix some ideas before introducing the central topic of the thesis, which will be discussed in Chapter 2. Moreover, we will make use, without giving proofs, of standard tools from homological algebra, in particular cohomology of sheaves. Detailed references are Voisin [46] and Huybrechts [31].

Furthermore, we introduce the general notion of Hodge structure of weight k on an abelian group $H_{\mathbb{Z}}$, pointing out the equivalent definitions as a decomposition of $H_{\mathbb{C}}$, a filtration of $H_{\mathbb{C}}$ or a real algebraic representation of the Deligne torus. This last group-theoretic point of view allows us to associate to a Hodge structure its so-called Mumford-Tate group. Notice that in this work we will *only* consider *pure* Hodge structures of some weight k, therefore we will always omit the adjective "pure".

We will follow here a *complex-analytic* perspective, in particular introducing Hodge decompostion for any compact Kähler manifold, while a more *algebraic* point of view will be discussed in the interlude at the end of the Chapter.

1.1 Hodge decomposition

Let X be a connected compact complex manifold of dimension n. Given a point $x \in X$, the complexified tangent space $T_xX_{\mathbb{C}} = T_xX_{\mathbb{R}} \otimes \mathbb{C}$ carries a complex structure, in particular it admits a decomposition

$$T_xX_{\mathbb{C}} = T_x^{1,0}X \oplus T_x^{0,1}X \tag{1.1}$$

where, once fixed a local holomorphic chart $z_1, \dots, z_n$ around x, a basis of $T_x^{1,0}X$ is given by $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$. This decomposition naturally extends to the complexified tangent bundle $TX_{\mathbb{C}}$, which is a $\mathscr{C}^\infty$ vector bundle on X, and the $(1,0)$-part canonically identifies with the holomorphic tangent bundle $\mathscr{T}_X$.

Moreover, the decomposition (1.1) is inherited by all tensor constructions obtained from $TX_{\mathbb{C}}$, in particular the sheaf $\mathscr{A}_X^k$ of complex $\mathscr{C}^\infty$ differential k-forms on X decomposes as

$$\mathscr{A}_X^k = \bigoplus_{p+q=k} \mathscr{A}_X^{p,q} \tag{1.2}$$

where $\mathscr{A}_X^{p,q}$ is the sheaf of $\mathscr{C}^\infty$ forms of type (p,q), namely the sheaf of $\mathscr{C}^\infty$ sections of the vector bundle $\bigwedge^p(T^{1,0}X)^* \otimes \bigwedge^q(T^{0,1}X)^*$.

Denote by Ω_X^p the sheaf of holomorphic p-forms on X, namely $\Omega_X^p = \ker(\overline{\partial} : \mathscr{A}_X^{p,0} \to \mathscr{A}_X^{p,1})$. Consider the following bicomplex of sheaves on X, known as Dolbeault bicomplex:

$$\begin{array}{ccccccc}
\mathscr{A}_X^{0,0} & \xrightarrow{\partial} & \mathscr{A}_X^{1,0} & \longrightarrow & \dots & \longrightarrow & \mathscr{A}_X^{n,0} \\
\downarrow{\scriptstyle \overline{\partial}} & & \downarrow{\scriptstyle -\overline{\partial}} & & & & \downarrow \\
\mathscr{A}_X^{0,1} & \xrightarrow{\partial} & \mathscr{A}_X^{1,1} & \longrightarrow & \dots & \longrightarrow & \mathscr{A}_X^{n,1} \\
\downarrow & & \downarrow & & & & \downarrow \\
\vdots & & \vdots & & & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow \\
\mathscr{A}_X^{0,n} & \longrightarrow & \mathscr{A}_X^{1,n} & \longrightarrow & \dots & \longrightarrow & \mathscr{A}_X^{n,n}
\end{array}$$

Here, thanks to the $\overline{\partial}$-Poincarè Lemma, the p-th column is an acyclic resolution of Ω_X^p, hence, if we introduce the Dolbeault cohomology

$$H_{\overline{\partial}}^{p,q}(X) = \frac{\ker(\overline{\partial} : \mathscr{A}_X^{p,q}(X) \to \mathscr{A}_X^{p,q+1}(X))}{\operatorname{im}(\overline{\partial} : \mathscr{A}_X^{p,q-1}(X) \to \mathscr{A}_X^{p,q}(X))}$$

we have a canonical isomorphism

$$H_{\overline{\partial}}^{p,q}(X) \cong H^q(X, \Omega_X^p).$$

The total complex of the Dolbeault bicomplex is the (complexified) $\mathscr{C}^\infty$ de Rham complex

$$\mathscr{A}_X^0 \xrightarrow{d} \mathscr{A}_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathscr{A}_X^{2n},$$

which provides an acyclic resolution of the constant sheaf of stalk $\mathbb{C}$, thus inducing a canonical isomorphism

$$H_{\mathrm{dR}}^k(X, \mathbb{C}) \cong H^k(X, \mathbb{C}),$$

where the left hand side denotes the de Rham cohomology of X with complex coefficients.

Furthermore $H^k(X,\mathbb{C})$ (respectively $H^k(X,\mathbb{Z})$) identifies with the singular cohomology of X with complex (respectively integral) coefficients and we have the isomorphism

$$H^k(X, \mathbb{Z}) \otimes \mathbb{C} \cong H^k(X, \mathbb{C}).$$

Remark 1.1.1. The construction of Dolbeault cohomology naturally extends to any rank r holomorphic vector bundle E on X. Let $\mathscr{A}^{p,q}(E)$ be the sheaf of $\mathscr{C}^\infty$ sections of the vector bundle $\bigwedge^p(T^{1,0}X)^* \otimes \bigwedge^q(T^{0,1}X)^* \otimes E$, namely the sheaf of differential forms of type (p,q) with values in E. In a local holomorphic trivialization $(\sigma_1, \dots, \sigma_r)$ of E, such a local section α can be written as $\alpha = \sum_{i=1}^r \alpha_i \otimes \sigma_i$, where each α_i is a local $\mathscr{C}^\infty$ section of $\mathscr{A}_X^{p,q}$. Then we define

$$\overline{\partial}_E(\alpha) = \sum_{i=1}^r \overline{\partial}(\alpha_i) \otimes \sigma_i.$$

Since the transition matrices between trivializations of E on different open subsets have holomorphic entries, this expression gives a well defined $\mathbb{C}$-linear morphism of sheaves $\overline{\partial}_E : \mathscr{A}_X^{p,q}(E) \to \mathscr{A}_X^{p,q+1}(E)$. In this way, we get a complex of sheaves

$$\dots \xrightarrow{\overline{\partial}_E} \mathscr{A}^{p,q+1}(E) \xrightarrow{\overline{\partial}_E} \mathscr{A}^{p,q}(E) \xrightarrow{\overline{\partial}_E} \mathscr{A}^{p,q+1}(E) \xrightarrow{\overline{\partial}_E} \dots$$

Taking global sections and then cohomology we obtain the Dolbeault cohomology of E, denoted $H^{p,q}_{\overline{\partial}}(X, E)$. As before, since this complex is an acyclic resolution of the sheaf $\Omega_X^p(E)$ of holomorphic p-forms with values in E, we obtain the isomorphism

$$H^{p,q}_{\overline{\partial}}(X, E) \cong H^q(X, \Omega_X^p(E)).$$

The question now is whether the cohomology $H^k(X, \mathbb{C})$ inherits the decomposition (1.2). It turns out that the answer is affirmative if X is a compact Kähler manifold, i.e. it carries a hermitian metric whose associated real (1,1)-form is closed.

Theorem 1.1.2. *Let X be a compact Kähler manifold. Then, if we donote by $H^{p,q}(X)$ the space of de Rham cohomology classes in $H^k(X, \mathbb{C})$ that have a representative of type (p, q), we have a decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

Moreover $H^{p,q}(X) = \overline{H^{q,p}(X)}$ and $H^{p,q}(X) \cong H^q(X, \Omega_X^p)$.

Proof. See Voisin [46], Theorem 5.23 and Corollary 6.10. □

As we will see in the next section, this deep result motivates the definition of an integral Hodge structure. However, before this, we will show that in this geometric situation, the Hodge decomposition carries an additional structure, namely a *polarization.* This notion will also be defined in a more general setting in the next section.

Let X be a compact Kähler manifold of dimension n and let ω be a Kähler form on X.

Definition 1.1.3. The Lefschetz operator $L_\omega : \mathscr{A}_X^k \to \mathscr{A}_X^{k+2}$ is defined as $L_\omega(\alpha) = \omega \wedge \alpha$ and induces a well defined map $H^k(X, \mathbb{R}) \to H^{k+2}(X, \mathbb{R})$. For $k \leq n$, we define the primitive cohomology spaces as

$$H^k_{\text{prim}}(X, \mathbb{R}) = \ker(L_\omega^{n-k+1} : H^k(X, \mathbb{R}) \to H^{2n-k+2}(X, \mathbb{R})).$$

An analogous definition can be done for $H^k_{\text{prim}}(X, \mathbb{C})$.

Now, let us define the following intersection form Q on $H^k(X, \mathbb{R})$, for $k \leq n$:

$$Q(\alpha, \beta) = \int_X \omega^{n-k} \wedge \alpha \wedge \beta.$$

Clearly, this form is symmetric if k is even, alternating otherwise and $h_k(\alpha, \beta) = i^k Q(\alpha, \overline{\beta})$ defines a hermitian form on $H^k(X, \mathbb{C})$. This hermitian form interacts with Hodge decomposition in a precise way, as stated in the following:

Theorem 1.1.4. *The hermitian form h_k defined above on $H^k(X, \mathbb{C})$ satisfies the following relations, known as Hodge-Riemann bilinear relations:*

1) the Hodge decomposition $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$ is orthogonal for h_k, *2) the form* $(-1)^{\frac{k(k-1)}{2}} i^{p-q-k} h_k$ *is positive definite on the complex subspace* $H^{p,q}_{\text{prim}}(X) = H^k_{\text{prim}}(X, \mathbb{C}) \cap H^{p,q}(X)$.

Proof. See Voisin [46], Theorem 6.32. □

In the formalism that we will develope in the next section, the previous Theorem translates to the statement that the Hodge structure on the primitive cohomology $H^k_{\text{prim}}(X, \mathbb{R})$ is polarized by the intersection form Q.

Remark 1.1.5. If X is a complex submanifold of the projective space, the restriction of the Fubini-Study metric gives X the structure of a compact Kähler manifold. Furthermore the Kähler class can be chosen to be integral, i.e. to lie in the image of $H^2(X, \mathbb{Z})$ inside $H^2(X, \mathbb{R})$. Hence, the Lefschetz operator acts on the integral cohomology, the primitive cohomology spaces are defined over $\mathbb{Z}$ and the intersection form Q takes integral values on integral classes. Thus, the polarization is defined on the primitive part of the integral Hodge structure on $H^k(X, \mathbb{Z})$.

1.2 Hodge structures

As we have anticipated in the previous section, Hodge decomposition on the cohomology of a compact Kähler manifold motivates the following definition.

Definition 1.2.1. An integral Hodge structure of weight $k \in \mathbb{Z}$ is the datum of a finitely generated abelian group $H_{\mathbb{Z}}$ together with a decomposition

$$H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$$

satisfying $H^{p,q} = \overline{H^{q,p}}$. Starting with a vector space over $\mathbb{Q}$, respectively $\mathbb{R}$, one can define analogously a rational, respectively real, Hodge structure.

We refer to $\{(p,q) : H^{p,q} \neq 0\}$ as the *type* of the Hodge structure.

Remark 1.2.2. A Hodge structure of weight k on $H_{\mathbb{Z}}$ is equivalent to the datum of a decreasing filtration $F^\bullet$ by complex subspaces of $H_{\mathbb{C}}$ satisfying

$$H_{\mathbb{C}} = F^p \oplus \overline{F^{k-p+1}}.$$

Indeed one can define F^p as $\bigoplus_{r \geq p} H^{r,k-r}$ and conversely, given such a filtration, the factors of the Hodge decomposition can be obtained as $H^{p,q} = F^p \cap \overline{F^q}$.

Definition 1.2.3. A morphism $f : (H_{\mathbb{Z}}, H^{p,q}) \to (H'_{\mathbb{Z}}, H'^{p,q})$ of integral Hodge structures of the same weight is a homomorphism of abelian groups $f : H_{\mathbb{Z}} \to H'_{\mathbb{Z}}$ such that the $\mathbb{C}$-linear map $f_{\mathbb{C}} : H_{\mathbb{C}} \to H'_{\mathbb{C}}$ is compatible with the decompositions.

Remark 1.2.4. If $H_{\mathbb{Z}}$ and $H'_{\mathbb{Z}}$ carry Hodge structures of weight k and k', then one can define a weight $k + k'$ Hodge structure on $H_{\mathbb{Z}} \otimes H'_{\mathbb{Z}}$ by $H_{\mathbb{C}} \otimes H'_{\mathbb{C}} = \bigoplus_{a+b=k+k'} T^{a,b}$, where

$$T^{a,b} = \bigoplus_{p+p'=a, q+q'=b} H^{p,q} \otimes H'^{p',q'}.$$

Similarly, we have a weight $k' - k$ Hodge structure on $\mathrm{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}})$ by defining the (a,b)-subspace of $\mathrm{Hom}_{\mathbb{C}}(H_{\mathbb{C}}, H'_{\mathbb{C}}) = \mathrm{Hom}_{\mathbb{Z}}(H_{\mathbb{Z}}, H'_{\mathbb{Z}}) \otimes \mathbb{C}$ as

$$\bigoplus_{p'-p=a, q'-q=b} \mathrm{Hom}_{\mathbb{C}}(H^{p,q}, H'^{p',q'}).$$

Hence, in general, starting with a weight k Hodge structure on $H_{\mathbb{Z}}$ we have Hodge structures on any tensor construction obtained from $H_{\mathbb{Z}}$. More precisely, given a collection $\nu = \{(a_i, b_i)\}_{1\leq i\leq t}$ with a_i, b_i non-negative integers,

$$T^\nu = \bigoplus_{i=1}^{t} H_{\mathbb{Z}}^{\otimes a_i} \otimes (H_{\mathbb{Z}}^*)^{\otimes b_i}$$

is a direct sum of Hodge structures of weight $(a_i - b_i)k$, for $i = 1, \dots, t$.

Example 1.2.5. The Tate Hodge structure $\mathbb{Z}(1)$ is the weight -2 Hodge structure on $2\pi i\mathbb{Z} \subseteq \mathbb{C}$ defined by $H_{\mathbb{C}} = H^{-1,-1}$. The m-tensor product $\mathbb{Z}(1)\otimes\cdots\otimes\mathbb{Z}(1)$ is a Hodge structure of weight $-2m$ on $(2\pi i)^m\mathbb{Z}$, denoted $\mathbb{Z}(m)$.

If $H_{\mathbb{Z}}$ carries a Hodge structure of weight k, its m-twist $H(m) = H_{\mathbb{Z}} \otimes (2\pi i)^m\mathbb{Z}$ is the Hodge structure of weight $k - 2m$ defined by $H(m)^{p,q} = H^{p+m,q+m}$.

We want now to give a more group-theoretic way of defining Hodge structures, pointing out a link with representation theory. This will allow us to associate to any rational Hodge structure its so-called Mumford-Tate group.

Definition 1.2.6. The Deligne torus is the real algebraic group $\mathbf{S}$ defined functorially as follows: for every $\mathbb{R}$-algebra R

$$\mathbf{S}(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R) : a - d = b + c = 0 \right\}.$$

From a more abstract point of view, $\mathbf{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbf{G}_{m,\mathbb{C}}$ is the Weil restriction to $\mathbb{R}$ of the multiplicative group over the complex numbers. The group of real points $\mathbf{S}(\mathbb{R})$ of the Deligne torus is isomorphic to the multiplicative group $\mathbf{G}_{m,\mathbb{C}}(\mathbb{C}) = \mathbb{C}^\times$ via the map $\mathbb{C}^\times \to \mathbf{S}(\mathbb{R})$ which sends $z = u + iv$ to $\begin{pmatrix} u & -v \\ v & u \end{pmatrix}$. This isomorphism gives a structure of a real algebraic group on $\mathbb{C}^\times$. Moreover, $\mathbf{S}(\mathbb{R}) = \mathbb{C}^\times$ is naturally embedded in the group $\mathbf{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$ of complex points through the map sending λ to $(\lambda, \overline{\lambda})$.

Notice that there is a natural embedding $w : \mathbf{G}_m \to \mathbf{S}$ of real algebraic groups, which on complex points is the diagonal embedding and on real points is the embedding $\mathbb{R}^\times \to \mathbb{C}^\times$.

Proposition 1.2.7. *A rational Hodge structure of weight k is equivalent to the datum of a rational vector space $H_{\mathbb{Q}}$ together with a real algebraic representation $\rho : \mathbf{S} \to \mathbf{GL}(H_{\mathbb{R}})$ such that $(\rho\circ w)(t)(v) = t^{-k}v$ for all $t \in \mathbb{R}^\times$, $v \in H_{\mathbb{R}}$, in particular $\rho\circ w : \mathbf{G}_m \to \mathbf{GL}(H_{\mathbb{R}})$ is defined over $\mathbb{Q}$.*

Proof. If $H_{\mathbb{R}}$ is a real vector space carrying a weight k Hodge structure, then we can define a representation ρ of $\mathbf{S}(\mathbb{R})$ on the complex vector space $H_{\mathbb{C}} = H_{\mathbb{R}} \otimes \mathbb{C}$ by

$$\rho(z)(v) = \sum_{p+q=k} z^{-p}\overline{z}^{-q}v^{p,q},$$

where $v = \sum_{p+q=k} v^{p,q}$ is the decomposition of $v \in H_{\mathbb{C}}$ with respect to the Hodge decomposition. Then, by the condition $H^{p,q} = \overline{H^{q,p}}$, we get

$$\overline{\rho(z)}(v) = \overline{\rho(z)(\overline{v})} = \overline{\sum_{p+q=k} z^{-p}\overline{z}^{-q}(\overline{v})^{p,q}} = \sum_{p+q=k} z^{-q}\overline{z}^{-p}v^{q,p} = \rho(z)(v),$$

so ρ actually takes values in $\mathbf{GL}(H_{\mathbb{R}})$. Moreover, one computes $(\rho \circ w)(t)(v) = t^{-k}v$, for all $v \in H_{\mathbb{R}}$, $t \in \mathbb{R}^{\times}$, and if $H_{\mathbb{R}} = H_{\mathbb{Q}} \otimes \mathbb{R}$ for a rational vector space $H_{\mathbb{Q}}$ we have that $\rho \circ w$ is defined over $\mathbb{Q}$.

Conversely, every finite dimensional representation of $\mathbf{S}(\mathbb{R})$ on a complex vector space $H_{\mathbb{C}}$ splits as a direct sum of one dimensional representations where $z \in \mathbf{S}(\mathbb{R})$ acts as multiplication by $z^{-p}\overline{z}^{-q}$, with $p, q \in \mathbb{Z}$. Therefore, such a representation ρ which is defined over $\mathbb{R}$, namely $\rho = \overline{\rho}$, induces a decomposition $H_{\mathbb{C}} = \bigoplus H^{p,q}$, such that $H^{p,q} = \overline{H^{q,p}}$. Furthermore, the condition $(\rho \circ w)(t)(v) = t^{-k}v$ for $v \in H_{\mathbb{R}}$, $t \in \mathbb{R}^{\times}$, implies that in the induced decomposition one has $p + q = k$.

For further details we refer to Peters-Steenbrink [39], Lemma 2.7. □

Finally, we add the additional structure of polarization, having in mind the example of the real Hodge structure on the primitive cohomology $H^k_{\mathrm{prim}}(X, \mathbb{R})$ of a compact Kähler manifold X or, more in particular, of the integral Hodge structure on the primitive integral cohomology of a complex projective manifold.

Definition 1.2.8. An integral Hodge structure of weight k on $H_{\mathbb{Z}}$ is polarized if there exists a non-degenerate bilinear form Q, defined on $H_{\mathbb{Z}}$, symmetric if k is even and alternating otherwise, such that its associated hermitian form on $H_{\mathbb{C}}$, defined by

$$h(v, u) = i^k Q(v, \overline{u}),$$

satisfies the Hodge-Riemann bilinear relations, as in Theorem 1.1.4.

1.3 Mumford-Tate and Hodge groups

Let us work now with rational Hodge structures. In particular, let us fix for the rest of the section a rational vector space V carrying a Hodge structure $V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$, which can also be described, in view of Proposition 1.2.7, by a real algebraic representation $\rho : \mathbf{S} \to \mathbf{GL}(V_{\mathbb{R}})$.

Definition 1.3.1. The Mumford-Tate group associated to the given Hodge structure on V, denoted $\mathbf{MT}(V)$ or $\mathbf{MT}(\rho)$, is the smallest algebraic subgroup $\mathbf{M}$ of $\mathbf{GL}(V)$, defined over $\mathbb{Q}$, such that ρ factors through the inclusion $\mathbf{M}_{\mathbb{R}} \subseteq \mathbf{GL}(V_{\mathbb{R}})$, where the subscript $\mathbb{R}$ denotes, as it is common, the base change to $\mathbb{R}$.

Now, consider the algebraic subgroup $\mathbf{S}^1$ of the Deligne torus whose set of $\mathbb{R}$-valued points is given by

$$\mathbf{S}^1(\mathbb{R}) = \{z \in \mathbb{C} : z\overline{z} = 1\} \subseteq \mathbb{C}^{\times}.$$

Definition 1.3.2. The Hodge group associated to the given Hodge structure on V, denoted $\mathbf{Hg}(V)$ or $\mathbf{Hg}(\rho)$, is the smallest algebraic subgroup $\mathbf{H}$ of $\mathbf{GL}(V)$, defined over $\mathbb{Q}$, such that the restriction of ρ to $\mathbf{S}^1$ factors through $\mathbf{H}_{\mathbb{R}} \subseteq \mathbf{GL}(V_{\mathbb{R}})$.

The essential property of the Mumford-Tate group is that it cuts out exactly the Hodge sub-structures and the Hodge classes in all tensor constructions obtained from V.

Definition 1.3.3. Given a rational Hodge structure of even weight $2k$ on a rational vector space V, a (rational) Hodge class is an element in $V \cap V^{k,k}$, that is a vector $v \in V$ such that its image $v \otimes 1$ in $V_{\mathbb{C}}$ belongs to the complex subspace $V^{k,k}$. Of course the same definition can be given for an integral Hodge structure on a finetely generated abelian group H, giving rise to integral Hodge classes.

Given a rational Hodge structure of any weight on a rational vector space V, a Hodge tensor is a Hodge class in any tensor construction (of even weight) obtained from V. Again, the same definition can be given for an integral Hodge structure.

Proposition 1.3.4. *Let V be a rational vector space carrying a weight k Hodge structure. Let ν be a collection of pairs of non-negative integers and let T^ν be the associated tensor construction, as in remark 1.2.4. If W is a subspace of T^ν, then W is a Hodge sub-structure if and only if it is stable under the action of $\mathbf{MT}(V)$ on T^ν. Moreover, an element $t \in T^\nu$ is a Hodge class if and only if it is invariant under the action of $\mathbf{MT}(V)$.*

Proof. Let $\mathbf{H} \subseteq \mathbf{GL}(V)$ be the stabilizer of W, seen as an algebraic subgroup defined over $\mathbb{Q}$. If W is a Hodge sub-structure of T^ν, then it is stable under the action of $\mathbf{S}$ through ρ, hence ρ factors through $\mathbf{H}_\mathbb{R}$, implying $\mathbf{MT}(V) \subseteq \mathbf{H}$. Conversely, if $\mathbf{MT}(V) \subseteq \mathbf{H}$, then ρ factors through $\mathbf{H}_\mathbb{R}$, so W is a Hodge sub-structure. The second assertion follows clearly from the first. □

Proposition 1.3.5. *Let V be a rational vector space carrying a weight k Hodge structure. Then:*

1) *the Mumford-Tate group $\mathbf{MT}(V)$ is connected,*
2) *if the Hodge structure has weight zero, $\mathbf{MT}(V) \subseteq \mathbf{SL}(V)$; otherwise $\mathbf{G}_m \cdot \mathrm{id} \subseteq \mathbf{MT}(V)$,*
3) *if the Hodge structure is polarized, $\mathbf{MT}(V)$ is reductive.*

Proof. The first two points follow immediately from the definition. To prove point (3) we recall the following criterion for reductive groups: a connected algebraic group over a field of characteristic zero is reductive if it has a faithful semi-simple representation (Proposition B.6). Then the assertion follows from the fact that the inclusion $\mathbf{MT}(V) \to \mathbf{GL}(V)$ is such a representation, since the category of rational polarizable Hodge structures is semi-simple. We refer to Peters-Steenbrink [39] (Corollary 2.12 and Theorem 2.19) for details. □

Interlude: the algebraic point of view I

A characteristic that makes Hodge theory so interesting is that it is at heart *not* an algebraic theory, but rather a transcendental one, however it is supposed to reflect, to some extents, the algebraic structure of projective algebraic varieties. In particular, the deep Hodge conjecture and Grothendieck period conjecture predict that the analytic character of Hodge theory should be constrained. Some of these constraints have been proven in the last decades. It is therefore interesting to analyze which form Hodge theory assumes when it is applied to an algebraic object, namely a projective algebraic variety, rather than to a general Kähler manifold. In this interlude we begin to introduce this perspective by giving an algebraic description of the objects involved in Hodge decomposition.

Let X be a smooth projective variety defined over $\mathbb{C}$ and consider its associated compact complex manifold X^{an}, which is Kähler. Indeed, the set of complex points of the projective space $\mathbb{P}^n$, seen as a scheme, has naturally the structure of a compact complex manifold, so every smooth projective variety over $\mathbb{C}$ is naturally a complex manifold. Furthermore, the restriction of the Fubini-Study metric gives a Kähler form on X^{an}.

Remark 1.1. This construction, called *analytification*, can be done for any algebraic variety X defined over $\mathbb{C}$. Indeed any Zariski locally closed subset of the complex affine space naturally admits the complex analytic topology, restricted from $\mathbb{A}^n_{\mathbb{C}}$, and can be endowed with its sheaf of holomorphic function. Applying this to an open affine cover of X one obtains a complex analytic variety, i.e. a locally ringed space which is locally isomorphic to a subset of $\mathbb{A}^n_{\mathbb{C}}$ cut out by a finite set of holomorphic functions. We refer to Serre [42] for further details.

To fix the notation let $\mathscr{O}_X$ be the algebraic structure sheaf on X, i.e. its sheaf of regular functions, and denote by $\mathscr{O}_{X^{\mathrm{an}}}$ the analytic structure sheaf of X^{an}, i.e. its sheaf of holomorphic functions. Clearly, the usual complex analytic topology on X^{an} is finer than the Zariski topology of X, hence the identity map on points $\mathrm{id} : X^{\mathrm{an}} \to X$ is continuous.

The procedure of analytification extends to any coherent sheaf $\mathscr{F}$ of $\mathscr{O}_X$-modules on X. Indeed one can construct the associated sheaf $\mathscr{F}^{\mathrm{an}}$ of $\mathscr{O}_{X^{\mathrm{an}}}$-modules as the pull-back of $\mathscr{F}$ via $\mathrm{id} : X^{\mathrm{an}} \to X$. It is easy to check that this construction gives an exact functor from the category of coherent algebraic sheaves on X to the category of coherent analytic sheaves on X^{an}. The remarkable results that follow, which constitute the heart of GAGA correspondence, state that if X is a projective algebraic variety, then this functor is actually an equivalence of categories inducing isomorphisms on cohomology.

GAGA Theorems. *Let X be a projective algebraic variety over $\mathbb{C}$. Then:*

1) *For any coherent algebraic sheaf $\mathscr{F}$ on X and for any $q \geq 0$, the analytification functor induces isomorphisms at the level of cohomology: $H^q(X, \mathscr{F}) \cong H^q(X^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}})$.*

2) *If $\mathscr{F}$ and $\mathscr{G}$ are coherent algebraic sheaves on X, then every analytic morphism from $\mathscr{F}^{\mathrm{an}}$ to $\mathscr{G}^{\mathrm{an}}$ comes from a unique algebraic morphism $\mathscr{F} \to \mathscr{G}$.*

3) *For each coherent analytic sheaf $\mathscr{M}$ on X^{an}, there exists, unique up to isomorphism, a coherent algebraic sheaf $\mathscr{F}$ on X such that $\mathscr{F}^{\mathrm{an}} \cong \mathscr{M}$.*

Proof. See Serre [42]. □

A direct application of point (3) of this theorem gives the following:

Chow's Theorem. *Any closed analytic subvariety of the complex projective space is the analytification of a projective algebraic variety.*

Now let us go back to our task of giving an algebraic description of the terms of Hodge decomposition for a projective variety X. Let Ω^1_X be the sheaf of Kähler differentials of X. It is a coherent algebraic sheaf on X and its analytification is the sheaf of holomorphic forms $\Omega^1_{X^{\mathrm{an}}}$ on the associated complex manifold. Of course one can take all wedge powers and construct the algebraic, respectively holomorphic, de Rham complex. By GAGA correspondence

$$H^q(X, \Omega^p_X) \cong H^q(X^{\mathrm{an}}, \Omega^p_{X^{\mathrm{an}}}).$$

Definition 1.2. The algebraic de Rham cohomology of a smooth algebraic variety X is the hypercohomology $\mathbf{H}^k(X, \Omega^\bullet_X)$ of the algebraic de Rham complex in the Zariski topology.

Algebraic de Rham Theorem. *Let X be a smooth projective variety over $\mathbb{C}$. Then there is a canonical isomorphism*

$$H^k(X^{\mathrm{an}}, \mathbb{C}) \cong \mathbf{H}^k(X, \Omega^\bullet_X).$$

Proof. By the holomorphic Poincaré Lemma there is a quasi isomorphism of complexes of sheaves on X^{an}

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\mathbb{C}} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{O}_{X^{\mathrm{an}}} & \longrightarrow & \Omega^1_{X^{\mathrm{an}}} & \longrightarrow & \Omega^2_{X^{\mathrm{an}}} & \longrightarrow & \dots \end{array}$$

inducing an isomorphism $H^k(X^{\mathrm{an}}, \mathbb{C}) \cong \mathbf{H}^k(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}})$. By general homological algebra, one has spectral sequences converging to $\mathbf{H}^k(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}})$, respectively $\mathbf{H}^k(X, \Omega^\bullet_X)$, with $E_1^{p,q} = H^q(X^{\mathrm{an}}, \Omega^p_{X^{\mathrm{an}}})$, respectively $H^q(X, \Omega^p_X)$. These terms are isomorphic by GAGA correspondence, so

$$\mathbf{H}^k(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}}) \cong \mathbf{H}^k(X, \Omega^\bullet_X) \tag{1.3}$$

and we are done. □

Finally, it is a remarkable fact that the Hodge filtration $F^\bullet$ on $H^k(X^{\mathrm{an}}, \mathbb{C})$ induced by Hodge decomposition, stated in Theorem 1.1.2, also admits an algebraic interpretation under the isomorphism $\mathbf{H}^k(X, \Omega^\bullet_X) \cong H^k(X^{\mathrm{an}}, \mathbb{C})$. Recall that the filtration is defined as

$$F^p H^k(X^{\mathrm{an}}, \mathbb{C}) = \bigoplus_{r \geq p} H^{r,k-r}(X^{\mathrm{an}}).$$

For every k let

$$F^p \mathscr{A}^k_{X^{\mathrm{an}}} = \bigoplus_{r \geq p} \mathscr{A}^{r,k-r}_{X^{\mathrm{an}}}.$$

Its space of global section $F^p \mathscr{A}^k_{X^{\mathrm{an}}}(X)$ is the space of complex differential k-forms which are sums of forms of type $(r, k-r)$ with $r \geq p$ at every point. This filtration clearly induces a filtration $F^p \mathscr{A}^\bullet_{X^{\mathrm{an}}}$ of the complexified $\mathscr{C}^\infty$ de Rham complex.

Proposition 1.3. *We have*

$$F^pH^k(X^{\mathrm{an}},\mathbb{C}) = \frac{\ker(d : F^p\mathscr{A}^k_{X^{\mathrm{an}}}(X) \to F^p\mathscr{A}^{k+1}_{X^{\mathrm{an}}}(X))}{\mathrm{im}(d : F^p\mathscr{A}^{k-1}_{X^{\mathrm{an}}}(X) \to F^p\mathscr{A}^k_{X^{\mathrm{an}}}(X))}$$

Proof. See Voisin [46], Proposition 7.5. □

Now, observe that the holomorphic de Rham complex is equipped with the naive filtration $\Omega^{\bullet\geq p}_{X^{\mathrm{an}}}$ and the same holds for the algebraic de Rham complex. The complex $(F^p\mathscr{A}^\bullet_{X^{\mathrm{an}}}, d)$ is acyclic and quasi isomorphic to $\Omega^{\bullet\geq p}_{X^{\mathrm{an}}}$. Indeed $(F^p\mathscr{A}^\bullet_{X^{\mathrm{an}}}, d)$ is the complex associated to the acyclic resolutions $(\mathscr{A}^{q,\bullet}_{X^{\mathrm{an}}}, \overline{\partial})$ of $\Omega^q_{X^{\mathrm{an}}}$, for $q \geq p$, that we recalled in Chapter 1. Combining this with the previous Proposition and with isomorphism (1.3) we obtain that the Hodge filtration is induced by the naive filtration of the algebraic de Rham complex.

Proposition 1.4. *Under the isomorphism $\mathbf{H}^k(X,\Omega^\bullet_X) \cong H^k(X^{\mathrm{an}},\mathbb{C})$ the Hodge filtration identifies as*

$$F^pH^k(X^{\mathrm{an}},\mathbb{C}) = \mathrm{im}(\mathbf{H}^k(X,\Omega^{\bullet\geq p}_X) \to H^k(X^{\mathrm{an}},\mathbb{C})).$$

Let us also emphasize that the Hodge decomposition gives the following purely algebraic statement, formulated in terms of the spectral sequence associated to the naive filtration of the algebraic de Rham complex. Indeed, as we have already recalled in the proof of the algebraic de Rham Theorem, we have a spectral sequence, known as Hodge to de Rham spectral sequence, whose first page reads

$$E_1^{p,q} = H^q(X,\Omega^p_X)$$

with differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ equal to the map $H^q(X,\Omega^p_X) \to H^q(X,\Omega^{p+1}_X)$ induced on cohomology by the de Rham derivation $\Omega^p_X \to \Omega^{p+1}_X$. This spectral sequence converges to the algebraic de Rham cohomology $\mathbf{H}^{p+q}(X,\Omega^\bullet_X)$.

Proposition 1.5. *If X is a smooth projective algebraic variety over $\mathbb{C}$, then the Hodge to de Rham spectral sequence degenerates at E_1.*

Proof. After identifying the algebraic de Rham cohomology with $H^k(X^{\mathrm{an}},\mathbb{C})$ we get, by convergence of the spectral sequence

$$E_r^{p,q} = F^pH^k(X^{\mathrm{an}},\mathbb{C})/F^{p+1}H^k(X^{\mathrm{an}},\mathbb{C})$$

for a sufficiently large r. Recall that $E_{i+1}^{p,q}$ is identified with the cohomology $\ker(d_i)/\mathrm{im}(d_i)$ in bidegree (p,q), where d_i denotes the differential at page E_i. So, $\dim E_{i+1}^{p,q} \leq \dim E_i^{p,q}$ with equality for all p,q if and only if $d_i = 0$. On the other hand, by Hodge decomposition,

$$\sum_{p+q=k} \dim E_1^{p,q} = \sum_{p+q=k} \dim H^q(X,\Omega^p_X) = \dim H^{p+q}(X^{\mathrm{an}},\mathbb{C}) = \sum_{p+q=k} \dim E_r^{p,q}$$

for sufficiently large r, therefore $\dim E_1^{p,q} = \dim E_r^{p,q}$ and $d_i = 0$ for all $i \geq 1$. Hence the spectral sequence degenerates at the first page. □

Remark 1.6. If we consider the filtration F^p on $H^k(X^{\mathrm{an}},\mathbb{C})$ as defined by the image of the naive filtration of the algebraic de Rham complex under the isomorphism given by the algebraic de Rham Theorem we have that, by the previous argument, the degeneration of the Hodge to de Rham spectral sequence is equivalent to the fact that

$$F^pH^k(X^{\mathrm{an}},\mathbb{C})/F^{p+1}H^k(X^{\mathrm{an}},\mathbb{C}) \cong H^q(X,\Omega^p_X)$$

as well as to the equality

$$\dim H^k(X^{\mathrm{an}}, \mathbb{C}) = \sum_{p+q=k} \dim H^q(X, \Omega_X^p).$$

However, it does not imply the symmetry of the Hodge numbers $h^{p,q} = h^{q,p}$, nor the Hodge decomposition at the level of complex vector spaces, namely

$$H^k(X^{\mathrm{an}}, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \quad H^{p,q}(X) = F^p \cap \overline{F^q}.$$

Hence it does not capture the whole phenomenon of Hodge decomposition. It has nevertheless the merit of being an algebraic statement, which can be extended outside complex algebraic geometry. Indeed, the proof that we have presented here relies fully on Theorem 1.1.2, which can be proved only in an analytic way, but there do exist purely algebraic proofs of the degeneration of the Hodge to de Rham spectral sequence, see for instance Deligne-Illusie [20].

Chapter 2

Variations of Hodge structures

In this Chapter we discuss how the Hodge structure on the cohomology groups of a family of compact Kähler manifolds varies over a base manifold S, extending what we have done in the previous Chapter to the *relative* case of a proper holomorphic submersion $f : X \to S$ with fibers being Kähler. In particular we will see that the Hodge filtration on the cohomology of each fiber induces a Hodge filtration of the holomorphic vector bundle $R^k f_* \mathbb{C} \otimes \mathscr{O}_S$, which satisfies a crucial additional condition, called Griffiths transversality. From the description of this geometric situation we will arrive to the abstract definition of a variation of Hodge structures and we will see how it can be described by means of a *period map*, taking values in an appropriate classifying space of Hodge structures. Finally, in the end of the Chapter, we will go back to the group-theoretic point of view that we introduced in Chapter 1 and we will describe, given a rational variation of Hodge structures over a complex manifold S, how the Mumford-Tate group of the induced Hodge structures on the fibers varies over S.

As in the previous Chapter, the discussion is here mainly *complex-analytic*, while an *algebraic* perspective will be provided in the interlude at the end of the Chapter.

The main references for the contents of this Chapter are Voisin [46], Cattani [16], as well as the original papers by Griffiths [25], [26].

2.1 Local systems and connections

Let S be a complex manifold.

Definition 2.1.1. A sheaf $\mathscr{L}$ over S is a local system (of complex vector spaces) if it is locally isomorphic to the constant sheaf with stalk $\mathbb{C}^n$ for some natural number n.

If $\mathscr{L}$ is a local system on S and we fix a base point $s_0 \in S$, then for any path $\gamma : [0,1] \to S$ such that $\gamma(0) = \gamma(1) = s_0$, the pull-back $\gamma^*(\mathscr{L})$ of $\mathscr{L}$ to the unit interval is locally constant, hence constant, since $[0,1]$ is contractible. Thus we obtain an automorphism of the stalk $\mathscr{L}_{s_0}$ as complex vector space, which depends only on the homotopy class of γ. Hence, we get a representation

$$\rho : \pi_1(S, s_0) \to \mathrm{GL}(\mathscr{L}_{s_0}) \cong \mathrm{GL}_n(\mathbb{C}).$$

If S is connected, as we will assume throughout this Chapter, this construction is independent, up to conjugation, of the choice of the base point.

Conversely, suppose we are given a representation $\rho : \pi_1(S, s_0) \to \mathrm{GL}_n(\mathbb{C})$, let $p : \tilde{S} \to S$ be the universal cover of S and consider the constant sheaf of stalk $\mathbb{C}^n$ on $\tilde{S}$, denoted $\underline{\mathbb{C}^n}$. The fundamental group of S acts on $\tilde{S}$ and can be identified with the group of covering transformations.

Then we can define a local system $\mathscr{L}$ on S by setting for each open U in S

$$\mathscr{L}(U) = \{s \in \Gamma(p^{-1}(U), \underline{\mathbb{C}^n}) : s(\gamma \cdot u) = \rho(\gamma) \cdot s(u), \forall u \in p^{-1}(U), \forall \gamma \in \pi_1(S, s_0)\}$$

i.e. taking the sections of $\underline{\mathbb{C}^n}$ which are equivariant under the action of the fundamental group.

Now, we will show how we can associate to a local system a holomorphic vector bundle on S together with a flat connection.

Recall that if E is a holomorphic vector bundle on S and we denote by $\mathscr{O}(E)$ its associated sheaf of holomorphic sections, a *connection* on E is a $\mathbb{C}$-linear morphism of sheaves

$$\nabla : \mathscr{O}(E) \to \Omega^1_S \otimes \mathscr{O}(E)$$

such that for any local holomorphic function f and any local holomorphic section σ of E one has

$$\nabla(f\sigma) = df \otimes \sigma + f\nabla(\sigma).$$

Given such a connection, one can extend this definition to $\nabla : \Omega^p_S \otimes \mathscr{O}(E) \to \Omega^{p+1}_S \otimes \mathscr{O}(E)$ as

$$\nabla(\alpha \otimes \sigma) = d\alpha \otimes \sigma + (-1)^p \alpha \wedge \nabla(\sigma).$$

Definition 2.1.2. A local holomorphic section σ of E is said to be flat if $\nabla(\sigma) = 0$. The connection ∇ is said to be flat if $\nabla^2 = 0$.

In terms of a local holomorphic frame $\sigma_1, \ldots \sigma_r$ of $\mathscr{O}(E)(U)$ for some open $U \subseteq S$, we can write

$$\nabla(\sigma_j) = \sum_{i=1}^{r} \theta_{ij} \otimes \sigma_i$$

where the $\theta_{ij} \in \Omega^1_S(U)$ are called *connection forms*. A connection is essentially a tool to differentiate sections of E in the direction of a local holomorphic vector field X on S. Indeed, given a local frame $\sigma_1, \ldots \sigma_r$ for E, we can define

$$\nabla_X \left(\sum_{j=1}^{r} f_j \sigma_j \right) = \sum_{i=1}^{r} \left(X(f_i) + \sum_{j=1}^{r} f_j \theta_{ij}(X) \right) \sigma_i.$$

Remark 2.1.3. One can prove that a connection is flat if and only if there exists a trivializing cover of S for which the corresponding frame for E consists of flat sections.

Now, if $\mathscr{L}$ is a local system on S, then its associated holomorphic vector bundle $\mathscr{L} \otimes \mathscr{O}_S$ admits a trivializing cover whose transition matrices have locally constant coefficients. Hence we can define a connection on $\mathscr{L} \otimes \mathscr{O}_S$ just by extending the de Rham differential, namely $\nabla(f\sigma) = df \otimes \sigma$ for any local holomorphic function f and any local holomorphic section σ of $\mathscr{L} \otimes \mathscr{O}_S$. This connection is clearly flat, since $d^2 = 0$.

Conversely, if E is a holomorphic vector bundle on S endowed with a flat connection ∇, then its sheaf of flat sections $\ker(\nabla)$ is locally constant with stalk of rank equal to the rank of E.

Summarizing the discussion of this section, we conclude that for a connected complex manifold S there is a equivalence of categories between the categories of:

(1) local systems of complex vector spaces on S,

(2) finite dimensional representations of $\pi_1(S, s_0)$,

(3) holomorphic vector bundles on S endowed with a flat connection.

2.2 Families of complex manifolds

Let S be a connected complex manifold. In this section we will consider families of complex manifolds varying holomorphically over S.

Definition 2.2.1. A family of complex manifolds over S is a proper holomorphic submersion $f : \mathcal{X} \to S$, namely f is surjective, proper and holomorphic with surjective differential.

In this situation each fiber $X_s = f^{-1}(s)$, for $s \in S$, is a compact complex manifold. The following theorem, due to Ehresmann, asserts that such a map f is a $\mathscr{C}^\infty$ fiber bundle.

Theorem 2.2.2. *For every $s_0 \in S$ there exists an open neighborhood U of s_0 and a $\mathscr{C}^\infty$ diffeomorphism $F : f^{-1}(U) \to U \times X_{s_0}$ such that $p_1 \circ F = f$, where p_1 is the projection onto the first factor. Furthermore, for every $x \in X_s$, the map $\sigma_x : U \to \mathcal{X}$ defined as $\sigma_x(s) = F^{-1}(s, x)$ is holomorphic.*

Remark 2.2.3. More precisely, one can prove that the family f trivializes over any contractible neighborhood of $s_0 \in S$. We refer to Voisin [46], Theorem 9.3, for details on the proof.

Now consider the derived pushforward of the constant sheaf on $\mathcal{X}$, namely $R^k f_* \mathbb{Z}$.

Remark 2.2.4. Recall that this is defined as the right derived functor of the pushforward $f_* : \mathrm{Sh}(\mathcal{X}) \to \mathrm{Sh}(S)$ between the categories of abelian sheaves on $\mathcal{X}$ and S. By considering the commutative square of functors

$$\begin{array}{ccc} \mathrm{Sh}(\mathcal{X}) & \xrightarrow{\text{for}} & \mathrm{PreSh}(\mathcal{X}) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{Sh}(S) & \xleftarrow[(\cdot)^{\mathrm{sh}}]{} & \mathrm{PreSh}(S) \end{array}$$

where $\mathrm{PreSh}(\cdot)$ denotes the category of abelian presheaves, the upper arrow is the forgetful functor and the bottom one the sheafification, we obtain that $R^k f_* \mathbb{Z}$ is the sheafification of the presheaf that associates to any open $U \subseteq S$ the cohomology group $H^k(f^{-1}(U), \mathbb{Z})$.

The previous Theorem implies that the cohomology groups $H^k(X_s, \mathbb{Z})$ are locally constant on S, so $R^k f_* \mathbb{Z}$ is a locally constant abelian sheaf on S, with stalk $(R^k f_* \mathbb{Z})_s \cong H^k(X_s, \mathbb{Z}) \cong H^k(X_{s_0}, \mathbb{Z})$ for a fixed $s_0 \in S$. Of course we can do the same replacing the constant sheaf of stalk $\mathbb{Z}$ with that of stalk $\mathbb{Q}$ or $\mathbb{C}$, obtaining that $R^k f_* \mathbb{C}$ is a local system of complex vector spaces on S. Recall from the discussion in the previous section that we can associate to it a representation

$$\rho : \pi_1(S, s_0) \to \mathrm{GL}(H^k(X_{s_0}, \mathbb{C}))$$

and that the holomorphic vector bundle $R^k f_* \mathbb{C} \otimes \mathscr{O}_S$ is naturally endowed with a flat connection, which is called in this situation Gauss-Manin connection.

2.3 The Kodaira-Spencer map

Let $f : \mathcal{X} \to S$ be a family of compact complex manifolds. Assume that S is a polydisk centered at zero and let $X = X_0$. By Theorem 2.2.2 we have a diffeomorphism $F : \mathcal{X} \to S \times X$, let G be the inverse. Then, for $s \in S$, the restriction

$$g_s = G|_{\{s\} \times X} : X \to X_s$$

is a $\mathscr{C}^\infty$ diffeomorphism that carries the complex structure on the real tangent space of X_s to a complex structure on the real tangent space of X. Therefore, we can think the family $f : \mathcal{X} \to S$ as a family of complex structures on a fixed differentiable manifold X. The Kodaira-Spencer map that we will define below can be interpreted as measuring the derivative at $s = 0$ of the association that maps $s \in S$ to the complex structure X_s on X.

Notice that the pushforward f_* induces an exact sequence of holomorphic vector bundles on X:

$$0 \longrightarrow \mathscr{T}_X \longrightarrow \mathscr{T}_{\mathcal{X}}|_X \xrightarrow{f_*} X \times \mathscr{T}_{S,0} \longrightarrow 0.$$

This sequence gives rise to an exact sequence of the corresponding sheaves of holomorphic sections and consequently to a long exact sequence in cohomology. In particular we have a map

$$H^0(X, \mathscr{O}(X \times \mathscr{T}_{S,0})) \to H^1(X, \mathscr{O}(\mathscr{T}_X)).$$

Since X is compact we have $H^0(X, \mathscr{O}(X \times \mathscr{T}_{S,0})) \cong \mathscr{T}_{S,0}$, while introducing a Dolbeault resolution as in Remark 1.1.1 leads to the isomorphism

$$H^1(X, \mathscr{O}(\mathscr{T}_X)) \cong H^{0,1}_{\overline{\partial}}(X, \mathscr{T}_X).$$

Definition 2.3.1. The map

$$\mathscr{T}_{S,0} \to H^1(X, \mathscr{O}(\mathscr{T}_X)) \cong H^{0,1}_{\overline{\partial}}(X, \mathscr{T}_X)$$

obtained above is called the Kodaira-Spencer map at zero of the family.

As noted above, the family $f : \mathcal{X} \to S$ gives rise to a family of complex structures on the differentiable manifold underlying X. Thus, for each point $x \in X$ and parameter $s \in S$, we have a splitting

$$T_x X_{\mathbb{C}} = (T_x)^+_s \oplus (T_x)^-_s$$

with $(T_x)^+_s = \overline{(T_x)^-_s}$ and $(T_x)^+_0 = T^{1,0}_x X \cong \mathscr{T}_{X,x}$. If s is small enough, the complex subspace $(T_x)^+_s$ is parametrized by a form $\alpha_s(x) \in (T_x X^*)^{0,1} \otimes T_x X^{1,0}$ (where $T_x X^*_{\mathbb{C}} = (T_x X^*)^{1,0} \oplus (T_x X^*)^{0,1}$ is the induced complex structure on the cotangent space), which is identified, up to sign, with the composition

$$T_x X^{0,1} \to (T_x)^-_s \to T_x X^{1,0},$$

where the first map is a section of the projection $(T_x)^-_s \to T_x X^{0,1}$ and the second one is the restriction of the projection $T_x X_{\mathbb{C}} \to T_x X^{1,0}$. So we can construct a form α_s in $\mathscr{A}^{0,1}_X(\mathscr{T}_X)$ parametrizing the complex structure of X_s.

Conversely, given such a form α_s, the subspace $(T_x)^-_s$ is generated by vectors of the form $v - \alpha_s(x)(v)$, with $v \in T_x X^{0,1}$.

Proposition 2.3.2. *The map* $\rho : \mathscr{T}_{S,0} \to H^0(X, \mathscr{A}^{0,1}_X(\mathscr{T}_X))$ *defined by*

$$\rho(v) = d_v(\alpha_s),$$

for each $v \in \mathscr{T}_{S,0}$*, has values in the space of* $\overline{\partial}$*-closed sections of* $\mathscr{A}^{0,1}_X(\mathscr{T}_X)$ *and for all* $v \in \mathscr{T}_{S,0}$ *the Dolbeault cohomology class of* $\rho(v)$ *in* $H^{0,1}_{\overline{\partial}}(X, \mathscr{T}_X)$ *coincides with the image of* v *under the Kodaira-Spencer map.*

Proof. Consider the trivialization $F : \mathcal{X} \to S \times X$. Since each submanifold $F^{-1}(S \times x)$, for $x \in X$, is a complex submanifold of $\mathcal{X}$, by pushing forward the tangent bundle of S along F^{-1}

we obtain a $\mathscr{C}^\infty$ complex subbundle of $T^{1,0}_{\mathcal{X}}$, which is isomorphic via f_* to $f^*\mathscr{T}_S$. Thus we have a $\mathscr{C}^\infty$ section

$$\sigma : f^*\mathscr{T}_S \to \mathscr{T}_{\mathcal{X}},$$

which provides, once restricted to X, a $\mathscr{C}^\infty$ splitting of the short exact sequence

$$0 \longrightarrow \mathscr{T}_X \longrightarrow \mathscr{T}_{\mathcal{X}}|_X \xrightarrow{f_*} X \times \mathscr{T}_{S,0} \longrightarrow 0.$$

By definition, the image in $H^{0,1}_{\overline{\partial}}(X, \mathscr{T}_X)$ of a tangent vector $v \in \mathscr{T}_{S,0}$ under the Kodaira-Spencer map is given by $\overline{\partial}(\sigma(v))$. Thus, we have to show that

$$\overline{\partial}(\sigma(v)) = d_v(\alpha_s)$$

in $H^0(X, \mathscr{A}^{0,1}_X(\mathscr{T}_X))$. We can check this equality locally, so let us assume that we have local holomorphic coordinates $s_1, \ldots, s_r$ centered at 0 on S, and functions $z_1, \ldots, z_n$ on $\mathcal{X}$ such that $z_1, \ldots, z_n, f^*s_1, \ldots, f^*s_r$ give a holomorphic coordinate system on $\mathcal{X}$. Composing the $\mathscr{C}^\infty$ trivialization $F : \mathcal{X} \to S \times X$ with the projection onto the second component we obtain a map $\pi : \mathcal{X} \to X$, given in these coordinates by an n-tuple of differentiable functions

$$(\pi_1(z_1, \ldots, z_n, s_1, \ldots, s_r), \ldots, \pi_n(z_1, \ldots, z_n, s_1, \ldots, s_r)),$$

which are holomorphic in the s_i. Now, given a point in $\mathcal{X}$ with coordinates (z, s), the subspace $(T_{\pi(z,s)})^-_s$, parametrizing the complex structure on X corresponding to $s \in S$, is generated by the vectors

$$\pi_*\left(\frac{\partial}{\partial \overline{z_i}}\right) = \sum_{j=1}^n \frac{\partial \pi_j}{\partial \overline{z_i}} \frac{\partial}{\partial z_j} + \sum_{j=1}^n \frac{\partial \overline{\pi_j}}{\partial \overline{z_i}} \frac{\partial}{\partial \overline{z_j}}.$$

Thus, by the definition of α_s, we get

$$\alpha_s\left(\sum_{j=1}^n \frac{\partial \overline{\pi_j}}{\partial \overline{z_i}} \frac{\partial}{\partial \overline{z_j}}\right) = -\sum_{j=1}^n \frac{\partial \pi_j}{\partial \overline{z_i}} \frac{\partial}{\partial z_j}$$

at the point $\pi(z, s)$. Clearly $\alpha_0 = 0$ and $\pi|_X = \mathrm{id}$, hence this expression gives, to the first order in s,

$$\alpha_s\left(\frac{\partial}{\partial \overline{z_i}}\right) = -\sum_{j=1}^n \frac{\partial \pi_j}{\partial \overline{z_i}} \frac{\partial}{\partial z_j}$$

at the point $(z, 0)$. Differentiating this along $\frac{\partial}{\partial s_k}$, we obtain

$$d_{\frac{\partial}{\partial s_k}|_{s=0}}(\alpha_s)\left(\frac{\partial}{\partial \overline{z_i}}\right) = \frac{\partial}{\partial s_k}(\alpha_s)|_{s=0}\left(\frac{\partial}{\partial \overline{z_i}}\right) = -\frac{\partial}{\partial \overline{z_i}}\left(\sum_{j=1}^n \frac{\partial \pi_j}{\partial s_k} \frac{\partial}{\partial z_j}\right). \tag{2.1}$$

Now we need to understand $\left(\overline{\partial}\sigma\left(\frac{\partial}{\partial s_k}\right)\right)\left(\frac{\partial}{\partial \overline{z_i}}\right)$. To do so, notice that the vector field $\sigma\left(\frac{\partial}{\partial s_k}\right)$ is the unique vector field of type $(1, 0)$ on $\mathcal{X}$ such that

$$\pi_*\sigma\left(\frac{\partial}{\partial s_k}\right) = 0, \quad f_*\sigma\left(\frac{\partial}{\partial s_k}\right) = \frac{\partial}{\partial s_k}.$$

But $\pi_* = \mathrm{id}$ along X and, since π is holomorphic in s_k,

$$\pi_*\left(\frac{\partial}{\partial s_k}\right) = \sum_{j=1}^{n} \frac{\partial \pi_j}{\partial s_k}\frac{\partial}{\partial z_j},$$

therefore we have

$$\sigma\left(\frac{\partial}{\partial s_k}\right) = \frac{\partial}{\partial s_k} - \sum_{j=1}^{n} \frac{\partial \pi_j}{\partial s_k}\frac{\partial}{\partial z_j}. \tag{2.2}$$

Now, $\left(\overline{\partial}\sigma\left(\frac{\partial}{\partial s_k}\right)\right)\left(\frac{\partial}{\partial \overline{z_i}}\right)$ is simply $\frac{\partial}{\partial \overline{z_i}}\left(\sigma\left(\frac{\partial}{\partial s_k}\right)\right)$, where we let $\frac{\partial}{\partial \overline{z_i}}$ act on the coefficients of $\sigma\left(\frac{\partial}{\partial s_k}\right)$. Using (2.2) we obtain

$$\frac{\partial}{\partial \overline{z_i}}\left(\sigma\left(\frac{\partial}{\partial s_k}\right)\right) = -\frac{\partial}{\partial \overline{z_i}}\left(\sum_{j=1}^{n} \frac{\partial \pi_j}{\partial s_k}\frac{\partial}{\partial z_j}\right). \tag{2.3}$$

Finally, comparing (2.1) and (2.3), we get

$$d_{\frac{\partial}{\partial s_k}|_{s=0}}(\alpha_s)\left(\frac{\partial}{\partial \overline{z_i}}\right) = \frac{\partial}{\partial \overline{z_i}}\left(\sigma\left(\frac{\partial}{\partial s_k}\right)\right),$$

which proves the desired equality. □

2.4 Geometric variations of Hodge structures

In this section we will finally describe how the Hodge structure on the cohomology of the fibers of a family of compact Kähler manifolds varies over the connected base manifold S. In particular we will see that the Hodge numbers $h^{p,q}(X_s) = \dim H^{p,q}(X_s)$ are constant in s and that the Hodge filtration induces a filtration of $R^k f_*\mathbb{C} \otimes \mathscr{O}_S$ by holomorphic subbundles.

Let $f : \mathcal{X} \to S$ be a family of complex manifolds, in the sense of Definition 2.2.1, such that all fibers are Kähler. Then, on each fiber we have a Hodge decomposition

$$H^k(X_s, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X_s).$$

Proposition 2.4.1. *The Hodge numbers $h^{p,q}(X_s) = \dim H^{p,q}(X_s)$ are constant.*

Proof. The crucial point here is that $\dim H^{p,q}(X_s)$ is upper semicontinuous in s. Indeed, the Hodge component $H^{p,q}(X_s)$ can be described as the space of $\overline{\partial}$-harmonic forms of bidegree (p, q), namely as the kernel of the Laplacian operator acting on complex differential forms of type (p, q). So, the claimed upper semicontinuity follows from a general result about elliptic operators, see Wells [48], Theorem 4.13. Hence,

$$\dim H^{p,q}(X_s) \leq \dim H^{p,q}(X_{s_0})$$

for s in a neighborhood of s_0. On the other hand, if we let $b_k(X_s) = \dim H^k(X_s, \mathbb{C})$ be the k-th Betti number of X_s, we have

$$\sum_{p+q=k} \dim H^{p,q}(X_s) = b^k(X_s) = b^k(X_{s_0}) = \sum_{p+q=k} \dim H^{p,q}(X_{s_0})$$

since the fibers are diffeomorphic. So, $\dim H^{p,q}(X_s)$ must be constant. □

Remark 2.4.2. While the Hodge numbers are constant, the interaction between Hodge decomposition and the rational structure on $R^k f_*\mathbb{C}$ given by $H^k(X_s, \mathbb{Q}) \otimes \mathbb{C} \cong H^k(X_s, \mathbb{C})$ is much more subtle. In particular, the dimension of the space of Hodge classes

$$H^{2k}(X_s, \mathbb{Q}) \cap H^{k,k}(X_s)$$

is not constant, in general. We will come back to this in section 2.7.

Now, recall that the Hodge decomposition can be described by its associated filtration defined as

$$F^p H^k(X_s, \mathbb{C}) = \bigoplus_{r \geq p} H^{r,k-r}(X_s).$$

Let $f(p) = \sum_{r \geq p} h^{r,k-r}(X_s)$, where the sum has the same value for all $s \in S$. Assume that S is contractible, so that f is $\mathscr{C}^\infty$ trivial over S. Then, for all $s \in S$, we have diffeomorphisms $g_s : X = X_{s_0} \to X_s$ inducing isomorphisms

$$g_s^* : H^k(X_s, \mathbb{C}) \to H^k(X, \mathbb{C}).$$

Then, denoting $\mathrm{Gr}(f(p), H^k(X, \mathbb{C}))$ the Grassmannian of $f(p)$-dimensional subspaces of $H^k(X, \mathbb{C})$, we can define a map

$$\Phi^p : S \to \mathrm{Gr}(f(p), H^k(X, \mathbb{C}))$$

putting $\Phi^p(s) = g_s^*(F^p H^k(X_s, \mathbb{C}))$.

Proposition 2.4.3. *The map $\Phi^p : S \to \mathrm{Gr}(f(p), H^k(X, \mathbb{C}))$ is holomorphic.*

Proof. First of all notice that, since the Hodge numbers are constant, the spaces $H^{p,q}(X_s)$ vary smoothly with s by a theorem of Kodaira, see Voisin [46], Proposition 9.22. Therefore, we need to show that the differential of Φ^p is $\mathbb{C}$-linear, namely that its complex linear extension to $T_{s_0} S_{\mathbb{C}}$ vanishes on vectors of type (0,1) for an arbitrary point $s_0 \in S$. Now, to lighten the notations, let $F^p(s) = g_s^*(F^p H^k(X_s, \mathbb{C})) \subseteq H^k(X, \mathbb{C})$, where $X = X_{s_0}$. The differential

$$d\Phi^p_{s_0} : T_{s_0} S_{\mathbb{C}} \to \mathrm{Hom}(F^p(s_0), H^k(X, \mathbb{C})/F^p(s_0))$$

can be computed by choosing, for $\alpha \in F^p(s_0)$, a smooth local section σ of $R^k f_*\mathbb{C} \otimes \mathscr{O}_S$ such that $\sigma(s_0) = \alpha$ and

$$\sigma(s) \in F^p H^k(X_s, \mathbb{C})$$

for s in some neighborhood of s_0. Hence, we may view $g_s^*(\sigma(s))$ as a curve in $H^k(X, \mathbb{C})$ such that $g_s^*(\sigma(s)) \in F^p(s)$. Then

$$d\Phi^p_{s_0}(u)(\alpha) = [u(g_s^* \sigma)] \mod F^p(s_0)$$

where we let the tangent vector $u \in T_{s_0} S_{\mathbb{C}}$ act on the coefficients (with respect to some local frame) of forms representing the cohomology classes $g_s^*(\sigma(s))$. More intrinsically, we can consider this action as the pull-back of the covariant derivative $\nabla_u(\sigma)$ evaluated at s_0. By a general theorem of complex geometry (see Voisin [46], Proposition 9.22) there exists a differential form Θ on $\mathcal{X}$ such that $\Theta \in F^p \mathscr{A}(\mathcal{X}) = \bigoplus_{r \geq p} \mathscr{A}^{r,k-r}(\mathcal{X})$, its restriction to X_s is closed and its cohomology class in $H^k(X_s, \mathbb{C})$ is precisely $\sigma(s)$. Now, taking a lift v of u to $\mathcal{X}$, namely a section of $f^* T S_{\mathbb{C}}$ such that $f_*(v) = u$, and applying the well known Cartan-Lie formula (see Voisin [46], Proposition 9.14), we have

$$d\Phi^p_{s_0}(u)(\alpha) = [\mathrm{int}_v(d\Theta)|_X] \mod F^p(s_0). \tag{2.4}$$

Here, if u is a tangent vector of type (0,1), then v is also of type (0,1) along X. Furthermore, $d\Theta \in F^p \mathscr{A}^{k+1}(\mathcal{X})$, so $\mathrm{int}_v(d\Theta) \in F^p \mathscr{A}^k(\mathcal{X})$ and the cohomology class of the restriction $\mathrm{int}_v(d\Theta)|_X$ lies in $F^p H^k(X, \mathbb{C}) = F^p(s_0)$. Therefore, for such u, $d\Phi^p_{s_0}(u)(\alpha) = 0$, so Φ^p is holomorphic. □

Actually, formula (2.4) implies much more than the fact that Φ^p is a holomorphic map: it implies the following property, known as *Griffiths transversality.*

Proposition 2.4.4. *The differential of* $\Phi^p : S \to \mathrm{Gr}(f(p), H^k(X,\mathbb{C}))$ *at* s_0 *has values in the subspace* $\mathrm{Hom}(F^pH^k(X,\mathbb{C}), F^{p-1}H^k(X,\mathbb{C})/F^pH^k(X,\mathbb{C}))$.

Proof. Keeping the same notations of the previous proof, we recall that

$$d\Phi^p_{s_0}(u)(\alpha) = [\mathrm{int}_v(d\Theta)|_X] \mod F^p(s_0).$$

Now, $\Theta \in F^p\mathscr{A}^k(\mathcal{X})$, so $d\Theta \in F^p\mathscr{A}^{k+1}(\mathcal{X})$ and $\mathrm{int}_v(d\Theta) \in F^{p-1}\mathscr{A}^k(\mathcal{X})$, whatever the type of v is. Thus

$$d\Phi^p_{s_0}(u)(\alpha) \in F^{p-1}H^k(X,\mathbb{C})/F^pH^k(X,\mathbb{C}).$$

□

Remark 2.4.5. By Proposition 2.4.4 we can consider the differential of Φ^p at s_0 as taking values in the space

$$\mathrm{Hom}(F^pH^k(X,\mathbb{C})/F^{p+1}H^k(X,\mathbb{C}), F^{p-1}H^k(X,\mathbb{C})/F^pH^k(X,\mathbb{C})),$$

which identifies, by the Dolbeault isomorphism $H^{p,q}(X) \cong H^q(X,\Omega^p_X)$, with

$$\mathrm{Hom}(H^q(X,\Omega^p_X), H^{q+1}(X,\Omega^{p-1}_X)).$$

We can then compute the differential $d\Phi^p_{s_0}$ as the composition of the Kodaira-Spencer map $\mathscr{T}_{S,s_0} \to H^1(X,\mathscr{T}_X)$ with the map

$$H^1(X,\mathscr{T}_X) \to \mathrm{Hom}(H^q(X,\Omega^p_X), H^{q+1}(X,\Omega^{p-1}_X))$$

given by interior product and the product in Čech cohomology. We refer to Voisin [46], Theorem 10.21, for details.

Remark 2.4.6. Clearly, we can consider this map Φ^p for every p to obtain a map

$$S \to \prod_p \mathrm{Gr}(f(p), H^k(X,\mathbb{C})),$$

which will be called period map, that sends a point $s \in S$ to the Hodge filtration of $H^k(X_s,\mathbb{C})$, seen as a filtration of the fixed vector space $H^k(X,\mathbb{C})$ via the identification g_s^*. However, let us remark that we have constructed these maps Φ^p over a contractible base S, so that $f : \mathcal{X} \to S$ is $\mathscr{C}^\infty$ trivial. If we put ourselves in the general situation where S may be not simply connected, in order to globalize this construction, we have to take care of the monodromy representation associated to the local system $R^kf_*\mathbb{C}$, since the identifications between the cohomologies of the fibers depend on the homotopy classes of paths connecting the corresponding points in S. This construction will be described, in a more general situation, in section 2.6.

Now, let $\mathscr{F}^p$ be the, a priori $\mathscr{C}^\infty$, subbundle of $R^kf_*\mathbb{C} \otimes \mathscr{O}_S$ whose fiber over $s \in S$ is $F^pH^k(X_s,\mathbb{C})$. By Proposition 2.4.3 it is a holomorphic subbundle of $R^kf_*\mathbb{C} \otimes \mathscr{O}_S$. Let us use here the common abuse of notations of using the same letter for a holomorphic vector bundle and its associated locally free sheaf of holomorphic sections. The subbundles $\mathscr{F}^p$ give a Hodge filtration of $R^kf_*\mathbb{C} \otimes \mathscr{O}_S$, i.e. they satisfy

$$R^kf_*\mathbb{C} \otimes \mathscr{O}_S = \mathscr{F}^p \oplus \overline{\mathscr{F}^{k-p+1}}$$

since this is true fiberwise by construction. The following result, which is just another way of expressing Griffiths transversality, gives a crucial constraint on how they interact with the Gauss-Manin connection.

Proposition 2.4.7. *With notations as above one has* $\nabla(\mathscr{F}^p) \subseteq \Omega^1_S \otimes \mathscr{F}^{p-1}$.

Proof. This is an immediate consequence of Proposition 2.4.4, since, given a holomorphic section σ of $\mathscr{F}^p$, one has for any $s \in S$ and $u \in \mathscr{T}_{S,s}$,

$$d\Phi^p_s(u)(\sigma(s)) = \nabla_u(\sigma) \mod F^p(s).$$

□

Finally, notice that if we have a family $f : \mathcal{X} \to S$ such that $\mathcal{X} \subseteq \mathbb{P}^N$, then every fiber X_s has an integral Kähler class $\omega_s \in H^{1,1}(X_s) \cap H^2(X_s, \mathbb{Z})$. These glue together into an element of $R^2 f_*\mathbb{Z}$. The cup product by this class gives a morphism of local systems $R^k f_*\mathbb{Z} \to R^{k+2} f_*\mathbb{Z}$, the restriction of the Gauss-Manin connection on the primitive cohomology remains flat and the integral non-degenerate forms Q_s, defined in Chapter 1, polarize the Hodge structure on the primitive cohomology $H^k_{\text{prim}}(X_s, \mathbb{Z})$ of each fiber.

2.5 The abstract definition

The geometric situation described in the previous section motivates, after taking the quotient by the torsion part of $R^k f_*\mathbb{Z}$ if necessary, the following general definition.

Definition 2.5.1. An integral variation of Hodge structures of weight k over a connected complex manifold S consists of a locally constant sheaf $\mathbb{V}$ of finitely generated free abelian groups on S and a decreasing filtration $\mathscr{F}^\bullet$ of the associated holomorphic vector bundle $\mathbb{V} \otimes \mathscr{O}_S$ by holomorphic subbundles satisfying:

(1) (Hodge symmetry) $\mathbb{V} \otimes \mathscr{O}_S = \mathscr{F}^p \oplus \overline{\mathscr{F}^{k-p+1}}$ as $\mathscr{C}^\infty$ bundles,

(2) (Griffiths transversality) $\nabla(\mathscr{F}^p) \subseteq \Omega^1_S \otimes \mathscr{F}^{p-1}$, where ∇ is the natural flat connection on $\mathbb{V} \otimes \mathscr{O}_S$.

Of course, an analogous definition can be made for rational and real variations of Hodge structures.

Clearly, such an integral variation of Hodge structures ($\mathbb{Z}$-VHS) induces an integral Hodge structure on each stalk $\mathbb{V}_s$, for $s \in S$.

Definition 2.5.2. We will say that a weight k integral variation of Hodge structures $(\mathbb{V}, \mathscr{F}^\bullet)$ is polarized by Q if Q is a flat non-degenerate bilinear form defined on $\mathbb{V}$, symmetric if k is even and alternating otherwise, such that for each $s \in S$ the Hodge structure on $\mathbb{V}_s$ is polarized in the sense of Definition 1.2.8.

2.6 Period domain and period map

Let $(\mathbb{V}, \mathscr{F}^\bullet)$ be an integral polarized variation of Hodge structures on S. Then, we can consider it as a family of Hodge structures on a fixed stalk $\mathbb{V}_{s_0}$, for $s_0 \in S$. Since the isomorphisms between the stalks of a locally constant sheaf are constructed using paths connecting the points and thus depend on the homotopy classes of these paths, the construction will be well defined modulo the action of $\pi_1(S, s_0)$. Let $p : \tilde{S} \to S$ be the universal cover of S, so that the pull-back of $\mathbb{V}$ along p is the constant sheaf with stalk $\mathbb{V}_{s_0}$. For each point $\tilde{s} \in \tilde{S}$ the pull-back of the Hodge structure on $\mathbb{V}_{p(\tilde{s})}$ gives a Hodge structure on $(p^*\mathbb{V})_{\tilde{s}}$, which induces a Hodge structure on $\mathbb{V}_{s_0}$ via the canonical isomorphism $(p^*\mathbb{V})_{\tilde{s}} \cong \mathbb{V}_{s_0}$. Then, we can construct a map from $\tilde{S}$ to an appropriate classifying space D of Hodge structures on $\mathbb{V}_{s_0}$, this will descend to a map $S \to \Gamma\backslash D$, where Γ is the *monodromy group* i.e. the image of the representation $\rho : \pi_1(S, s_0) \to \mathrm{GL}(\mathbb{V}_{s_0}, \mathbb{Z})$ associated to the locally constant sheaf $\mathbb{V}$ and already described in section 2.1.

Let us start with the general construction of such a classifying space D. Fix a finetely generated free abelian group $V_{\mathbb{Z}}$, a weight $k \in \mathbb{Z}$ and a collection of Hodge numbers $h^{p,q}$, for $p+q=k$, such that $h^{p,q} = h^{q,p}$ and $\sum h^{p,q} = \dim_{\mathbb{C}} V_{\mathbb{C}}$. Let $f(p) = \sum_{r \geq p} h^{r,k-r}$. Fix also an integral non-degenerate bilinear form Q on $V_{\mathbb{Z}}$, symmetric if k is even and alternating otherwise.

Let $\check{D}$ be the subset of the product of Grassmannians

$$\prod_p \mathrm{Gr}(f(p), V_{\mathbb{C}})$$

consisting of all decreasing filtrations $F^\bullet$ of $V_{\mathbb{C}}$ such that $\dim_{\mathbb{C}} F^p = f(p)$ and $Q(F^p, F^{k-p+1}) = 0$.

Remark 2.6.1. Notice that if $F^\bullet$ is the filtration associated to the Hodge decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$, with $V^{q,p} = \overline{V^{p,q}}$, then the condition $Q(F^p, F^{k-p+1}) = 0$ is equivalent to the first Hodge-Riemann relation for Q, that is the orthogonality of the decomposition with respect to the hermitian form h defined by $h(v,u) = i^k Q(v, \overline{u})$.

Indeed, if $v \in V^{p,q}$, $u \in V^{p',q'}$ and we assume $q' \geq q+1 = k-p+1$ (the other case is analogous), then $Q(F^p, F^{k-p+1}) = 0$ implies that $Q(v, \overline{u}) = 0$, since v belongs to F^p and $\overline{u}$ lies in

$$\overline{V^{p',q'}} = V^{q',p'} \subseteq \bigoplus_{r \geq k-p+1} V^{r,k-r} = F^{k-p+1},$$

so $h(v,u) = 0$.

Conversely, if $v \in \bigoplus_{r \geq p} V^{r,k-r} = F^p$, $u \in \bigoplus_{s \geq k-p+1} V^{s,k-s} = F^{k-p+1}$ and the decomposition is orthogonal for h, then $h(v, \overline{u}) = 0$, as

$$\overline{u} \in \bigoplus_{s \geq k-p+1} \overline{V^{s,k-s}} = \bigoplus_{t \leq p-1} V^{t,k-t},$$

so $Q(v,u) = 0$.

Since $\check{D}$ is given by closed conditions in a product of Grassmannians, which are projective varieties, it is a projective variety, more precisely what is usually called a (partial) flag variety. Moreover, the open subset of $\check{D}$ consisting of Hodge filtrations satisfying the additional positivity condition given the second Hodge-Riemann relation is the space D classifying all Hodge structures on $V_{\mathbb{Z}}$ with weight k, Hodge numbers $h^{p,q}$ and polarized by Q. It is called *period domain.*

Theorem 2.6.2. *Both D and $\check{D}$ are complex manifolds. In fact, $\check{D}$ is a homogeneous space $\check{D} \cong G_{\mathbb{C}}/B$, where $G_{\mathbb{C}} = \mathrm{Aut}(V_{\mathbb{C}}, Q)$ is the subgroup of $\mathrm{GL}(V_{\mathbb{C}})$ of automorphisms that preserve the bilinear form Q and B is the stabilizer of a given flag $F^\bullet \in \check{D}$. The open subset D of $\check{D}$ is an orbit of the real group $G_{\mathbb{R}} = \mathrm{Aut}(V_{\mathbb{R}}, Q)$ and $D \cong G_{\mathbb{R}}/K$, where $K = G_{\mathbb{R}} \cap B$ is a compact subgroup.*

Proof. See Griffiths [25], Theorem 4.3. □

Now, the tangent bundle of the homogeneous space $\check{D}$ can be described in terms of the Lie algebra $\mathfrak{g}$ of $G_{\mathbb{C}}$, i.e.

$$\mathfrak{g} = \{A \in \mathfrak{gl}(V_{\mathbb{C}}) : Q(Au, v) + Q(u, Av) = 0, \forall u, v \in V_{\mathbb{C}}\}.$$

The choice of a Hodge filtration $F_0^\bullet \in \check{D}$ induces a Hodge filtration of $\mathfrak{g}$ defined as

$$F^a\mathfrak{g} = \{A \in \mathfrak{g} : A(F_0^p) \subseteq F_0^{p+a}\}.$$

Equivalently,

$$\mathfrak{g}^{a,-a} = F^a\mathfrak{g} \cap \overline{F^{-a}\mathfrak{g}} = \{A \in \mathfrak{g} : A(V^{p,q}) \subseteq V^{p+a,q-a}\}$$

defines a weight zero Hodge structure on $\mathfrak{g}$. Then the Lie algebra $\mathfrak{b}$ of the stabilizer B of the chosen filtration is $F^0\mathfrak{g}$. Now, let $\mathfrak{g}_0$ be the Lie algebra of $G_\mathbb{R}$, i.e. $\mathfrak{g}_0 = \mathfrak{g} \cap \mathfrak{gl}(V_\mathbb{R})$. Then the Lie algebra of K is $\mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{g}_0 \cap F^0\mathfrak{g} = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}$.

Since $\check{D} \cong G_\mathbb{C}/B$ and B is the stabilizer of the chosen flag $F_0^\bullet$, the holomorphic tangent space of $\check{D}$ at $F_0^\bullet$ is $\mathfrak{g}/\mathfrak{b} \cong \bigoplus_{r<0} \mathfrak{g}^{r,-r}$ and the holomorphic tangent space at any other point is obtained via the action of $G_\mathbb{C}$. More precisely, the holomorphic tangent bundle

$$\mathscr{T}\check{D} \cong \check{D} \times_B \mathfrak{g}/\mathfrak{b}$$

is the space of equivalence classes of couples $(F^\bullet, A + \mathfrak{b})$ in $\check{D} \times \mathfrak{g}/\mathfrak{b}$ where, for $g \in G_\mathbb{C}$, $(F^\bullet, A+\mathfrak{b})$ is equivalent to $(g \cdot F^\bullet, A' + \mathfrak{b})$ if and only if

$$A + \mathfrak{b} = \mathrm{Ad}(g^{-1})(A' + \mathfrak{b})$$

for the adjoint action of $G_\mathbb{C}$ on $\mathfrak{g}$. Since $[F^0\mathfrak{g}, F^a\mathfrak{g}] \subseteq F^a\mathfrak{g}$, the adjoint action leaves invariant the subspaces $F^a\mathfrak{g}$, thus we can consider the subbundle $\mathscr{T}^{-1,1}\check{D}$ of $\mathscr{T}\check{D}$ associated with the subspace

$$F^{-1}\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{g}^{-1,1}.$$

It is called the *horizontal subbundle*. As D is open in $\check{D}$, it restricts to a holomorphic bundle over D.

Remark 2.6.3. We can describe a tangent vector in $\mathscr{T}^{-1,1}\check{D}$ at a point $F^\bullet \in \check{D}$ as follows. It consists of a class $[X] \in \mathfrak{g}/\mathfrak{b}$ such that, if $F^\bullet = g \cdot F_0^\bullet \in \check{D}$, we have $\mathrm{Ad}(g^{-1})(X) \in F^{-1}\mathfrak{g}$, which means, if we see $\mathfrak{g}$ as a Lie algebra of endomorphisms of $V_\mathbb{C}$,

$$(g^{-1} \cdot X \cdot g)(F_0^p) \subseteq F_0^{p-1}.$$

Equivalently, thinking of X as the $G_\mathbb{C}$-equivariant vector field generated by $[X] \in \mathfrak{g}/\mathfrak{b}$, we may write $X(F^p) \subseteq F^{p-1}$.

Now that we have given a description of the space of Hodge structures on a lattice $V_\mathbb{Z}$ with given Hodge numbers and polarization, we can go back to the purpose of constructing, given a polarized $\mathbb{Z}$-VHS $(\mathbb{V}, \mathscr{F}^\bullet)$ on S, a map sending a point $s \in S$ to its correspoding Hodge structure. In this way, applying the previous construction with $V_\mathbb{Z} = \mathbb{V}_{s_0}$, we obtain a map

$$\tilde{S} \longrightarrow D \xhookrightarrow{\text{open}} \check{D} \xhookrightarrow{\text{closed}} \textstyle\prod_p \mathrm{Gr}(f(p), \mathbb{V}_{s_0} \otimes \mathbb{C})$$

sending a point $\tilde{s} \in \tilde{S}$ to the corresponding Hodge filtration, seen as a filtration of $\mathbb{V}_{s_0} \otimes \mathbb{C}$ as explained in the beginning of this section. Since the construction is equivariant under the action of $\pi_1(S, s_0)$ on $\tilde{S}$ by deck transformations, this descends to

$$\begin{array}{ccc} \tilde{S} & \longrightarrow & D \\ \big\downarrow{\scriptstyle p} & & \big\downarrow \\ S & \xrightarrow{\Phi} & \Gamma\backslash D \end{array}$$

where Γ is the monodromy group, namely the image of the representation

$$\rho : \pi_1(S, s_0) \to \mathrm{GL}(\mathbb{V}_{s_0}, \mathbb{Z})$$

associated with the locally constant abelian sheaf $\mathbb{V}$. Notice that the action of Γ on D is properly discontinuous, namely given K_1, K_2 compact subsets of D, one has $\gamma \cdot K_1 \cap K_2 \neq \emptyset$ for at most finitely many $\gamma \in \Gamma$. To show this, notice that the projection map $G_{\mathbb{R}} \to D \cong G_{\mathbb{R}}/B$ is proper, since B is a compact subgroup. Thus, the preimages $\tilde{K_i}$ of K_i under the projection map are compact and so is the product $L = \tilde{K_2} \cdot \tilde{K_1}^{-1}$. But then $\gamma \cdot K_1 \cap K_2 \neq \emptyset$ implies $\gamma \in L \cap \Gamma$, i.e. γ lies in the intersection between a compact subset and a discrete subset in $G_{\mathbb{R}}$, hence in a finite set. Now, the quotient $\Gamma \backslash D$ has then the structure of a complex manifold at those points where the action of Γ is free and the structure, at points which are fixed by some $\gamma \in \Gamma$, of a quotient of an open subset of $\mathbb{C}^N$ by a finite group. Thus, $\Gamma \backslash D$ is a complex analytic variety.

We refer to Φ as the *period map* of the given polarized $\mathbb{Z}$-VHS and to $\Gamma \backslash D$ as the *Hodge variety*.

Theorem 2.6.4. *The period map Φ has local liftings to D which are holomorphic and horizontal, namely the differential takes values in the horizontal subbundle $\mathscr{T}^{-1,1}D$.*

Proof. In view of remark 2.6.3, these properties are just a restatement of properties (1) and (2) in the definition of a variation of Hodge structures. □

Remark 2.6.5. We will provide a detailed construction of a Hodge variety in the case of Hodge structures of type $(1,0), (0,1)$ in Appendix A, where we will underline the relation between such Hodge structures and complex abelian varieties. However, let us already emphasize that this will be a highly peculiar case, where the Hodge variety $\Gamma \backslash D$ is actually algebraic, namely a Shimura variety. This will be the case essentially because the period domain D of such Hodge structures is a Hermitian symmetric domain, but for more complicated Hodge structures, a Hodge variety will typically be *not* biholomorphic to the analytification of an algebraic veriety, as proved in Griffiths-Robles-Toledo [27].

2.7 Mumford-Tate groups in families and Hodge loci

We have already remarked that, given a family of compact Kähler manifolds, while the Hodge numbers are constant, the set of Hodge classes is not constant. Let us now analyze this phenomenon in more details, describing how the Mumford-Tate group of the fibers of a rational (or integral) VHS varies over the base S. So, let $(\mathbb{V}, \mathscr{F}^\bullet)$ be a rational (or integral) variation of Hodge structures on S and let $p : \tilde{S} \to S$ be the universal covering.

Consider all tensor constructions arising from $\mathbb{V}$, namely for any collection of couples of non negative integers $\nu = \{(a_i, b_i)\}_{1 \leq i \leq t}$ let

$$\mathbb{T}^\nu = \bigoplus_{i=1}^{t} \mathbb{V}^{\otimes a_i} \otimes (\mathbb{V}^*)^{\otimes b_i}$$

which inherits a natural structure of $\mathbb{Q}$-VHS, as pointed out in Remark 1.2.4. Consider those $\mathbb{T}^\nu$ which inherit Hodge structures of even weights $2k(\nu)$.

Now, for such ν, consider the pull-back $p^*\mathbb{T}^\nu$: it is a constant sheaf, hence any element in the stalk of $p^*\mathbb{T}^\nu$ at some point extends uniquely to a global section of $p^*\mathbb{T}^\nu$, called horizontal continuation. Let t be a global section of $p^*\mathbb{T}^\nu$ and define

$$\tilde{Y}(t) = \{\tilde{s} \in \tilde{S} : t_{\tilde{s}} \text{ is a Hodge class in } (p^*\mathbb{T}^\nu)_{\tilde{s}}\}.$$

Notice that $t_{\tilde{s}}$ is a Hodge class if and only if it lands, under the map $(p^*\mathbb{T}^\nu)_{\tilde{s}} \to (p^*\mathbb{T}^\nu)_{\tilde{s}} \otimes \mathbb{C}$, in the subspace $F_\nu^{k(\nu)}(\tilde{s})$, namely in the piece of degree $k(\nu)$ of the Hodge filtration (of weight $2k(\nu)$) induced on $(p^*\mathbb{T}^\nu)_{\tilde{s}} \otimes \mathbb{C}$. Indeed, if we denote by $T_\nu^{p,q}(\tilde{s})$ the (p,q) component of the Hodge decomposition induced on $(p^*\mathbb{T}^\nu)_{\tilde{s}} \otimes \mathbb{C}$, we have by Hodge symmetry,

$$(p^*\mathbb{T}^\nu)_{\tilde{s}} \cap F_\nu^{k(\nu)}(\tilde{s}) = (p^*\mathbb{T}^\nu)_{\tilde{s}} \cap \left(\bigoplus_{r \geq k(\nu)} T_\nu^{r,2k(\nu)-r}(\tilde{s}) \right) = (p^*\mathbb{T}^\nu)_{\tilde{s}} \cap T_\nu^{k(\nu),k(\nu)}(\tilde{s}).$$

Therefore, denoting by $\mathscr{F}_\nu^\bullet$ the induced Hodge filtration of $p^*\mathbb{T}^\nu \otimes \mathscr{O}_{\tilde{S}}$, so that the fiber of $\mathscr{F}_\nu^{k(\nu)}$ at $\tilde{s}$ is $F_\nu^{k(\nu)}(\tilde{s})$, we obtain that $\tilde{Y}(t)$ is the zero locus of the section of $(p^*\mathbb{T}^\nu \otimes \mathscr{O}_{\tilde{S}})/\mathscr{F}_\nu^{k(\nu)}$ given by t. Hence, $\tilde{Y}(t)$ is the whole $\tilde{S}$ or a closed analytic subvariety of $\tilde{S}$.

This leads to consider

$$\tilde{S}^{\mathrm{exc}} = \bigcup_{\nu,t} \tilde{Y}(t)$$

where ν runs over all collections as above and t runs over all global sections of $p^*\mathbb{T}^\nu$ such that $\tilde{Y}(t)$ is *not* the whole $\tilde{S}$. So, we have isolated those points in $\tilde{S}$ where the fiber of the given $\mathbb{Q}$-VHS has exceptional Hodge tensors, namely those points $\tilde{s}$ where, for some tensor construction $\mathbb{T}^\nu$ arising from $\mathbb{V}$ and some Hodge class $t_{\tilde{s}}$ in the stalk of $p^*\mathbb{T}^\nu$ at $\tilde{s}$, the horizontal continuation of $t_{\tilde{s}}$ is not everywhere a Hodge class. Since in the definition of $\tilde{S}^{\mathrm{exc}}$ the union runs over a countable set, we have that $\tilde{S}^{\mathrm{exc}}$ is countable union of closed analytic subvarieties of $\tilde{S}$.

Finally, the construction of $\tilde{S}^{\mathrm{exc}}$ is stable under the action of $\pi_1(S, s_0)$ by deck transformations, thus induces the definition of a subset of S, denoted $\mathrm{HL}(S, \mathbb{V}^\otimes)$.

Now recall that we can see the $\mathbb{Q}$-VHS on $p^*\mathbb{V}$ as a family of Hodge structures on a fixed stalk $\mathbb{V}_s$. Consequently, we can consider the Mumford-Tate groups of the induced Hodge structures on every stalk of $p^*\mathbb{V}$ as subgroups of the fixed algebraic group $\mathbf{GL}(\mathbb{V}_s)$.

Proposition 2.7.1. *The Mumford-Tate group of the stalks of a rational (or integral) variation of Hodge structures on S is locally constant outside the subset $\mathrm{HL}(S, \mathbb{V}^\otimes) \subseteq S$, which a countable union of closed analytic subvarieties of S.*

Proof. The proof is an immediate consequence of the above discussion together with the fact, stated in Proposition 1.3.4, that the Mumford-Tate group associated to a Hodge structure on $\mathbb{V}_s$ is precisely the algebraic subgroup of $\mathbf{GL}(\mathbb{V}_s)$ that fixes all Hodge tensors appearing in tensor constructions arising from $\mathbb{V}_s$. □

Definition 2.7.2. The subset $\mathrm{HL}(S, \mathbb{V}^\otimes) \subseteq S$ is called the Hodge locus of the given rational (or integral) variation of Hodge structures. Any point $s \in S \smallsetminus \mathrm{HL}(S, \mathbb{V}^\otimes)$ is called Hodge generic and the Mumford-Tate group $\mathbf{MT}(\mathbb{V}_s)$ for such a Hodge generic point s is called generic Mumford-Tate group of the given VHS, denoted $\mathbf{MT}^{\mathrm{gen}}$.

Clearly, for all $s \in S$, we have $\mathbf{MT}(\mathbb{V}_s) \subseteq \mathbf{MT}^{\mathrm{gen}}$.

We conclude this section with a strong algebraization result on the Hodge locus and with a Theorem of André relating the generic Mumford-Tate group of a VHS with the algebraic monodromy group. For these results, we need to restrict our situation to an integral polarized variation of Hodge structures on a smooth connected quasi-projective algebraic variety over $\mathbb{C}$.

Theorem 2.7.3. *Let $(\mathbb{V}, \mathscr{F}^\bullet)$ be an integral polarized variation of Hodge structures on a smooth connected quasi-projective algebraic variety S. Then the Hodge locus $\mathrm{HL}(S, \mathbb{V}^\otimes)$ is a countable union of closed irreducible algebraic subvarieties of S, called special subvarieties.*

Proof. For the original proof we refer to Cattani-Deligne-Kaplan [17], while a different and more recent approach, which uses techniques of o-minimal geometry, can be found in Bakker-Klingler-Tsimerman [5]. □

Definition 2.7.4. Given a locally constant abelian sheaf $\mathbb{V}$ with stalk $V_{\mathbb{Z}}$ on a smooth connected quasi-projective algebraic variety S, the algebraic monodromy group $\mathbf{H}$ of $\mathbb{V}$ is the connected component of the identity of the Zariski closure of the image of the associated representation

$$\pi_1(S^{\mathrm{an}}, s_0) \to \mathbf{GL}(V_{\mathbb{Q}}),$$

seen as an algebraic group over $\mathbb{Q}$.

Theorem 2.7.5. *Let S be a smooth connected quasi-projective algebraic variety and let $\mathbb{V}$ be a locally constant abelian sheaf on S^{an} carrying an integral polarized VHS. Let $\mathbf{M}$ be its associated generic Mumford-Tate group and let $\mathbf{H}$ be the algebraic monodromy group of $\mathbb{V}$. Then:*

1) $\mathbf{H}$ *is a normal subgroup of the derived subgroup* $\mathbf{M}^{\mathrm{der}}$*,*

2) *if there exists $s \in S^{\mathrm{an}}$ is such that $\mathbf{MT}(\mathbb{V}_s)$ is a torus, then $\mathbf{H} = \mathbf{M}^{\mathrm{der}}$.*

Proof. See André [1]. □

Interlude: the algebraic point of view II

We plan to discuss here a more algebraic point of view on some topics of the previous Chapter, extending what we have done in the first interlude to the *relative* situation of a smooth projective morphism of smooth connected algebraic varieties over $\mathbb{C}$.

Let us start by giving a more algebraic construction of the Kodaira-Spencer map, in a particular but significant case, which gives a different point of view to interpret the Kodaira-Spencer map as a classifying map for the first order deformations of an algebraic variety (more general deformations of schemes are treated for instance in Sernesi [41]). Given an algebraic variety X over $\mathbb{C}$, a *first order deformation* of X is a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\mathbb{C}) & \longrightarrow & \operatorname{Spec}(\mathbb{C}[\epsilon]/\epsilon^2) \end{array}$$

where the morphism $\mathcal{X} \to \operatorname{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$ is flat. This construction can be done over any field but since we are interested in complex algebraic varieties here, let us keep working over $\mathbb{C}$.

Lemma 2.1. *Every first order deformation $\mathcal{X}$ of a smooth affine variety X is trivial, namely $\mathcal{X} \cong X \times_{\operatorname{Spec}(\mathbb{C})} \operatorname{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$.*

Proof. First of all let us prove that $\mathcal{X}$ is affine as well. By the cohomological characterization of affine schemes (see for instance Hartshorne [29], Theorem 3.7) it is enough to prove that every coherent sheaf of ideals on $\mathcal{X}$ has vanishing first cohomology group. Let $\mathscr{F}$ be such a coherent sheaf of $\mathscr{O}_{\mathcal{X}}$-modules. Since the morphism $\iota : X \to \mathcal{X}$ is a square zero thickening, namely it is the identity on topological spaces and is defined by a square zero ideal sheaf $\mathscr{I} \subseteq \mathscr{O}_{\mathcal{X}}$, we can construct, by intersecting with $\mathscr{I}$, a subsheaf $\mathscr{F}' \subseteq \mathscr{F}$ such that both $\mathscr{F}'$ and $\mathscr{F}/\mathscr{F}'$ are annihilated by $\mathscr{I}$. But any quasi-coherent sheaf $\mathscr{G}$ of $\mathscr{O}_{\mathcal{X}}$-modules which is annihilated by $\mathscr{I}$ is the pushforward along ι of a quasi-coherent sheaf on X, since the natural morphism

$$\mathscr{G} \to \iota_* \iota^* \mathscr{G} \cong \mathscr{G} \otimes \iota_* \mathscr{O}_X \cong \mathscr{G} \otimes \mathcal{O}_{\mathcal{X}}/\mathscr{I}$$

is an isomorphism if $\mathscr{I}\mathscr{G} = 0$. Thus, $\mathscr{F}'$ and $\mathscr{F}/\mathscr{F}'$ are pushforward of quasi-coherent sheaves on the affine scheme X, hence, since cohomology is preserved under ι_*, we have $H^1(\mathcal{X}, \mathscr{F}') = H^1(\mathcal{X}, \mathscr{F}/\mathscr{F}') = 0$, from which we immediately deduce $H^1(\mathcal{X}, \mathscr{F}) = 0$.

Now, let $X = \operatorname{Spec}(A)$ and $\mathcal{X} = \operatorname{Spec}(B)$, so that, by definition of first order deformation, we have $A = B \otimes_{\mathbb{C}[\epsilon]/\epsilon^2} \mathbb{C}$. Let also $A[\epsilon] = A \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]/(\epsilon^2)$. The morphism $\mathcal{X} \to \operatorname{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$

is smooth since flat and smooth on the unique fiber X, in other words B is a smooth $\mathbb{C}[\epsilon]/(\epsilon^2)$-algebra. Applying the infinitesimal lifting criterion for smoothness to the commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & A \\ \uparrow & \searrow & \uparrow \\ \mathbb{C}[\epsilon]/(\epsilon^2) & \longrightarrow & A[\epsilon] \end{array}$$

we obtain a morphism $\varphi : B \to A[\epsilon]$ making the triangles commutative: we claim that it is an isomorphism. Applying $(- \otimes_{\mathbb{C}[\epsilon]/\epsilon^2} \mathbb{C})$ to the exact sequence

$$B \longrightarrow A[\epsilon] \longrightarrow \operatorname{coker}(\varphi) \longrightarrow 0$$

and using $A = B \otimes_{\mathbb{C}[\epsilon]/\epsilon^2} \mathbb{C}$, we obtain that $\operatorname{coker}(\varphi) \otimes_{\mathbb{C}[\epsilon]/\epsilon^2} \mathbb{C} = 0$. Thus we have $\operatorname{coker}(\varphi) = 0$ by Nakayama's Lemma (which is applied to the nilpotent ideal $(\epsilon) \subseteq \mathbb{C}[\epsilon]/(\epsilon^2)$ hence does not require $\operatorname{coker}(\varphi)$ to be finetely generated as $\mathbb{C}[\epsilon]/(\epsilon^2)$-module), so φ is surjective. Finally, applying again $(- \otimes_{\mathbb{C}[\epsilon]/\epsilon^2} \mathbb{C})$ to the short exact sequence

$$0 \longrightarrow \ker(\varphi) \longrightarrow B \longrightarrow A[\epsilon] \longrightarrow 0$$

and using that $\operatorname{Tor}_1^{\mathbb{C}[\epsilon]/\epsilon^2}(B, \mathbb{C}) = \operatorname{Tor}_1^{\mathbb{C}[\epsilon]/\epsilon^2}(A[\epsilon], \mathbb{C}) = 0$ by flatness, we obtain $\ker(\varphi) \otimes_{\mathbb{C}[\epsilon]/\epsilon^2} \mathbb{C} = 0$. So we can conclude $\ker(\varphi) = 0$ by Nakayama's Lemma. This completes the proof. □

We want to show now that there is a natural way to associate to a first order deformation of a smooth projective variety X a cohomology class in $H^1(X, \mathscr{T}_X)$, where $\mathscr{T}_X$ is the tangent sheaf to X, which can be defined in algebraic terms as the dual of the sheaf of Kähler differentials on X, i.e. $\mathscr{T}_X = \mathscr{H}om_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X)$. Its analytification is the sheaf of holomorphic sections of the holomorphic tangent bundle of X^{an}, justifying the use of the same symbol.

Lemma 2.1.1. *Let R be a $\mathbb{C}$-algebra and let $R[\epsilon] = R \otimes_{\mathbb{C}} \mathbb{C}[\epsilon]/(\epsilon^2)$. We have $R[\epsilon] = R \oplus \epsilon R$ as a complex vector space and for $i = 1, 2$ we denote by π_i the projection of $R[\epsilon]$ onto the i-th factor of this splitting. Denote by $\operatorname{Aut}^0(R[\epsilon])$ the group of $\mathbb{C}[\epsilon]/(\epsilon^2)$-algebra automorphisms φ of $R[\epsilon]$ such that $\pi_1 \circ \varphi|_R = \mathrm{id}_R$ and let $\operatorname{Der}_{\mathbb{C}}(R)$ be the group of $\mathbb{C}$-derivations of R into itself. Then there is a natural isomorphism of groups*

$$\operatorname{Aut}^0(R[\epsilon]) \cong \operatorname{Der}_{\mathbb{C}}(R).$$

Proof. For each $\varphi \in \operatorname{Aut}^0(R[\epsilon])$ we can write $\varphi = \varphi_1 + \epsilon \cdot \varphi_2$, with $\varphi_i = \pi_i \circ \varphi$. We claim that the restriction of φ_2 to $R \subseteq R[\epsilon]$ is a $\mathbb{C}$-derivation of R into itself. Indeed it is clearly $\mathbb{C}$-linear and expanding

$$\varphi(rs) = \varphi(r)\varphi(s) = (\varphi_1(r) + \epsilon\varphi_2(r))(\varphi_1(s) + \epsilon\varphi_2(s)) = rs + \epsilon(r\varphi_2(s) + s\varphi_2(r))$$

for $r, s \in R$, we obtain the Leibniz rule $\varphi_2(rs) = r\varphi_2(s) + s\varphi_2(r)$.

Conversely, if $d : R \to R$ is a derivation over $\mathbb{C}$, one can easily check that

$$\varphi = \pi_1 + \epsilon \cdot ((d \circ \pi_1) + \pi_2) \in \operatorname{Aut}^0(R[\epsilon])$$

providing the inverse construction. Finally, it is immediate to check that these maps respect the group structures given by composition on $\operatorname{Aut}^0(R[\epsilon])$ and addition on $\operatorname{Der}_{\mathbb{C}}(R)$. □

Now, given a first order deformation $f : \mathcal{X} \to \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$ of X, we can find an affine open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X trivializing the morphism f, i.e. such that for each i there exists an isomorphism

$$\theta_i : U_i \times_{\mathrm{Spec}(\mathbb{C})} \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2) \to \mathcal{X}|_{U_i}$$

over $\mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$. Putting $U_{ij} = U_i \cap U_j$, we thus obtain isomorphisms

$$\theta_{ij} = \theta_i^{-1} \circ \theta_j : U_{ij} \times_{\mathrm{Spec}(\mathbb{C})} \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2) \xrightarrow{\cong} U_{ij} \times_{\mathrm{Spec}(\mathbb{C})} \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2).$$

Each U_{ij} is affine, say $U_{ij} = \mathrm{Spec}(R_{ij})$ for some $\mathbb{C}$-algebra R_{ij}, so the isomorphisms θ_{ij} correspond to automorphisms of $R_{ij}[\epsilon] = R_{ij} \otimes \mathbb{C}[\epsilon]/(\epsilon^2)$. By the previous lemma, this gives elements

$$d_{ij} \in \mathrm{Der}_{\mathbb{C}}(R_{ij}) \cong \Gamma(\mathrm{Spec}(R_{ij}), \mathscr{T}_X)$$

where the last isomorphism follows from the universal property of the sheaf of Kähler differentials, since

$$\Gamma(\mathrm{Spec}(R_{ij}), \mathscr{T}_X) = \mathrm{Hom}_{\mathscr{O}_{U_{ij}}}(\Omega^1_X|_{U_{ij}}, \mathscr{O}_{U_{ij}}) \cong \mathrm{Der}_{\mathbb{C}}(R_{ij}).$$

Clearly we have $\theta_{ij} \circ \theta_{jk} \circ \theta_{ik}^{-1} = \mathrm{id}$, hence $d_{ij} + d_{jk} - d_{ik} = 0$. Therefore, the elements $\{d_{ij}\}$ form a $\mathscr{T}_X$-valued 1-cocycle for the affine cover $\mathcal{U}$, giving rise to a class in the first Čech cohomology group of $\mathscr{T}_X$. It is easy to show that, if $\mathcal{X}' \to \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$ is a deformation which is isomorphic to f, the above procedure gives the same cohomology class in $\check{H}^1(X, \mathscr{T}_X)$. To summarize, we have associated to a first order deformation $f : \mathcal{X} \to \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$ of X a class in $H^1(X, \mathscr{T}_X)$ represented by the Čech cocycle $\{d_{ij}\}$.

Proposition 2.2. *In the above situation, the class represented by $\{d_{ij}\}$ coincides with the image under the Kodaira-Spencer map of the tangent vector $\frac{\partial}{\partial \epsilon}$ to $\mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$.*

Proof. We refer to Voisin [46] (section 9.1.2) for details, let us nevertheless sketch the argument here. Let α be the image of $\frac{\partial}{\partial \epsilon}$ under the Kodaira-Spencer map. Recall that this map is defined (adapting Definition 2.3.1 to our morphism $f : \mathcal{X} \to \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$) as the connecting morphism $\mathscr{T}_{\mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2),0} \to H^1(X, \mathscr{T}_X)$ arising from the short exact sequence of sheaves

$$0 \longrightarrow \mathscr{T}_X \longrightarrow \mathscr{T}_{\mathcal{X}}|_X \longrightarrow f^*\mathscr{T}_{\mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)}|_X \longrightarrow 0.$$

It follows that a 1-cocycle representing α in the first Čech cohomology group $\check{H}^1(\mathcal{U}, \mathscr{T}_X)$ with respect to the affine cover $\mathcal{U}$ of X can be computed by finding local liftings of $\frac{\partial}{\partial \epsilon}$ to vector fields on $\mathcal{X}|_{U_i}$ and applying the boundary map

$$d : \bigoplus_i \mathscr{T}_{\mathcal{X}}(U_i) \to \bigoplus_{i<j} \mathscr{T}_{\mathcal{X}}(U_{ij}), \quad d(\{\beta_i\}_i) = \{\beta_i - \beta_j\}_{i,j}$$

in the Čech complex. But such liftings are exactly induced by the trivializations

$$\theta_i : U_i \times_{\mathrm{Spec}(\mathbb{C})} \mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2) \to \mathcal{X}|_{U_i},$$

which provide splittings over U_i of the above short exact sequence. Following the construction of the $\mathscr{T}_X$-valued 1-cocycle $\{d_{ij}\}$ we thus obtain that α is represented by $\{d_{ij}\}$. □

Now, let us go back to the geometric situation where $f : \mathcal{X} \to S$ is a smooth projective morphism of smooth connected algebraic varieties over $\mathbb{C}$. Clearly, its analytification f^{an} is a proper holomorphic submersion of complex manifolds, whose fibers are Kähler, thus we can apply the machinery developed in the previous Chapter to construct a Hodge filtration of the holomorphic vector bundle $R^k f_*^{\mathrm{an}}\mathbb{Z} \otimes \mathscr{O}_{S^{\mathrm{an}}}$. We want to give an algebraic description of this construction, starting with showing that $R^k f_*^{\mathrm{an}}\mathbb{Z} \otimes \mathscr{O}_{S^{\mathrm{an}}}$ is the analytification of a coherent algebraic sheaf. To do so, let $\Omega^1_{\mathcal{X}/S}$ be the sheaf of *relative* Kähler differentials associated with the morphism f. It is a locally free sheaf on $\mathcal{X}$ sitting in the short exact sequence

$$0 \longrightarrow f^*\Omega^1_S \longrightarrow \Omega^1_{\mathcal{X}} \longrightarrow \Omega^1_{\mathcal{X}/S} \longrightarrow 0. \tag{2.5}$$

One can take wedge powers and construct the relative de Rham complex $\Omega^\bullet_{\mathcal{X}/S}$. Notice that we have

$$\Omega^p_{\mathcal{X}/S} = \bigwedge^p \Omega^1_{\mathcal{X}/S} \cong \frac{\Omega^p_X}{f^*\Omega^1_S \wedge \Omega^{p-1}_X}.$$

By the relative holomorphic Poincaré Lemma, we have a quasi isomorphism of complexes of sheaves

$$f^{-1}\mathscr{O}_{S^{\mathrm{an}}} \to (\Omega^{\mathrm{an}}_{\mathcal{X}/S})^\bullet$$

which gives, taking the hyperderived functors of f_*^{an},

$$\mathbf{R}^k f_*^{\mathrm{an}}(\Omega^{\mathrm{an}}_{\mathcal{X}/S})^\bullet \cong R^k f_*^{\mathrm{an}}(f^{-1}\mathscr{O}_{S^{\mathrm{an}}}).$$

Here, the first term is isomorphic to $(\mathbf{R}^k f_*\Omega^\bullet_{\mathcal{X}/S})^{\mathrm{an}}$ by (a relative version of) GAGA correspondence and the second term is isomorphic to $R^k f_*^{\mathrm{an}}\mathbb{Z}\otimes\mathscr{O}_{S^{\mathrm{an}}}$ by a derived version of the projection formula, therefore we get

$$(\mathbf{R}^k f_*\Omega^\bullet_{\mathcal{X}/S})^{\mathrm{an}} \cong R^k f_*^{\mathrm{an}}\mathbb{Z} \otimes \mathscr{O}_{S^{\mathrm{an}}}. \tag{2.6}$$

Hence $\mathbf{R}^k f_*\Omega^\bullet_{\mathcal{X}/S}$ is a locally free algebraic sheaf on S, whose fiber at a point s is

$$\mathbf{H}^k(X_s, \Omega^\bullet_{\mathcal{X}/S}|_{X_s}) \cong \mathbf{H}^k(X_s, \Omega^\bullet_{X_s}) \cong H^k(X_s^{\mathrm{an}}, \mathbb{C})$$

by the algebraic de Rham theorem.

Let us now construct a natural connection on $\mathbf{R}^k f_*\Omega^\bullet_{\mathcal{X}/S}$. Consider the Koszul filtration of the algebraic de Rham complex $\Omega^\bullet_{\mathcal{X}}$ defined by

$$\mathrm{Koz}^q\Omega^\bullet_{\mathcal{X}} = \mathrm{im}(f^*\Omega^q_S \wedge \Omega^\bullet_{\mathcal{X}}[-q] \to \Omega^\bullet_{\mathcal{X}})$$

where the map is the wedge product. Using the locally split exact sequence (2.5) we can compute its degree q graded part:

$$\mathrm{Gr}^q_{\mathrm{Koz}}\Omega^\bullet_{\mathcal{X}} = \mathrm{Koz}^q/\mathrm{Koz}^{q+1} \cong f^*\Omega^q_S \otimes \Omega^\bullet_{\mathcal{X}/S}[-q].$$

Therefore, the self-evident short exact sequence of complexes

$$0 \longrightarrow \mathrm{Gr}^1_{\mathrm{Koz}} \longrightarrow \mathrm{Koz}^0/\mathrm{Koz}^2 \longrightarrow \mathrm{Gr}^0_{\mathrm{Koz}} \longrightarrow 0$$

takes the form

$$0 \longrightarrow f^*\Omega^1_S \otimes \Omega^\bullet_{\mathcal{X}/S}[-1] \longrightarrow \mathrm{Koz}^0/\mathrm{Koz}^2 \longrightarrow \Omega^\bullet_{\mathcal{X}/S} \longrightarrow 0$$

which reduces, if $\dim S = 1$, to the short exact sequence

$$0 \longrightarrow f^*\Omega^1_S \otimes \Omega^\bullet_{\mathcal{X}/S}[-1] \longrightarrow \Omega^\bullet_{\mathcal{X}} \longrightarrow \Omega^\bullet_{\mathcal{X}/S} \longrightarrow 0$$

which can be deduced from (2.5) by taking wedge powers. Applying the hyperderived functors of f_* we get the connecting morphism

$$\mathbf{R}^k f_* \Omega^\bullet_{\mathcal{X}/S} \to \mathbf{R}^{k+1}(f^*\Omega^1_S \otimes \Omega^\bullet_{\mathcal{X}/S}[-1]) \cong \Omega^1_S \otimes \mathbf{R}^k f_* \Omega^\bullet_{\mathcal{X}/S}$$

where the last isomorphism is an application of the projection formula.

Proposition 2.3. *This connecting morphism coincides, after passing to analytification, with the usual Gauss-Manin connection.*

Proof. See Katz-Oda [32], or Bertin-Peters [11], Theorem 2.1. □

Now, recall from Proposition 1.4 that for each $s \in S$ the Hodge filtration on $H^k(X_s^{\mathrm{an}}, \mathbb{C})$ corresponds, under the isomorphism $\mathbf{H}^k(X_s, \Omega^\bullet_{X_s}) \to H^k(X_s^{\mathrm{an}}, \mathbb{C})$, to the naive filtration of the algebraic de Rham complex $\Omega^\bullet_{X_s}$. Clearly this extends to our relative case, so that, considering the naive filtration $\Omega^{\bullet \geq p}_{\mathcal{X}/S}$ of the relative de Rham complex for each p, we obtain coherent algebraic subsheaves $\mathbf{R}^k f_* \Omega^{\bullet \geq p}_{\mathcal{X}/S}$ of $\mathbf{R}^k f_* \Omega^\bullet_{\mathcal{X}/S}$, which induce, under analytification and isomorphism (2.6), the holomorphic subbundles $\mathscr{F}^p$ of $R^k f_*^{\mathrm{an}} \mathbb{Z} \otimes \mathscr{O}_{S^{\mathrm{an}}}$ defined in section 2.4. Indeed, the natural morphism of sheaves $\mathbf{R}^k f_* \Omega^{\bullet \geq p}_{\mathcal{X}/S} \to \mathbf{R}^k f_* \Omega^\bullet_{\mathcal{X}/S}$ coincides at the level of fibers with the natural inclusion $\mathbf{H}^k(X_s, \Omega^{\bullet \geq p}_{X_s}) \to \mathbf{H}^k(X_s, \Omega^\bullet_{X_s})$, so it is injective and under isomorphism (2.6) it identifies on the fiber at $s \in S$ with $F^p H^k(X_s^{\mathrm{an}}, \mathbb{C}) \to H^k(X_s^{\mathrm{an}}, \mathbb{C})$.

Now, observe that we have a morphism of short exact sequences of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^*\Omega^1_S \otimes (\Omega^{\bullet \geq p-1}_{\mathcal{X}/S})[-1] & \longrightarrow & (\mathrm{Koz}^0/\mathrm{Koz}^2)^{\geq p} & \longrightarrow & \Omega^{\bullet \geq p}_{\mathcal{X}/S} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & f^*\Omega^1_S \otimes \Omega^\bullet_{\mathcal{X}/S}[-1] & \longrightarrow & \mathrm{Koz}^0/\mathrm{Koz}^2 & \longrightarrow & \Omega^\bullet_{\mathcal{X}/S} & \longrightarrow & 0 \end{array}$$

This induces a commutative diagram for the connecting morphisms:

$$\begin{array}{ccc} \mathbf{R}^k f_* \Omega^{\bullet \geq p}_{\mathcal{X}/S} & \longrightarrow & \Omega^1_S \otimes \mathbf{R}^k f_* \Omega^{\bullet \geq p-1}_{\mathcal{X}/S} \\ \downarrow & & \downarrow \\ \mathbf{R}^k f_* \Omega^\bullet_{\mathcal{X}/S} & \xrightarrow{\nabla} & \Omega^1_S \otimes \mathbf{R}^k f_* \Omega^\bullet_{\mathcal{X}/S} \end{array}$$

Griffiths transversality $\nabla(\mathscr{F}^p) \subseteq \Omega^1_{S^{\mathrm{an}}} \otimes \mathscr{F}^{p-1}$ is now an immediate consequence of the commutativity of this diagram, once identified $\mathbf{R}^k f_* \Omega^{\bullet \geq p}_{\mathcal{X}/S}$ with $\mathscr{F}^p \subseteq R^k f_*^{\mathrm{an}} \mathbb{Z} \otimes \mathscr{O}_{S^{\mathrm{an}}}$ as described above.

To summarize, we have shown that the datum $(\mathbb{V}, \mathscr{F}^\bullet, \nabla)$ of an integral variation of Hodge structures coming from a smooth projective morphism $f : X \to S$ of smooth connected algebraic varieties is actually a collection of algebraic data: the holomorphic vector bundle associated with the locally constant sheaf $\mathbb{V} = R^k f_*^{\mathrm{an}} \mathbb{Z}$ and its subbundles $\mathscr{F}^p$ are the analytification of coherent algebraic sheaves on S and the flat connection ∇ associated with $\mathbb{V}$ is algebraic. Remarkably the same holds for any integral polarized VHS $(\mathbb{V}, \mathscr{F}^\bullet)$, not necessarily coming from geometry, on a smooth connected quasi-projective variety S. The strategy to prove this is to consider a smooth projective compactification Y of S such that $Y \setminus S$ is a divisor with simple normal crossing (such Y exists by Hironaka's resolution of singularites) and to show that the holomorphic vector bundle $\mathbb{V} \otimes \mathscr{O}_{S^{\mathrm{an}}}$ and its subbundles $\mathscr{F}^p$ extend to coherent analytic sheaves on the whole Y. This argument can be found in Schmid [40] and allows one to conclude the algebraicity of $\mathbb{V} \otimes \mathscr{O}_{S^{\mathrm{an}}}$ and of its subbundles $\mathscr{F}^p$ by applying GAGA correspondence to their extensions to Y.

Chapter 3

Density of the Hodge locus

In this third Chapter we focus on an interesting aspect of the study of the geometry of the Hodge locus of an integral polarized variation of Hodge structures on a smooth quasi-projective variety S. In particular we will present the proof of a criterion for the analytic density of the Hodge locus, recently proved by Khelifa-Urbanik [33].

This criterion fits into a more general line of work which was developed in the last years. The essential point of view that has been adopted to work in this direction consists in seeing Hodge loci as intersection loci: indeed the period map $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$ sends irreducible components of the Hodge locus to irreducible components of the intersection, inside $\Gamma\backslash D$, of the image of the period map with *special* subvarieties of $\Gamma\backslash D$ which arise as quotients of period sub-domains of D corresponding to Mumford-Tate groups that are smaller than the generic one. This perspective is introduced in section 3.1 and suggests to relate the geometry of special subvarieties of S with the geometry of the corresponding intersection in $\Gamma\backslash D$. This leads to the dichotomy between typical and atypical Hodge locus and to the (still open) Zilbert-Pink conjectures, that are stated at the end of the first section. Even though we will not cover any of the work that has been done in the direction of proving these conjectures, they provide a crucial inspiration for the density criterion we will discuss; furthermore, we will explain how they take form concretely in the applications that are presented in the final part of the Chapter.

Let us also mention here that an essential driving force behind the recent progress in the study of the Hodge locus was provided by the introduction of techniques of o-minimal geometry, which constitute, roughly speaking, a framework that is intermediate between complex analytic geometry and algebraic geometry. It turns out that the geometric objects arising from variations of Hodge structures fit in this framework, for instance, Hodge varieties and period maps are in general not algebraic, but they are definable in some appropriate o-minimal structure. That being said, developing o-minimal geometry would be a goal beyond the aims of the present work. For the interested reader we refer to Klingler's survey [34]. The only result that we use in the proof of the density criterion and whose proof strongly requires an o-minimal approach is essentially Ax-Schanuel Theorem for VHS. We will present it without proof in section 3.2, but we will discuss the heuristic behind this crucial result.

In section 3.3 we give the proof of the density criterion due to Khelifa-Urbanik and in section 3.4 we discuss some applications. In particular, we will show how this criterion can be used to obtain results on the density of the Hodge locus of some universal families of smooth projective hypersurfaces and complete intersections and on the density in $\mathcal{M}_g$ of families of curves with non-simple Jacobian.

The main references for this Chapter are Baldi-Klingler-Ullmo [9] and Klingler-Otwinowska

[35] for the discussion on Hodge loci as intersection loci, Bakker-Tsimerman [7] for the Ax-Schanuel Theorem and clearly Khelifa-Urbanik [33] for the density criterion we will prove.

3.1 Hodge loci as intersection loci

Notation 3.1.1. If $\mathbf{G}$ is a rational algebraic group, we denote by $\mathbf{G}^{\mathrm{ad}}$ its adjoint group and by $\mathbf{G}(\mathbb{R})^+$ the preimage under the natural morphism $\mathbf{G} \to \mathbf{G}^{\mathrm{ad}}$ of the identity connected component (for the real analytic topology) of $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$. Moreover, we define $\mathbf{G}(\mathbb{Q})^+ = \mathbf{G}(\mathbb{R})^+ \cap \mathbf{G}(\mathbb{Q})$.

Let us start by giving a generalization of the construction of period domains and Hodge varieties, described in section 2.6. If V is a rational vector space, then a Hodge structure on V can be described, by Proposition 1.2.7, as a real algebraic representation $\rho : \mathbf{S} \to \mathbf{GL}(V_{\mathbb{R}})$ of the Deligne torus. Let us fix such a representation ρ and let $\mathbf{MT}(\rho)$ be its associated Mumford-Tate group. The associated Mumford-Tate domain is the orbit of ρ in $\mathrm{Hom}(\mathbf{S}, \mathbf{MT}(\rho)_{\mathbb{R}})$ under the identity connected component $\mathbf{MT}(\rho)(\mathbb{R})^+$ of the real Lie group of $\mathbb{R}$-valued points of $\mathbf{MT}(\rho)$.

Notice that any Mumford-Tate domain associated with a rational polarized Hodge structure ρ embeds in the full period domain, constructed in section 2.6, classifying Hodge structures on V with the same Hodge numbers and polarization Q as the given one, which is, as stated in Theorem 2.6.2, an orbit of the group $G = \mathrm{Aut}(V_{\mathbb{R}}, Q)$. More precisely, since the Lie algebra of the group of real points of the Mumford-Tate group $\mathbf{MT}(\rho)$ is a Hodge sub-structure of the Hodge structure induced by ρ on the Lie algebra of G, the Mumford-Tate domain of $\mathbf{MT}(\rho)$ embeds as a complex submanifold of the full period domain (for details see Carlson-Peters-Müller-Stach [15], Proposition 15.3.2)

We can summarize this situation in the following definition, that we will use for the rest of the Chapter.

Definition 3.1.2. 1) A *Hodge datum* is a pair $(\mathbf{G}, D)$ where $\mathbf{G}$ is the Mumford-Tate group of some Hodge structure and D is its associated Mumford-Tate domain.

2) A morphism of Hodge data $(\mathbf{G}, D) \to (\mathbf{G}', D')$ is a morphism of rational algebraic groups $\mathbf{G} \to \mathbf{G}'$ sending D to D'.

3) A Hodge sub-datum of $(\mathbf{G}, D)$ is a Hodge datum $(\mathbf{G}', D')$ such that $\mathbf{G}'$ is a rational algebraic subgroup of $\mathbf{G}$ and the inclusion $\mathbf{G}' \to \mathbf{G}$ induces a morphism of Hodge data $(\mathbf{G}', D') \to (\mathbf{G}, D)$.

4) A *Hodge variety* is a quotient variety of the form $\Gamma \backslash D$ for some Hodge datum $(\mathbf{G}, D)$ and some torsion-free arithmetic lattice $\Gamma \subseteq \mathbf{G}(\mathbb{Q})^+$, that is a torsion-free subgroup commensurable with $\mathbf{G}(\mathbb{Z})^+$.

Notice that what we are defining here as *Hodge datum* is sometimes called in literature *connected Hodge datum*. Since we will only consider connected Hodge data, we found convenient to omit the adjective "connected" in these definitions as well as in the rest of the thesis.

Now, let S be a smooth connected quasi-projective variety over $\mathbb{C}$ and let $(\mathbb{V}, \mathscr{F}^\bullet)$ be an integral polarized variation of Hodge structures on S^{an}. As we have already discussed, the Mumford-Tate group of the induced Hodge structures on the stalks of $\mathbb{V}$ is locally constant outside a countable union of irreducible algebraic subvarieties of S, called special subvarieties. Fixing a Hodge generic point $s \in S^{\mathrm{an}} \smallsetminus \mathrm{HL}(S, \mathbb{V}^{\otimes})$ and its associated (generic) Mumford-Tate group $\mathbf{G}$ we can construct the Hodge datum $(\mathbf{G}, D)$, that we will call *generic Hodge datum* of the given polarized $\mathbb{Z}$-VHS.

If now Y is an algebraic subvariety of S, we can apply the same construction to the restriction $\mathbb{V}|_Y$ and obtain the generic Hodge datum $(\mathbf{G}_Y, D_{G_Y})$ of Y for $(\mathbb{V}, \mathscr{F}^\bullet)$. Clearly, we can consider it as a Hodge sub-datum of $(\mathbf{G}, D)$.

In view of the definitions we get immediately the following:

Proposition 3.1.3. *An irreducible algebraic subvariety Y of S is special if and only if it is maximal for the inclusion among irreducible algebraic subvarieties of S whose generic Mumford-Tate group is $\mathbf{G}_Y$.*

Clearly, here we are considering S itself as a special subvariety, thus we will refer to the irreducible components of the Hodge locus as *strict* special subvarieties.

Now recall that we can describe the given variation of Hodge structures by means of its period map $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$, where Γ is the monodromy group. Let π be the quotient map $D \to \Gamma\backslash D$. We notice that we can characterise special subvarieties of S as preimages under Φ of some specific analytic subvarieties of the Hodge variety $\Gamma\backslash D$, namely those which are Hodge varieties with respect to a Hodge sub-datum of $(\mathbf{G}, D)$:

Definition 3.1.4. Given a Hodge variety $\Gamma\backslash D$, a *special subvariety* of $\Gamma\backslash D$ is a subvariety of the form $\Gamma'\backslash D'$ for a Hodge sub-datum $(\mathbf{G}', D')$ of $(\mathbf{G}, D)$ and $\Gamma' = \Gamma \cap \mathbf{G}'(\mathbb{Q})^+$.

If Y is an irreducible algebraic subvariety of S with generic Hodge datum $(\mathbf{G}_Y, D_{G_Y})$ and $y \in Y$ is Hodge generic for the restriction of $(\mathbb{V}, \mathscr{F}^\bullet)$ to Y, we have that D_{G_Y} is the $\mathbf{G}_Y(\mathbb{R})^+$-orbit of z in D, where $z \in D$ is such that $\pi(z) = \Phi(y)$. Clearly, the quotient $\Gamma_{\mathbf{G}_Y}\backslash D_{G_Y}$, for $\Gamma_{\mathbf{G}_Y} = \Gamma \cap \mathbf{G}_Y(\mathbb{Q})^+$, is the smallest special subvariety of $\Gamma\backslash D$ containing $\Phi(Y)$, so Y is special if and only if it is an irreducible component of $\Phi^{-1}(\Gamma_{\mathbf{G}_Y}\backslash D_{G_Y})$. We thus have obtained:

Proposition 3.1.5. *An irreducible algebraic subvariety Y of S is special if and only if Y is an irreducible component of the preimage under Φ of a special subvariety of the Hodge variety $\Gamma\backslash D$.*

This characterisation allows us to see special subvarieties of S as coming from the intersection of the image of the period map with particular kind of analytic subvarieties of the Hodge variety $\Gamma\backslash D$. This turns out to be a very fruitful perspective. In particular it suggests to refine the description of the geometry of special subvarieties of S in terms of the type of their corresponding intersection in $\Gamma\backslash D$.

Definition 3.1.6. A special subvariety Y of S with generic Hodge datum $(\mathbf{G}_Y, D_{G_Y})$ is said to be *atypical* if $\Phi(S^{\mathrm{an}})$ and $\Gamma_{\mathbf{G}_Y}\backslash D_{G_Y}$ intersect along $\Phi(Y^{\mathrm{an}})$ with dimension bigger than expected, namely

$$\mathrm{codim}_{\Gamma\backslash D}\Phi(Y^{\mathrm{an}}) < \mathrm{codim}_{\Gamma\backslash D}\Phi(S^{\mathrm{an}}) + \mathrm{codim}_{\Gamma\backslash D}\Gamma_{\mathbf{G}_Y}\backslash D_{G_Y}$$

Otherwise, it is called *typical.*

Remark 3.1.7. This distinction can be found in Baldi-Klingler-Ullmo [9] with the difference that special subvarieties of S with image under Φ contained in the singular locus of $\Phi(S^{\mathrm{an}})$ were considered always atypical. Khelifa-Urbanik [33] (Remark 2.3) point out that this is a technical assumption, needed in order to consider tanget spaces, that can be harmlessly removed.

Then, the Hodge locus splits as

$$\mathrm{HL}(S, \mathbb{V}^{\otimes}) = \mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{typ}} \cup \mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{atyp}}$$

where the typical Hodge locus $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{typ}}$ (respectively $\mathrm{HL}(S, \mathbb{V}^{\otimes})_{\mathrm{atyp}}$) is the union of the *strict* typical (respectively atypical) special subvarieties of S. These two parts are expected to behave much differently. In particular, Baldi-Klingler-Ullmo formulated the following conjecture, known as Zilbert-Pink conjecture for variations of Hodge structures.

Conjecture 3.1.8. *Let* $(\mathbb{V}, \mathscr{F}^\bullet)$ *be an integral polarized variation of Hodge structures on a smooth connected quasi-projective variety* S*. Then:*

1) *the typical Hodge locus for* $(\mathbb{V}, \mathscr{F}^\bullet)$ *is either empty or dense in* S^{an} *for the complex analytic topology,*

2) *the atypical Hodge locus is algebraic, namely the set of atypical special subvarieties of* S *has finitely many maximal elements for the inclusion.*

This conjecture is still open. We refer to Baldi-Klingler-Ullmo [9] for the work that has been done in this direction, which is however limited, as for today, to the Hodge locus of positive period dimension.

3.2 Weakly special subvarieties and the Ax-Schanuel Theorem

As before, let $(\mathbb{V}, \mathscr{F}^\bullet)$ be an integral polarized variation of Hodge structures on a smooth connected quasi-projective variety S with generic Hodge datum $(\mathbf{G}, D)$ and period map $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$. Let $\mathbf{H}$ be the algebraic monodromy group of $\mathbb{V}$, defined in section 2.7. More generally, if Y is an irreducible subvariety of S, we can define the algebraic monodromy group $\mathbf{H}_Y$ of Y for $\mathbb{V}$ as the algebraic monodromy group of the restriction of $\mathbb{V}$ to the smooth locus of Y.

In analogy with Proposition 3.1.3, we define:

Definition 3.2.1. An irreducible algebraic subvariety Y of S is said to be *weakly special* for $\mathbb{V}$ if it is maximal for the inclusion among irreducible algebraic subvarieties of S which have algebraic monodromy group $\mathbf{H}_Y$.

Now, consider the Mumford-Tate domain D_H associated with the algebraic monodromy group $\mathbf{H}$, namely the $\mathbf{H}(\mathbb{R})^+$-orbit of ρ in $\mathrm{Hom}(\mathbf{S}, \mathbf{MT}(\rho)_{\mathbb{R}})$, for a representation ρ corresponding to the induced Hodge structure on the stalk $\mathbb{V}_s$ at some Hodge generic point $s \in S^{\mathrm{an}}$. In other words, we obtain a sub-domain of D by allowing conjugation of ρ only by elements of $\mathbf{H}$ instead of the whole $\mathbf{MT}(\rho)$. Notice that some authors refers to such a sub-domain of D as a *weak* Mumford-Tate domain. Let $\Gamma_{\mathbf{H}} = \Gamma \cap \mathbf{H}(\mathbb{Q})^+$.

Now, since $\mathbf{G}$ is reductive, we have that the derived subgroup $\mathbf{G}^{\mathrm{der}}$ is semisimple (see Appendix B, Proposition B.8), hence Theorem 2.7.5 implies that $\mathbf{G}^{\mathrm{der}}$ factors as an almost direct product $\mathbf{G}^{\mathrm{der}} = \mathbf{H} \cdot \mathbf{L}$. Therefore, we can consider the projection of the Hodge variety $\Gamma\backslash D$ onto the two factors $\Gamma_{\mathbf{H}}\backslash D_H$ and $\Gamma_{\mathbf{L}}\backslash D_L$.

Lemma 3.2.2. *The composition of the period map* $\Phi : S^{\mathrm{an}} \to \Gamma\backslash D$ *with the projection onto* $\Gamma_{\mathbf{L}}\backslash D_L$ *is constant equal to some Hodge generic point* $t_L \in \Gamma_{\mathbf{L}}\backslash D_L$.

Proof. See Klingler-Otwinowska [35], Lemma 3.12. □

Thus we can simply write $\Phi : S^{\mathrm{an}} \to \Gamma_{\mathbf{H}}\backslash D_H$ for the period map to mean $\Phi : S^{\mathrm{an}} \to \Gamma_{\mathbf{H}}\backslash D_H \times \{t_L\} \subseteq \Gamma\backslash D$.

Lemma 3.2.2 allows us to characterize weakly special subvarieties of S as preimages under the period map of a certain type of analytic subvarieties of $\Gamma\backslash D$.

Definition 3.2.3. Given a Hodge variety $\Gamma\backslash D$, a *weakly special subvariety* of $\Gamma\backslash D$ is either a special subvariety or the image of a subvariety of the form

$$\Gamma_{\mathbf{M}}\backslash D_M \times \{t\} \subseteq \Gamma_{\mathbf{M}}\backslash D_M \times \Gamma_{\mathbf{L}}\backslash D_L$$

under the morphism of Hodge varieties $\Gamma_{\mathbf{M}}\backslash D_M \times \Gamma_{\mathbf{L}}\backslash D_L \to \Gamma\backslash D$, where $(\mathbf{M}\times\mathbf{L}, D_M \times D_L)$ is a Hodge sub-datum of the adjoint Hodge datum $(\mathbf{G}^{\mathrm{ad}}, D)$, and t is a Hodge generic point of $\Gamma_{\mathbf{L}}\backslash D_L$.

Now, if the algebraic monodromy group of $\mathbb{V}$ shrinks along an irreducible subvariety Y of S, then Y is mapped by Φ into a weakly special subvariety of $\Gamma\backslash D$ by Lemma 3.2.2, hence, if Y is maximal among irreducible subvarieties of S having algebraic monodromy group $\mathbf{H}_Y$, then it is an irreducible component of the preimage of a weakly special subvariety of $\Gamma\backslash D$. Conversely, since intersections of weakly special subvarieties of a Hodge variety are again weakly special, an irreducible component of the preimage of a weakly special subvariety of $\Gamma\backslash D$ is weakly special in S. Thus, we have the following Proposition, we refer to Klingler-Otwinowska [35] (Definition 3.5 and Corollary 3.14) for more details.

Proposition 3.2.4. *An irreducible algebraic subvariety Y of S is weakly special if and only if it is an irreducible component of the preimage under Φ of a weakly special subvariety of the Hodge variety $\Gamma\backslash D$.*

The proof of the density criterion of Khelifa-Urbanik relies on a crucial result, which is the application of the so-called Ax-Schanuel principle to the context of variations of Hodge structures. In order to state it, we need to define a notion of an algebraic subvariety of a Mumford-Tate domain $D = D_G$. Such D is in general not algebraic, however, as we have shown for a full period domain in section 2.6, it admits an open immersion in a partial flag variety $\check{D}$, that is a quotient of $\mathbf{G}(\mathbb{C})$ by a parabolic subgroup (for this general case see for instance Green-Griffiths-Kerr [24], section II.B). Since $\check{D}$ is a projective algebraic variety, we can give the following definition:

Definition 3.2.5. An irreducible algebraic subvariety of a Mumford-Tate domain D is an analytic irreducible component of the intersection $D \cap Z^{\mathrm{an}}$ of D with an algebraic subvariety Z of the projective variety $\check{D}$.

Analogously, we can define an algebraic subvariety of $S \times D$ to be an irreducible component of the intersection of $S \times D$ with an algebraic subvariety of the algebraic variety $S \times \check{D}$.

Now, in the setting above, consider the fiber product diagram

$$\begin{array}{ccc} S \times_{\Gamma_{\mathbf{H}}\backslash D_H} D_H & \longrightarrow & D_H \\ \big\downarrow{\scriptstyle p} & & \big\downarrow \\ S & \xrightarrow{\ \Phi\ } & \Gamma_{\mathbf{H}}\backslash D_H \end{array}$$

Then we have the following result (we will refer to it as Ax-Schanuel Theorem), whose proof, due to Bakker-Tsimerman [6], uses techniques from o-minimal geometry which are beyond the aims of this work.

Theorem 3.2.6. *Let $W \subseteq S \times D_H$ be an algebraic subvariety and let U be an irreducible analytic component of $W \cap (S \times_{\Gamma_{\mathbf{H}}\backslash D_H} D_H)$ such that*

$$\mathrm{codim}_{S\times D_H} U < \mathrm{codim}_{S\times D_H} W + \mathrm{codim}_{S\times D_H}(S \times_{\Gamma_{\mathbf{H}}\backslash D_H} D_H).$$

Then the projection of U to S is contained in a strict weakly special subvariety of S for $\mathbb{V}$.

3.2.1 Functional transcendence, or the heuristic behind Ax-Schanuel Theorem

We want to discuss here the heuristic behind the previous Theorem. In particular, we will show, using a simple explicit example, how we can see Ax-Schanuel Theorem as a statement of *functional transcendence*. To provide the natural context, we have to start with the following:

Definition 3.2.7. A bi-algebraic structure on a complex connected algebraic variety S is a pair

$$(\tilde{\Phi} : \widetilde{S^{an}} \to X^{an}, \rho : \pi_1(S^{an}) \to \mathrm{Aut}(X))$$

where $\pi : \widetilde{S^{an}} \to S^{an}$ is the universal cover of S^{an}, X is a complex algebraic variety, $\mathrm{Aut}(X)$ is its group of algebraic automorphisms, ρ is a group morphism and $\tilde{\Phi}$ is a ρ-equivariant holomorphic map.

This datum allows to emulate an algebraic structure on the universal cover of S^{an}.

Definition 3.2.8. Let S be a complex connected algebraic variety endowed with a bi-algebraic structure $(\tilde{\Phi}, \rho)$.

1) An irreducible analytic subvariety Z of $\widetilde{S^{an}}$ is said to be a closed irreducible algebraic subvariety of $\widetilde{S^{an}}$ if Z is an analytic irreducible component of $\tilde{\Phi}^{-1}(\overline{\tilde{\Phi}(Z)}^{Zar})$, where $\overline{\tilde{\Phi}(Z)}^{Zar}$ denotes the Zariski closure of $\tilde{\Phi}(Z)$ in the algebraic variety X.

2) A closed irreducible algebraic (in the sense of point (1)) subvariety Z of $\widetilde{S^{an}}$ is said to be bi-algebraic if $\pi(Z)$ is a closed algebraic subvariety of S.

3) A closed irreducible algebraic subvariety Y of S is said to be bi-algebraic if any (equivalently one) analytic irreducible component of $\pi^{-1}(Y^{an})$ is a closed irreducible algebraic subvariety of $\widetilde{S^{an}}$ (in the sense of point (1)).

Now, if we put ourselves in the usual situation where $(\mathbb{V}, \mathscr{F}^\bullet)$ is a polarized $\mathbb{Z}$-VHS on a smooth connected quasi-projective algebraic variety S, with generic Hodge datum $(\mathbf{G}, D)$ and period map $\Phi : S^{an} \to \Gamma\backslash D$, we have a canonical bi-algebraic structure on S given by

$$\tilde{\Phi} : \widetilde{S^{an}} \to \check{D}^{an}.$$

Recall that the period map $\tilde{\Phi}$ (at the level of the universal cover) actually takes value in the Mumford-Tate domain D, here we are composing it with the open embedding of D in the flag variety $\check{D}$.

Definition 3.2.9. Let $(\mathbb{V}, \mathscr{F}^\bullet)$ be a polarized $\mathbb{Z}$-VHS on a smooth connected quasi-projective algebraic variety S. The bi-algebraic structure on S defined by $(\mathbb{V}, \mathscr{F}^\bullet)$ is the pair

$$(\tilde{\Phi} : \widetilde{S^{an}} \to \check{D}^{an}, \rho = (\Phi)_* : \pi_1(S^{an}) \to \Gamma \subseteq \mathbf{G}(\mathbb{C})).$$

The following result of Klingler-Otwinowska ([35], Proposition 3.20) relates this bi-algebraic structure with weakly special subvarieties of S.

Proposition 3.2.10. *In the above situation, the weakly special subvarieties of S for $(\mathbb{V}, \mathscr{F}^\bullet)$ are exactly the bi-algebraic subvarietes of S for the bi-algebraic structure on S defined by $(\mathbb{V}, \mathscr{F}^\bullet)$.*

Proof. Let Y be a weakly special subvariety of S. Then, by Proposition 3.2.4, it is an irreducible component of the preimage under Φ of a weakly special subvariety of the Hodge variety $\Gamma\backslash D$, that is the image in $\Gamma\backslash D$ of $\Gamma_{\mathbf{M}}\backslash D_M \times \{t\}$, for a Hodge sub-datum $(\mathbf{M} \times \mathbf{L}, D_M \times D_L)$ of the adjoint Hodge datum of the generic $(\mathbf{G}, D)$. The morphism

$$\Gamma_{\mathbf{M}}\backslash D_M \times \{t\} \to \Gamma\backslash D$$

is defined at the level of universal cover by a closed analytic immersion of the Mumford-Tate domains $D_M \hookrightarrow D$, that is the restriction of a closed algebraic immersion of their associated flag varieties $\iota : \check{D}_M \hookrightarrow \check{D}$. Thus, if Z is an irreducible analytic component of $\pi^{-1}(Y^{\mathrm{an}}) \subseteq \widetilde{S^{\mathrm{an}}}$, then Z is an irreducible component of the preimage under $\tilde{\Phi}$ of

$$\iota(\check{D}_M) = \overline{\iota(D_M)}^{\mathrm{Zar}} = \overline{\tilde{\Phi}(Z)}^{\mathrm{Zar}},$$

where the Zariski closure is taken in $\check{D}$. Hence, Z is algebraic in $\widetilde{S^{\mathrm{an}}}$ in the sense of Definition 3.2.8 and Y is bi-algebraic in S.

Conversely, assume that Y is a bi-algebraic subvariety of S. By Lemma 3.2.2, the restriction of Φ to Y factors through the weakly special subvariety $\Gamma_{\mathbf{H}_Y} \backslash D_{H_Y} \times \{t\}$ of $\Gamma\backslash D$, where $\mathbf{H}_Y$ is the algebraic monodromy group of the restriction of $\mathbb{V}$ to Y. Let Z be an irreducible component of $\pi^{-1}(Y^{\mathrm{an}})$ and consider the lifting $\tilde{\Phi}|_Z : Z \to D_{H_Y}$. Since Z is algebraic in $\widetilde{S^{\mathrm{an}}}$, we have that Z is an irreducible component of $\tilde{\Phi}^{-1}(\overline{\tilde{\Phi}(Z)}^{\mathrm{Zar}})$, hence $\overline{\tilde{\Phi}(Z)}^{\mathrm{Zar}}$ has to be stable under the action of the monodromy group $\mathbf{H}_Y(\mathbb{C})$. But then $\overline{\tilde{\Phi}(Z)}^{\mathrm{Zar}} = \check{D}_{H_Y}$. Therefore Z is an irreducible component of $\tilde{\Phi}^{-1}(D_{H_Y})$ and Y is an irreducible component of $\Phi^{-1}(\Gamma_{\mathbf{H}_Y}\backslash D_{H_Y} \times \{t\})$, so it is weakly special. □

The heuristic idea of the Ax-Schanuel principle is that intersections of algebraic subvarieties of the product $S \times D$ with $S \times_{\Gamma\backslash D} D$ that have bigger dimension than expected are controlled by the bi-algebraic (equivalently, weakly special) subvarieties of S. The following toy example illustrates how this fact can be seen as a statement of functional transcendence.

Consider the universal cover $\pi : \mathbb{C}^n \to (\mathbb{C}^\times)^n$, where

$$\pi(z_1, \dots, z_n) = (e^{z_1}, \dots, e^{z_n}),$$

and the bi-algebraic structure on $(\mathbb{C}^\times)^n$ given by the identity on $\mathbb{C}^n$. Even though this is not the bi-algebraic structure induced by a *pure* variation of Hodge structures it is a significant example to show our point. We will see how the natural analog of the Ax-Schanuel Theorem for this example follows from the following result.

Proposition 3.2.11. *Let $f_1, \dots, f_n$ be elements in the ring of formal power series $\mathbb{C}[[z_1, \dots, z_m]]$ which are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$, that is there is no non zero linear relation of the form*

$$r_1 f_1 + \cdots + r_n f_n = w$$

with $r_1, \dots, r_n \in \mathbb{Q}$ and $w \in \mathbb{C}$. Then

$$\mathrm{trdeg}_{\mathbb{C}} \mathbb{C}(f_1, \dots, f_n, e^{f_1}, \dots, e^{f_n}) \geq n + \mathrm{rk}(J(f_\bullet)),$$

where $J(f_\bullet) = \left(\frac{\partial f_i}{\partial z_j}\right)_{i,j}$ is the Jacobian matrix of $f_1, \dots, f_n$, seen as a matrix with entries in $\mathbb{C}((z_1, \dots, z_m))$.

Remark 3.2.12. This result, proved by Ax [3] (Theorem 3), can be seen as the functional analog of the famous (and still open) Schanuel's conjecture: if $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ are $\mathbb{Q}$-linearly independent, then

$$\mathrm{trdeg}_{\mathbb{Q}} \mathbb{Q}(\alpha_1, \dots, \alpha_n, e^{\alpha_1}, \dots, e^{\alpha_n}) \geq n.$$

Now, Ax-Schanuel Theorem in our simple example takes the following form:

Theorem 3.2.13. *Let V be the graph of $\pi : \mathbb{C}^n \to (\mathbb{C}^\times)^n$ in $\mathbb{C}^n \times (\mathbb{C}^\times)^n$ and let W be an algebraic subvariety of $\mathbb{C}^n \times (\mathbb{C}^\times)^n$. Suppose V and W intersect along some irreducible analytic subvariety U with bigger dimension than expected, namely*

$$\mathrm{codim}_{\mathbb{C}^n \times (\mathbb{C}^\times)^n} U < \mathrm{codim}_{\mathbb{C}^n \times (\mathbb{C}^\times)^n} W + \mathrm{codim}_{\mathbb{C}^n \times (\mathbb{C}^\times)^n} V.$$

Then the projection of U on $\mathbb{C}^n$ is contained in a strict bi-algebraic subvariety of $\mathbb{C}^n$.

Proof. Let us argue by contradiction, so assume that $p_1(U)$ is not contained in any strict bi-algebraic subvariety of $\mathbb{C}^n$, where p_1 is the projection onto the first component. Let $f = (f_1, \ldots, f_n) : \Delta^m \to p_1(U) \subseteq \mathbb{C}^n$ be a local holomorphic parametrization of $p_1(U)$, where Δ denotes the unit disk in $\mathbb{C}$. Then we clearly have

$$\dim U = \dim p_1(U) = \mathrm{rk}(J(f_\bullet)).$$

Furthermore, our contradiction hypothesis implies that $f_1, \ldots, f_n$ are $\mathbb{Q}$-linearly independent modulo $\mathbb{C}$. Indeed, any non zero relation of the form

$$r_1 f_1 + \cdots + r_n f_n = w$$

for $r_1, \ldots, r_n \in \mathbb{Q}$ and $w \in \mathbb{C}$ would give a non trivial algebraic relation between the exponentials, namely

$$\prod_{i=1}^{n} e^{r_i f_i} = e^w,$$

thus $p_1(U)$ would be contained in the linear subvariety $V(r_1 X_1 + \cdots + r_n X_n - w)$ of $\mathbb{C}^n$, which is bi-algebraic since π maps it to the algebraic subvariety $V(\prod_{i=1}^{n} Y_i^{r_i} - e^w)$ of $(\mathbb{C}^\times)^n$.

Consider the Zariski closure $\overline{U}^{\mathrm{Zar}}$ of U in $\mathbb{C}^n \times (\mathbb{C}^\times)^n$. Since W is algebraic we have $\overline{U}^{\mathrm{Zar}} \subseteq W$, so $\dim W \geq \dim \overline{U}^{\mathrm{Zar}}$. Now, it is a basic fact in Algebraic Geometry that the dimension of an irreducible algebraic variety is equal to the transcendence degree (over the base field) of its field of rational functions. In our case, we have that

$$\mathbb{C}(f_1, \ldots, f_n, e^{f_1}, \ldots, e^{f_n})$$

is contained in the field of rational functions on $\overline{U}^{\mathrm{Zar}}$, since $f_1, \ldots, f_n$ parametrize locally the projection of U to $\mathbb{C}^n$ and their exponentials parametrize its projection on $(\mathbb{C}^\times)^n$, as U lies in the graph of π. Therefore

$$\dim W \geq \dim \overline{U}^{\mathrm{Zar}} \geq \mathrm{trdeg}_{\mathbb{C}} \mathbb{C}(f_1, \ldots, f_n, e^{f_1}, \ldots, e^{f_n}).$$

By Proposition 3.2.11 we thus obtain

$$\dim W \geq n + \mathrm{rk}(J(f_\bullet)) = n + \dim U = \dim U + \dim(\mathbb{C}^n \times (\mathbb{C}^\times)^n) - \dim V,$$

which is precisely, once we pass to codimensions, the opposite inequality of that in the assumption of the Theorem, namely it imposes that V and W intersect along U with expected dimension. We thus have reached a contradiction and the Theorem is proven. □

3.3 A density criterion

In this section we will finally state and prove an effective criterion for the density of the Hodge locus, due to Khelifa-Urbanik [33]. As always, let $(\mathbb{V}, \mathscr{F}^\bullet)$ be an integral polarized variation of Hodge structures on a smooth connected quasi-projective variety S with generic Hodge datum $(\mathbf{G}, D)$ and period map Φ. Denote by π the quotient map $D \to \Gamma\backslash D$.

Definition 3.3.1. Let $(\mathbf{M}, D_M)$ be a strict Hodge sub-datum of $(\mathbf{G}, D)$. The Hodge locus of type $\mathbf{M}$ is

$$\mathrm{HL}(S, \mathbb{V}^\otimes, \mathbf{M}) = \{s \in S^{\mathrm{an}} : \mathbf{MT}(\mathbb{V}_s) \subseteq g\mathbf{M}g^{-1} \text{ for some } g \in \mathbf{G}(\mathbb{Q})^+\} \subseteq \mathrm{HL}(S, \mathbb{V}^\otimes).$$

Notice that, if Y is an irreducible component of the Hodge locus of type $\mathbf{M}$, for some strict Hodge sub-datum $(\mathbf{M}, D_M)$ of $(\mathbf{G}, D)$, and $(\mathbf{G}_Y, D_{G_Y})$ is its generic Hodge datum, then we have that $\mathbf{G}_Y$ is contained in some rational translate of $\mathbf{M}$ and

$$\operatorname{codim}_{\Gamma\backslash D}\Phi(Y^{\mathrm{an}}) \leq \operatorname{codim}_{\Gamma\backslash D}\Phi(S^{\mathrm{an}}) + \operatorname{codim}_{\Gamma\backslash D}\Gamma_{\mathbf{G}_Y}\backslash D_{G_Y},$$

which is equivalent to

$$\dim \Phi(Y^{\mathrm{an}}) \geq \dim \Phi(S^{\mathrm{an}}) + \dim D_{G_Y} - \dim D,$$

with equality if and only if Y is typical. It follows, since $\dim D_M \geq \dim D_{G_Y}$, that a necessary condition for the existence of typical special subvarieties of type $\mathbf{M}$ is the inequality

$$\dim \Phi(S^{\mathrm{an}}) + \dim D_M - \dim D \geq 0.$$

Therefore, having in mind Conjecture 3.1.8, we expect this inequality to be a necessary condition for the density of the Hodge locus of type $\mathbf{M}$. In the following criterion we will show that this is not far from being also a sufficient condition.

Let us first state the theorem in a simplified case, in order to fix the ideas. This version of the theorem will actually be enough for the geometric applications we will discuss in the next section.

Theorem 3.3.2. *Let us put ourselves in the setting above. Furthermore, assume that the algebraic monodromy group of $\mathbb{V}$ is $\mathbf{H} = \mathbf{G}^{\mathrm{der}}$ and is $\mathbb{Q}$-simple. Let $(\mathbf{M}, D_M)$ be a strict Hodge sub-datum of $(\mathbf{G}, D)$ such that*

$$\dim \Phi(S^{\mathrm{an}}) + \dim D_M - \dim D \geq 0. \tag{3.1}$$

Then the Hodge locus of type $\mathbf{M}$ is analytically dense in S^{an}.

Proof. Let $s \in S^{\mathrm{an}}$ be a Hodge generic point, fix $z \in D$ such that $\pi(z) = \Phi(s) = x$ and let $\Gamma_{\mathbf{M}} = \Gamma \cap \mathbf{M}(\mathbb{Q})^+$. By Definition 3.3.1 and Proposition 3.1.5, any point in S^{an} which lies in the preimage under Φ of a rational translate of the special subvariety $\Gamma_{\mathbf{M}}\backslash D_M$ belongs to the Hodge locus of type $\mathbf{M}$. Thus we have to show that for every neighborhood V of x in $\Gamma\backslash D$ there exists a rational translate of $\Gamma_{\mathbf{M}}\backslash D_M$ intersecting $V \cap \Phi(S^{\mathrm{an}})$. Since Hodge generic points are clearly dense in S^{an}, this implies the density of the Hodge locus of type $\mathbf{M}$, which is our claim.

Now, since $\mathbf{G}(\mathbb{R})^+$ acts transitively on D, there exists $g \in \mathbf{G}(\mathbb{R})^+$ such that $z \in g \cdot D_M$, hence $x \in \pi(g \cdot D_M)$. Let U be an irreducible analytic component of the intersection

$$\pi(g \cdot D_M) \cap \Phi(S^{\mathrm{an}})$$

containing x. We will show that the Theorem follows from the following:

Claim 3.3.3. *The analytic variety U has the expected dimension as irreducible component of the intersection $\pi(g \cdot D_M) \cap \Phi(S^{\mathrm{an}})$ inside $\Gamma\backslash D$, namely*

$$\dim U = \dim \Phi(S^{\mathrm{an}}) + \dim D_M - \dim D \geq 0 \tag{3.2}$$

Let us first deduce the Theorem from this. Notice that the condition (3.2) is an open condition in $g \in \mathbf{G}(\mathbb{R})^+$, namely there exists an open neighborhood $\mathcal{V}$ of g in $\mathbf{G}(\mathbb{R})^+$ such that $\pi(h \cdot D_M)$ intersects $\Phi(S^{\mathrm{an}})$ with expected dimension along some irreducible analytic component for all $h \in \mathcal{V}$. Now, by Borel [13], Theorem 18.2, $\mathbf{G}$ is unirational as algebraic variety over $\mathbb{Q}$, hence $\mathbf{G}(\mathbb{Q})^+$ is dense in $\mathbf{G}(\mathbb{R})^+$. So, $\mathbf{G}(\mathbb{Q})^+ \cap \mathcal{V}$ is non empty and for every $h \in \mathbf{G}(\mathbb{Q})^+ \cap \mathcal{V}$ we have that $\pi(h \cdot D_M)$ intersects $\Phi(S^{\mathrm{an}})$ with expected (non-negative) dimension along some irreducible analytic component. In this way, we can construct $\mathbf{G}(\mathbb{Q})^+$-translates of D_M intersecting $\Phi(S^{\mathrm{an}})$ with expected (non-negative) dimension in the neighborhood of any Hodge generic point. Indeed, if V is any open neighborhood of x in $\Gamma\backslash D$, then, taking $\mathcal{V}$ small enough, $\pi(h \cdot D_M)$ will intersect $\Phi(S^{\mathrm{an}}) \cap V$ for every $h \in \mathbf{G}(\mathbb{Q})^+ \cap \mathcal{V}$, namely

$$\pi(h \cdot D_M) \cap V \cap \Phi(S^{\mathrm{an}})$$

is non empty, as we wanted. □

It remains to prove Claim 3.3.3.

Proof. Assume by contradiction that there exists an irreducible component U of the intersection $\pi(g \cdot D_M) \cap \Phi(S^{\mathrm{an}})$ such that $x \in U$ and

$$\dim U > \dim \Phi(S^{\mathrm{an}}) + \dim D_M - \dim D.$$

Consider the subvariety $W = S \times g \cdot D_M$ of $S \times D$, which is algebraic in the sense of Definition 3.2.5. Let $\tilde{U}$ be the irreducible analytic component of the intersection $W \cap (S \times_{\Gamma\backslash D} D)$ containing (s, z) such that $\Phi(p_S(\tilde{U})) = U$, where p_S denotes the projection onto S and we have kept the same notation as before. Using the definition of W and our contradiction hypotheses, we get

$$\begin{aligned}\dim U &> \dim \Phi(S^{\mathrm{an}}) + \dim D_M - \dim D \\ &= \dim \Phi(S^{\mathrm{an}}) + \dim \pi(g \cdot D_M) - \dim D \\ &= \dim \Phi(S^{\mathrm{an}}) + \dim W - \dim S - \dim D \\ &= \dim(S \times_{\Gamma\backslash D} D) + \dim W - \dim(S \times D)\end{aligned}$$

which implies, since $\Phi(p_S(\tilde{U})) = U$, $\dim \tilde{U} > \dim W + \dim(S \times_{\Gamma\backslash D} D) - \dim(S \times D)$. Passing to the codimension and recalling that we are assuming $\mathbf{H} = \mathbf{G}^{\mathrm{der}}$, this is exactly the assumption of Theorem 3.2.6, hence we obtain that $p_S(\tilde{U})$ is contained in a strict weakly special subvariety Y. Let $\mathbf{G}_Y$ be its associated Mumford-Tate group and $\mathbf{H}_Y$ be its algebraic monodromy group: then, by Theorem 2.7.5, we have

$$\mathbf{H}_Y \trianglelefteq \mathbf{G}_Y^{\mathrm{der}} = \mathbf{G}^{\mathrm{der}} = \mathbf{H}$$

where the last equality is in our assumptions and $\mathbf{G}_Y = \mathbf{G}$ since Y contains by construction the Hodge generic point s. Hence, since Y is a strict weakly special subvariety, $\mathbf{H}_Y$ must be a proper connected normal subgroup of $\mathbf{H}$, which is $\mathbb{Q}$-simple by assumption. Thus $\mathbf{H}_Y = 0$, so the image of Y under the period map is a point. But $\Phi(Y)$ contains U by construction, so this contradicts

$$\dim U > \dim \Phi(S^{\mathrm{an}}) + \dim D_M - \dim D \geq 0.$$

□

Let us now deal with the general case, namely removing the assumptions $\mathbf{H} = \mathbf{G}^{\mathrm{der}}$ and $\mathbf{H}$ being $\mathbb{Q}$-simple. In general, the algebraic monodromy group $\mathbf{H}$ decomposes as the almost direct product of its $\mathbb{Q}$-simple factors: $\mathbf{H} = \mathbf{H}_1 \cdots \mathbf{H}_n$. Up to replacing S by a finite étale cover, this decomposition induces a factorization of the period map

$$\Phi : S^{\mathrm{an}} \to \Gamma_{\mathbf{H}}\backslash D_H = \Gamma_1\backslash D_1 \times \cdots \times \Gamma_n\backslash D_n$$

where we have denoted by D_i the Mumford-Tate domain associated with $\mathbf{H}_i^{\mathrm{ad}}$ and $\Gamma_i = \Gamma \cap \mathbf{H}_i^{\mathrm{ad}}(\mathbb{Q})^+$. For each $I \subseteq \{1, \ldots, n\}$, let $D_I = \prod_{i \in I} D_i$, let

$$p_I : \Gamma \backslash D \to \Gamma_I \backslash D_I = \prod_{i \in I} \Gamma_i \backslash D_i$$

be the projection and put $\Phi_I = p_I \circ \Phi$, $\pi_I = p_I \circ \pi$.

To give the general version of Theorem 3.3.2, we will assume that the condition (3.1) holds on each factor of this decomposition. The following example shows that we should expect this to be a necessary adjustment of the criterion.

Example 3.3.4. Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g (see Appendix A for details on its construction). One has

$$\dim \mathcal{A}_g = \frac{g(g+1)}{2}$$

for which we refer again to Appendix A. Consider a Hodge generic curve C in $\mathcal{A}_3$ and the integral variation of Hodge structures with period map the inclusion

$$\mathcal{A}_3 \times C \hookrightarrow \mathcal{A}_3 \times \mathcal{A}_3.$$

Let $(\mathbf{M}, D_M)$ be the strict Hodge sub-datum whose associated Mumford-Tate domain is

$$\Gamma_{\mathbf{M}} \backslash D_M = \mathcal{A}_3 \times (\mathcal{A}_1 \times \mathcal{A}_2) \hookrightarrow \mathcal{A}_3 \times \mathcal{A}_3.$$

We have

$$\dim(\mathcal{A}_3 \times C) + \dim(\mathcal{A}_3 \times (\mathcal{A}_1 \times \mathcal{A}_2)) - \dim(\mathcal{A}_3 \times \mathcal{A}_3) = 7 + 10 - 12 = 5$$

so $(\mathbf{M}, D_M)$ satisfies condition (3.1). However, if the Hodge locus of type $\mathbf{M}$ in $\mathcal{A}_3 \times C$ were analytically dense, then we would have that the curve C intersect rational translates of $\mathcal{A}_1 \times \mathcal{A}_2$ in a dense subset. But now, consider the variation of Hodge structures on C with period map the inclusion $C \hookrightarrow \mathcal{A}_3$: we have $\operatorname{codim}_{\mathcal{A}_3} C = 5$ and $\operatorname{codim}_{\mathcal{A}_3}(\mathcal{A}_1 \times \mathcal{A}_2) = 2$, so every special subvariety of C for this VHS (which is just a point in C) has to be atypical in the sense of Definition 3.1.6. Hence the density in C of the intersections of C with rational translates of $\mathcal{A}_1 \times \mathcal{A}_2$ would violate Conjecture 3.1.8, as the Hodge locus in C is conjectured to be algebraic, therefore not analytically dense.

Thus, in the general case we replace assumption (3.1) with the following stronger condition.

Definition 3.3.5. A strict Hodge sub-datum $(\mathbf{M}, D_M)$ of $(\mathbf{G}, D)$ is said to be factorwise $\mathbb{V}$-admissible if for every non empty set of indexes $I \subseteq \{1, \ldots, n\}$ we have

$$\dim \Phi_I(S^{\mathrm{an}}) + \dim \pi_I(D_M) - \dim D_I \geq 0.$$

Definition 3.3.6. If the derived subgroup of the generic Mumford-Tate group factors as an almost direct product $\mathbf{G}^{\mathrm{der}} = \mathbf{H} \cdot \mathbf{L}$, then a strict Hodge sub-datum $(\mathbf{M}, D_M)$ of $(\mathbf{G}, D)$ is said to be full on the $\mathbf{L}$-factor if $\mathbf{L}$ is contained in $\mathbf{M}$.

Let us now reformulate Definition 3.3.5 in a more technical way, which will turn out to be more suited to the proof. Write the decomposition of the derived Mumford-Tate group as $\mathbf{G}^{\mathrm{der}} = \mathbf{H} \cdot \mathbf{L}$.

Lemma 3.3.7. *Let $(\mathbf{M}, D_M)$ be a strict Hodge sub-datum of $(\mathbf{G}, D)$, let $I \subseteq \{1, \dots, n\}$ be a non empty set of indexes and denote by I^c its complement. Pick $g \in \mathbf{G}(\mathbb{R})^+$ and $t \in D_{I^c} \times D_L$ a Hodge generic point such that*

$$\Phi(S^{\mathrm{an}}) \cap \pi(g \cdot D_M) \cap \pi(D_I \times \{t\})$$

is not empty. Then, up to replacing S by some non empty Zariski open subset, the quantity

$$d_I(\mathbf{M}, D_M) = \dim((g \cdot D_M) \cap (D_I \times \{t\})) + \dim(\Phi(S^{\mathrm{an}}) \cap \pi(D_I \times \{t\})) - \dim D_I \tag{3.3}$$

only depends on I and not on g and t.

Proof. Take any other choice of $h \in \mathbf{G}(\mathbb{R})^+$ and t'. First, let us prove

$$\dim((g \cdot D_M) \cap (D_I \times \{t\})) = \dim((h \cdot D_M) \cap (D_I \times \{t'\})).$$

Take $z \in (g \cdot D_M) \cap (D_I \times \{t\})$ and $z' \in (h \cdot D_M) \cap (D_I \times \{t'\})$. Since $h\mathbf{M}(\mathbb{R})^+ h^{-1}$ acts transitively on $h \cdot D_M$, we can find $m \in h\mathbf{M}(\mathbb{R})^+ h^{-1}$ sending $(hg^{-1}) \cdot z$ to z'. Then

$$z' \in (mhg^{-1}) \cdot ((g \cdot D_M) \cap (D_I \times \{t\})) = (h \cdot D_M) \cap (mhg^{-1}) \cdot (D_I \times \{t\}).$$

Now, $(mhg^{-1}) \cdot (D_I \times \{t\})$ and $D_I \times \{t'\}$ are translate of each other both containing z', hence they must coincide. Therefore

$$(mhg^{-1}) \cdot ((g \cdot D_M) \cap (D_I \times \{t\})) = (h \cdot D_M) \cap (D_I \times \{t'\}),$$

proving the equality of dimensions

$$\dim((g \cdot D_M) \cap (D_I \times \{t\})) = \dim((h \cdot D_M) \cap (D_I \times \{t'\})).$$

Thus we have proven that $d_I(\mathbf{M}, D_M)$ does not depend on the choice of $g \in \mathbf{G}(\mathbb{R})^+$. Proving that, up to replacing S by some Zariski open subset, one has

$$\dim(\Phi(S^{\mathrm{an}}) \cap \pi(D_I \times \{t\})) = \dim(\Phi(S^{\mathrm{an}}) \cap \pi(D_I \times \{t'\}))$$

is more subtle. The key point here is that $\Phi(S^{\mathrm{an}})$ has a natural structure of algebraic variety and the subset Z of $\Phi(S^{\mathrm{an}})$ where $\Phi(S^{\mathrm{an}}) \cap \pi(D_I \times \{t\})$, letting t vary, has dimension strictly greater than the minimum, which is of course closed in the complex analytic topology, is actually algebraic. Then the lemma follows by replacing S with the Zariski open subset $\Phi^{-1}(\Phi(S^{\mathrm{an}}) \smallsetminus Z)$. The details here require an approach via o-minimal geometry, so we omit them, referring to Khelifa-Urbanik [33] for them. □

Thanks to this lemma we can consider (3.3) as the definition for the quantity $d_I(\mathbf{M}, D_M)$. By convention, let us set $d_\emptyset(\mathbf{M}, D_M) = 0$ and write $d(\mathbf{M}, D_M)$ for $d_{\{1,\dots,n\}}(\mathbf{M}, D_M)$.

Lemma 3.3.8. *A strict Hodge sub-datum $(\mathbf{M}, D_M)$ of $(\mathbf{G}, D)$ is factorwise $\mathbb{V}$-admissible if and only if for every set of indexes $I \subsetneq \{1, \dots, n\}$, one has*

$$d(\mathbf{M}, D_M) \geq d_I(\mathbf{M}, D_M).$$

Proof. Given a non empty strict set of indexes I, we can pick $g \in \mathbf{G}(\mathbb{R})^+$ and $t \in D_I$ such that

$$\pi(g \cdot D_M) \cap \Phi(S^{\mathrm{an}}) \cap \pi(D_{I^c} \times \{t\})$$

is not empty. Then we clearly have

$$\begin{aligned}
&\dim \pi_I(g\cdot D_M) = \dim \pi((g\cdot D_M)\cap D_H) - \dim(\pi(g\cdot D_M)\cap \pi(D_{I^c}\times\{t\})),\\
&\dim \Phi_I(S^{\mathrm{an}}) = \dim \Phi(S^{\mathrm{an}}) - \dim(\Phi(S^{\mathrm{an}})\cap \pi(D_{I^c}\times\{t\})),\\
&\dim D_I = \dim D_H - \dim D_{I^c},
\end{aligned}$$

thus

$$\dim \Phi_I(S^{\mathrm{an}}) + \dim \pi_I(D_M) - \dim D_I = d(\mathbf{M}, D_M) - d_{I^c}(\mathbf{M}, D_M)$$

from which the thesis follows immediately. □

We can finally state the general version of Theorem 3.3.2.

Theorem 3.3.9. *Let us once again put ourselves in the setting above and let us write the derived subgroup of the Mumford-Tate group as* $\mathbf{G}^{\mathrm{der}} = \mathbf{H}\cdot\mathbf{L}$*,* $\mathbf{H}$ *being the algebraic monodromy group of* $\mathbb{V}$*. If* $(\mathbf{M}, D_M)$ *is a strict Hodge sub-datum of* $(\mathbf{G}, D)$ *which is full on the* $\mathbf{L}$*-factor and factorwise* $\mathbb{V}$*-admissible, then the Hodge locus of type* $\mathbf{M}$ *is analytically dense in* S^{an}*.*

Proof. Looking at the proof of Theorem 3.3.2 we notice that we can deduce the Theorem from Claim 3.3.3 in the exact same way as before. So we just need to adjust the proof of Claim 3.3.3 so that it works in the general setting of Theorem 3.3.9.

Let us use the same notation as in the proof of Theorem 3.3.2 and let us again argue by contradiction, so assume that there exists an irreducible component U of the intersection $\pi(g\cdot D_M)\cap\Phi(S^{\mathrm{an}})$ such that $x\in U$ and

$$\dim U > \dim \Phi(S^{\mathrm{an}}) + \dim D_M - \dim D.$$

Since $(\mathbf{M}, D_M)$ is full on the $\mathbf{L}$-factor, we have

$$\dim D - \dim D_M = \dim D_H - \dim(D_M\cap D_H).$$

It follows that $\pi(g\cdot D_M)$ and $\Phi(S^{\mathrm{an}})$ intersect along U with bigger dimension than expected also inside the monodromy orbit $\Gamma_{\mathbf{H}}\backslash D_H$, namely

$$\mathrm{codim}_{\Gamma_{\mathbf{H}}\backslash D_H} U < \mathrm{codim}_{\Gamma_{\mathbf{H}}\backslash D_H} \Phi(S^{\mathrm{an}}) + \mathrm{codim}_{\Gamma_{\mathbf{H}}\backslash D_H} \pi((g\cdot D_M)\cap D_H). \tag{3.4}$$

Consider, similarly as before, the algebraic subvariety $W = S\times((g\cdot D_M)\cap D_H)$ of $S\times D_H$ and let $\tilde{U}$ be the complex analytic component of $W\cap(S\times_{\Gamma_{\mathbf{H}}\backslash D_H} D_H)$ containing (s,z) such that $\Phi(p_S(\tilde{U})) = U$. Argung precisely as before we obtain that inequality (3.4) implies

$$\mathrm{codim}_{S\times D_H}\tilde{U} < \mathrm{codim}_{S\times D_H} W + \mathrm{codim}_{S\times D_H}(S\times_{\Gamma_{\mathbf{H}}\backslash D_H} D_H),$$

thus by the Ax-Schanuel Theorem 3.2.6, the projection of $\tilde{U}$ to S is contained in a strict weakly special subvariety Y for $\mathbb{V}$. We take Y to be minimal for the inclusion among weakly special subvarieties Z of S such that $U\subseteq\Phi(Z^{\mathrm{an}})$. The generic Mumford-Tate group of Y is $\mathbf{G}$ since Y contains the Hodge generic point s, thus by Theorem 2.7.5 the algebraic monodromy group $\mathbf{H}_Y$ is a proper normal subgroup of $\mathbf{H}$. Let us write $\mathbf{H}_Y = \prod_{i\in I}\mathbf{H}_i$, so that Y^{an} is an irreducible component of the preimage by Φ of some sub-domain of $\Gamma\backslash D$ of the form $\pi(D_I\times\{t\})$ for some $I\subseteq\{1,\dots,n\}$. Then it follows from the construction that U is an irreducible component of the intersection

$$\pi((g\cdot D_M)\cap(D_I\times\{t\}))\cap\Phi(Y^{\mathrm{an}}).$$

Claim 3.3.10. *The terms $\pi((g \cdot D_M) \cap (D_I \times \{t\}))$ and $\Phi(Y^{\mathrm{an}})$ intersect inside $\pi(D_I \times \{t\})$ with expected dimension along U, namely*

$$\dim U = \dim(\Phi(Y^{\mathrm{an}}) \cap \pi(D_I \times \{t\})) + \dim \pi((g \cdot D_M) \cap (D_I \times \{t\})) - \dim \pi(D_I \times \{t\})$$

Proof. The proof of this claim is straightforward. Assume by contradiction that the above intersection has dimension bigger that expected along U. Then we can repeat the same argument as above for the restriction of the given VHS $(\mathbb{V}, \mathscr{F}^\bullet)$ to Y. The Ax-Schanuel principle thus implies that there exists a strict weakly special subvariety Y' of Y for $\mathbb{V}$ such that U is contained in the image of Y' under the period map restricted to Y. Clearly, the subvariety Y' in S is also weakly special with respect to the whole VHS $(\mathbb{V}, \mathscr{F}^\bullet)$ on S, so this contradicts the minimality of Y. □

In order to conclude, we will apply Lemma 3.3.8, thus we may need to replace S by some Zariski open subset, so that we can consider the quantities $d_I(\mathbf{M}, D_M)$ defined in Lemma 3.3.7. The conclusion of the Theorem does not change thanks to the following:

Lemma 3.3.11. *Let X be an algebraic variety over $\mathbb{C}$. If V is a Zariski open subset of X that is Zariski dense, then V^{an} is analytically dense in X^{an}.*

Proof. The question is local, since we have to show that V^{an} intersects the analytic neighborhoods of each point in X^{an}. Thus, it is not restrictive to assume that X is affine, say $X \subseteq \mathbb{A}^n_{\mathbb{C}}$. Assume by contradiction that V^{an} is not analytically dense, then there exists a point $x \in X^{\mathrm{an}}$ and an analytic neighborhood W of x in X^{an} which is fully contained in the complement of V^{an}, which is by assumption a Zariski closed subset Z of X. Let us denote by $\mathscr{O}_x$ the stalk at x of the algebraic structure sheaf of $\mathbb{A}^n_{\mathbb{C}}$, namely $\mathscr{O}_x = \mathbb{C}[z_1, \dots, z_n]_{\mathfrak{m}}$, where $\mathfrak{m}$ is the maximal ideal corresponding to the point x. Then we have that the algebraic germ of X (resp. Z) at x corresponds to an ideal I (resp. J) in the local ring $\mathscr{O}_x$. Since every regular function is holomorphic we have a natural morphism $\eta : \mathscr{O}_x \to \mathscr{O}_x^{\mathrm{an}}$, where the target is the stalk at x of the analytic structure sheaf of $\mathbb{A}^n_{\mathbb{C}}$. Since Z^{an} and X^{an} coincide in some analytic neighborhood of x in X^{an}, that we can take as intersection of X^{an} with an analytic open subset of $\mathbb{A}^n_{\mathbb{C}}$, we have

$$I\mathscr{O}_x^{\mathrm{an}} = J\mathscr{O}_x^{\mathrm{an}}$$

and the same holds passing to the completions of the local rings. But clearly η induces an isomorphism

$$\widehat{\mathscr{O}_x} \cong \widehat{\mathscr{O}_x^{\mathrm{an}}} \cong \mathbb{C}[[z_1, \dots, z_n]],$$

hence

$$I\widehat{\mathscr{O}_x} = J\widehat{\mathscr{O}_x}.$$

This implies that $I = J$ in $\mathscr{O}_x$ since the completion morphism

$$\mathscr{O}_x = \mathbb{C}[z_1, \dots, z_n]_{\mathfrak{m}} \to \mathbb{C}[[z_1, \dots, z_n]] = \widehat{\mathscr{O}_x}$$

is faithfully flat. But then X and Z coincide in some Zariski open subset of $\mathbb{A}^n_{\mathbb{C}}$, contradicting the density of $V = X \smallsetminus Z$ in X for the Zariski topology. □

Let us now finish the proof of Theorem 3.3.9. Since S is irreducible, every Zariski open subset of S is Zariski dense, so analytically dense by the previous lemma. So we can apply Lemma 3.3.7 and consider the well defined quantities $d_I(\mathbf{M}, D_M)$. Now, notice that

$$\begin{aligned} \dim U &> \dim((g \cdot D_M) \cap D_H) + \dim \Phi(Y^{\mathrm{an}}) - \dim D_H \\ &= d(\mathbf{M}, D_M) \end{aligned}$$

by an immediate restatement of inequality (3.4) and by definition of $d(\mathbf{M}, D_M)$. On the other hand, $U \subseteq \pi(D_I \times \{t\})$, so

$$\begin{aligned}\dim U &= \dim(U \cap \pi(D_I \times \{t\})) \\ &= \dim(\Phi(Y^{\mathrm{an}}) \cap \pi(D_I \times \{t\})) + \dim \pi((g \cdot D_M) \cap (D_I \times \{t\})) - \dim \pi(D_I \times \{t\}) \\ &= \dim(\Phi(Y^{\mathrm{an}}) \cap \pi(D_I \times \{t\})) + \dim((g \cdot D_M) \cap (D_I \times \{t\})) - \dim D_I \\ &= d_I(\mathbf{M}, D_M)\end{aligned}$$

where the second equality is Claim 3.3.10, the third is obvious and the last one is the definition of $d_I(\mathbf{M}, D_M)$. Putting this together we obtain

$$d(\mathbf{M}, D_M) - d_I(\mathbf{M}, D_M) < \dim U - \dim(U \cap \pi(D_I \times \{t\})) = 0.$$

However, by factorwise $\mathbb{V}$-admissibility of $(\mathbf{M}, D_M)$ and Lemma 3.3.8, one has

$$d(\mathbf{M}, D_M) - d_I(\mathbf{M}, D_M) \geq 0$$

so we have a contradiction. □

Remark 3.3.12. Recall that by Conjecture 3.1.8 we expect that the analytic density of the Hodge locus comes exclusively from its typical part. Hence we should expect that the same criterion stated in Theorem 3.3.9 ensures the density of the typical Hodge locus. This has not been proven yet, as far as we know, nevertheless Khelifa-Urbanik showed that strengthening slightly the factorwise admissibility criterion one actually has density of the typical Hodge locus. Namely they proved the following:

Theorem 3.3.13. *Let us once again put ourselves in the situation at the beginning of the section and assume $\mathbf{H} = \mathbf{G}^{\mathrm{der}}$, $\mathbf{H}$ being the algebraic monodromy group of $\mathbb{V}$. Let $(\mathbf{M}, D_M)$ be a strict Hodge sub-datum of $(\mathbf{G}, D)$ which is factorwise strongly $\mathbb{V}$-admissible, namely*

$$\dim \Phi_I(S^{\mathrm{an}}) + \dim \pi_I(D_M) - \dim D_I > 0$$

for every non empty set of indexes I as in Definition 3.3.5. Then the typical Hodge locus of type $\mathbf{M}$ is analytically dense in S^{an}.

3.4 Applications

In this section we collect some applications of Khelifa-Urbanik criterion for the density of the Hodge locus. As we have already said, the version of Theorem 3.3.2 will be enough.

3.4.1 Noether-Lefschetz locus of hypersurfaces in $\mathbb{P}^3$

As a first application, let us give, following Khelifa-Urbanik [33], a simple proof of the density of the Noether-Lefschetz locus of smooth degree d hypersurfaces in $\mathbb{P}^3$, for $d \geq 5$. This is a very classical result, originally proved by Ciliberto-Harris-Miranda [18].

Recall that for a compact complex manifold X, we have the exponential sequence, namely the short exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_X^* \longrightarrow 1.$$

Taking the associated long exact cohomology sequence we obtain the connecting morphism

$$\mathrm{Pic}(X) \cong H^1(X, \mathscr{O}_X^*) \to H^2(X, \mathbb{Z}).$$

Definition 3.4.1. The rank of the image of the map $\mathrm{Pic}(X) \to H^2(X,\mathbb{Z})$ is called the Picard number of X, denoted $\rho(X)$.

Now assume that X is compact Kähler. Working again with the cohomology sequence associated with the exponential sequence we can show that the composition

$$\mathrm{Pic}(X) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}) \to H^2(X,\mathscr{O}_X) \cong H^{0,2}(X)$$

is the zero map. Therefore, by Hodge symmetry, the image of the composition $\mathrm{Pic}(X) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$ lies in $H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$, where we used the common abuse of notation of denoting by $H^2(X,\mathbb{Z})$ its image in $H^2(X,\mathbb{C})$.

A classical result by Lefschetz (see for instance Huybrechts [31], Proposition 3.3.2) shows that

$$\mathrm{Pic}(X) \to H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$$

is surjective, thus the Picard number of X coincides with the rank of $H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ or, equivalently, with the dimension of the space of Hodge classes of type (1,1) in $H^2(X,\mathbb{C})$.

From now and throughout this section we will denote with the same letter an algebraic variety and its analytification, as there is no need to separate the two concepts.

Let us consider the universal family of smooth hypersurfaces of degree d in $\mathbb{P}^n$, for $n \geq 3$. It is given by a holomorphic submersion $f : \mathcal{X} \to U_{n,d}$, where $U_{n,d}$ is the open subset of $\mathbb{P}(H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d)))$ consisting of degree d polynomials in $n+1$ variables whose vanishing locus in $\mathbb{P}^n$ is smooth. Notice that, if X is such an hypersurface we easily obtain from the ideal sheaf sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n}(-d) \longrightarrow \mathscr{O}_{\mathbb{P}^n} \longrightarrow \mathscr{O}_X \longrightarrow 0 \tag{3.5}$$

that $H^1(X,\mathscr{O}_X) = 0$, so the exponential sequence gives an injective map $\mathrm{Pic}(X) \hookrightarrow H^2(X,\mathbb{Z})$ and the Picard number of X coincides with the rank of the Picard group.

We can consider the integral polarized variation of Hodge structures given by the primitive part of $R^{n-1}f_*\mathbb{Z}$. Notice that there is no need to quotient out the torsion part since we can deduce from the weak Lefschetz Theorem (see Voisin [47], Theorem 1.23) and the universal coefficients Theorem that $R^{n-1}f_*\mathbb{Z}$ is torsion-free. We recall the following classical result, due to Donagi [21], which is necessary in order to apply Theorem 3.3.2 to this situation.

Theorem 3.4.2. *The period map of the polarized $\mathbb{Z}$-VHS associated with the universal family $f : \mathcal{X} \to U_{n,d}$ of smooth hypersurfaces of degree d in $\mathbb{P}^n$ is generically injective, except possibly for the following cases:*

1) $n = d = 3$ *(cubic surfaces),*

2) $n + 1 \equiv 0 \pmod{d}$,

3) $d = 4$ *and* $n - 1 \equiv 0 \pmod{4}$,

4) $d = 6$ *and* $n - 2 \equiv 0 \pmod{6}$.

Now, let us fix $n = 3$ and consider the universal family of smooth degree d hypersurfaces in $\mathbb{P}^3$, for $d \geq 5$. Choosing the Fubini-Study metric in $\mathbb{P}^3$, we have an integral Kähler class $[\omega]$ on X which coincides with the first Chern class of the line bundle $\mathscr{O}_X(1)$. The classical Noether-Lefschetz Theorem (see for instance Voisin [47], section 3.3) asserts that for a generic $s \in U_{3,d}$ one has $\mathrm{Pic}(X_s) \cong \mathbb{Z}$. In other words, the space of Hodge classes of type (1,1) in $H^2(X_s,\mathbb{C})$ is one dimensional, generated by the Kähler class $[\omega]$.

Remark 3.4.3. Notice that the condition $\mathrm{Pic}(X) \cong \mathbb{Z}$ has a concrete geometric interpretation. Indeed it means that every curve in the hypersurface X is the complete intersection of X with another hypersurface in $\mathbb{P}^3$. To show this, observe that, by the injection $\mathrm{Pic}(X) \hookrightarrow H^2(X,\mathbb{Z})$, the Hodge class of a curve $Y \hookrightarrow X$ is a multiple of the first Chern class of $\mathscr{O}_X(1)$ if and only if its associated line bundle $\mathscr{O}_X(Y)$ is isomorphic to $\mathscr{O}_X(k)$ for some $k \in \mathbb{Z}$. But then Y is the vanishing locus of a section of $\mathscr{O}_X(k)$. Hence, to show that in this case Y is the complete intersection of X with another hypersurface in $\mathbb{P}^3$ it is enough to show that the restriction map

$$H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(k)) \to H^0(X, \mathscr{O}_X(k))$$

is surjective. But this can be easily deduced from the cohomology sequence associated with the ideal sheaf sequence (3.5) twisted by $\mathscr{O}_{\mathbb{P}^3}(k)$, using the fact that $H^1(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(k-d)) = 0$.

Since the generic smooth hypersurface of degree d in $\mathbb{P}^3$ has Picard number 1, we can define the *Noether-Lefschetz locus* of smooth degree d hypersurfaces in $\mathbb{P}^3$ as

$$\mathrm{NL}(d) = \{s \in U_{3,d} : \rho(X_s) > 1\}.$$

The condition $\rho(X_s) > 1$ is clearly equivalent to the existence of an element in $H^2(X_s,\mathbb{Z}) \cap H^{1,1}(X_s)$ which is not a multiple of $[\omega]$. Recalling that $H^2(X_s,\mathbb{Z})_{\mathrm{prim}}$ is defined as the kernel of the Lefschetz operator (see Definition 1.1.3), we thus have that the condition $\rho(X_s) > 1$ is equivalent to $H^2(X_s,\mathbb{Z})_{\mathrm{prim}}$ containing a non zero Hodge class.

Consider the integral polarized variation of Hodge structure $\mathbb{V} = (R^2 f_* \mathbb{Z})_{\mathrm{prim}}$ on $U_{3,d}$. We clearly have

$$\mathrm{NL}(d) \subseteq \mathrm{HL}(U_{3,d}, \mathbb{V}^{\otimes}).$$

Denote by $Q : \mathbb{V} \otimes \mathbb{V} \to \underline{\mathbb{Z}}$ the usual polarization, given fiberwise as in Definition 1.1.3.

Proposition 3.4.4. *The Noether-Lefschetz locus* $\mathrm{NL}(d)$ *is analytically dense in* $U_{3,d}$ *for* $d \geq 5$.

Proof. By a theorem of Beauville [10] (Theorem 2) the algebraic monodromy group of $\mathbb{V}$ is the full orthogonal group $\mathrm{Aut}(\mathbb{V}_s, Q_s)$, for a general $s \in U_{3,d}$. Thus the generic Hodge datum of this polarized $\mathbb{Z}$-VHS is $(\mathbf{G}, D)$, where $\mathbf{G} = \mathrm{GAut}(\mathbb{V}_s, Q_s)$ and D is the full period domain of polarized Hodge structures with the same Hodge numbers $(h^{2,0}, h^{1,1}, h^{0,2})$ as $\mathbb{V}_s$, for any $s \in U_{3,d}$. Consider the Hodge sub-datum $(\mathbf{M}, D_M)$ where $\mathbf{M}$ is the fixator of a single Hodge vector. Then, by what we observed above, we have

$$\mathrm{HL}(U_{3,d}, \mathbb{V}^{\otimes}, \mathbf{M}) \subseteq \mathrm{NL}(d) \subseteq \mathrm{HL}(U_{3,d}, \mathbb{V}^{\otimes}).$$

The corresponding sub-domain D_M is cut out in D by $h^{2,0}$ equations. Thus

$$\dim D - \dim D_M = h^{2,0} = \binom{d-1}{3}$$

where the computation of $h^{2,0} = h^{0,2} = \dim H^2(X_s, \mathscr{O}_{X_s})$ is done immediately taking the long exact cohomology sequence associated with the ideal sheaf sequence (3.5).

Now, $U_{3,d}$ is open in the projective space of the vector space of degree d polynomials in 4 variables, thus $\dim U_{3,d} = \binom{d+3}{3} - 1$. Moreover, by Donagi's Theorem 3.4.2, the period map Φ is generically injective, of course modulo the action of the group $\mathrm{PGL}_4(\mathbb{C})$ of automorphisms of $\mathbb{P}^3$, which has dimension 15. Thus

$$\dim \Phi(U_{3,d}) = \binom{d+3}{3} - 16.$$

Therefore we can see that condition (3.1)

$$\dim \Phi(U_{3,d}) + \dim D_M - \dim D = \binom{d+3}{3} - 16 - \binom{d-1}{3} \geq 0 \tag{3.6}$$

holds for $d \geq 5$, so Theorem 3.3.2 gives the density of the Hodge locus of type $\mathbf{M}$, hence also of the Noether-Lefschetz locus. □

Remark 3.4.5. Let Y be an irreducible component of the Noether-Lefschetz locus. Then, keeping the notations as in the proof, we have

$$\operatorname{codim}_{U_{3,d}} Y \leq \operatorname{codim}_D D_M = h^{2,0}.$$

It is natural to ask whether the density of the Noether-Lefschetz locus is given only by subvarieties having maximal codimension $h^{2,0}$ in $U_{3,d}$. This is a question of typical intersections. Let $(\mathbf{G}_Y, D_{G_Y})$ be the generic Hodge datum associated with an irreducible component Y of $\mathrm{NL}(d)$. Then Y is a typical special subvariety if and only if

$$\operatorname{codim}_{U_{3,d}} Y = \operatorname{codim}_D D_{G_Y}$$

by an immediate restatement of Definition 3.1.6. By construction, $(\mathbf{G}_Y, D_{G_Y})$ is a Hodge sub-datum of $(\mathbf{M}, D_M)$ (notations as in the proof), hence

$$\operatorname{codim}_D D_{G_Y} = \operatorname{codim}_D D_M + \operatorname{codim}_{D_M} D_{G_Y} = h^{2,0} + \operatorname{codim}_{D_M} D_{G_Y}.$$

Thus, Y is typical if and only if

$$\operatorname{codim}_{U_{3,d}} Y = h^{2,0} + \operatorname{codim}_{D_M} D_{G_Y}$$

holds (and necessarily $\operatorname{codim}_{D_M} D_{G_Y} = 0$).

Now, since (3.6) is actually a *strict* inequality, we see that $(\mathbf{M}, D_M)$ is a strongly admissible Hodge sub-datum, hence we can apply Theorem 3.3.13 to conclude that the *typical* Hodge locus of type $\mathbf{M}$ is dense, so we have the density in $U_{3,d}$ of components of the Noether-Lefschetz locus with maximal codimension $h^{2,0}$.

On the other hand, Conjecture 3.1.8 would give that, on the contrary, the union of the components with codimension strictly less than $h^{2,0}$ is not (even Zariski) dense in $U_{3,d}$. For the case of the Noether-Lefschetz locus, this actually has been proven in Baldi-Klingler-Ullmo [8]. The key point here is that in order to prove it, one does not need the whole Zilbert-Pink conjecture, but just some version for special subvarieties of positive period dimension.

3.4.2 Another universal family of hypersurfaces

Now that we have seen that the Hodge locus of the family of smooth hypersurfaces of degree $d \geq 5$ in $\mathbb{P}^3$ is analytically dense, it is natural to ask whether the same happens for smooth hypersurfaces in $\mathbb{P}^n$, for $n \geq 4$. So, consider the integral polarized VHS on $U_{d,n}$ given by $(R^{n-1} f_* \mathbb{Z})_{\text{prim}}$.

Actually, we expect that the situation for $n \geq 4$ is much different. Indeed, Baldi-Klingler-Ullmo [9] (Theorem 3.3 and Corollary 1.6) showed the following:

Theorem 3.4.6. *The typical Hodge locus of the universal family of smooth degree d hypersurfaces in $\mathbb{P}^n$ is empty for $n \geq 4, d \geq 5$, except for $n = d = 5$. In particular, for $n \geq 4, d \geq 5$ and $(n, d) \neq (5, 5)$, the Hodge locus of positive period dimension is algebraic.*

So, unconditionally we know the non density of the Hodge locus of positive period dimension, but by Conjecture 3.1.8 we expect the whole Hodge locus to be algebraic, therefore not (even Zariski) dense.

Let us then consider a low degree exception to this phenomenon, in particular let us put $n = d = 5$. In order to apply an argument in the same spirit as in the proof of Proposition 3.4.4 we need to compute the Hodge numbers of a smooth hypersurface in $\mathbb{P}^5$, so this is our first task. Let us start with some general results which hold in general for a smooth hypersurface X of degree d in $\mathbb{P}^n$. To fix the notations, let

$$h^{p,q}(X) = \dim H^q(X, \Omega_X^p) = h^{q,p}(X).$$

Lemma 3.4.7. *For $p + q < n - 1$ one has*

$$H^q(X, \Omega_X^p) \cong H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^p) \cong \delta_{pq}\mathbb{C}$$

where δ_{pq} is the Kronecker symbol. The same holds for $n - 1 < p + q \leq 2n - 2$.

Proof. The cohomology of the projective space is well known (one can for instance compute $H^k(\mathbb{P}^n, \mathbb{C})$ by induction on n using a Mayer-Vietoris sequence and deduce $H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^p) \cong \delta_{pq}\mathbb{C}$ by Hodge decomposition). Then the isomorphism $H^q(X, \Omega_X^p) \cong H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^p)$ follows directly from the weak Lefschetz Theorem (see for instance Voisin [47], Theorem 1.23) together with Hodge decomposition. The second part follows from the first by Serre duality. □

Lemma 3.4.8. *The Euler-Poincaré characteristic of the sheaf Ω_X^p is*

$$\chi(X, \Omega_X^p) = \begin{cases} (-1)^p h^{p,p}(X) & \text{if } p = n - 1 - p \\ (-1)^p + (-1)^{n-1-p} h^{p,n-1-p}(X) & \text{if } p \neq n - 1 - p \end{cases}$$

Proof. By definition of the Euler-Poincarè characteristic and Lemma 3.4.7, we have

$$\begin{aligned} \chi(X, \Omega_X^p) &= \sum_{j=0}^{n-1} (-1)^j \dim H^j(X, \Omega_X^p) \\ &= (-1)^{n-1-p} h^{p,n-1-p}(X) + \sum_{j=0, j\neq n-1-p}^{n-1} (-1)^j \dim H^j(X, \Omega_X^p) \\ &= (-1)^{n-1-p} h^{p,n-1-p}(X) + \sum_{j=0, j\neq n-1-p}^{n-1} (-1)^j \delta_{jp} \end{aligned}$$

from which the result follows immediately. □

Lemma 3.4.9. *One has*

$$\chi(X, \Omega_X^1) = \chi(X, \mathscr{O}_X(-1)^{n+1}) - \chi(X, \mathscr{O}_X) - \chi(X, \mathscr{O}_X(-d)).$$

Proof. Denote by ι the embedding of X in $\mathbb{P}^n$. Then, applying the pull-back along ι to the well known Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^n}^1 \longrightarrow \mathscr{O}_{\mathbb{P}^n}(-1)^{n+1} \longrightarrow \mathscr{O}_{\mathbb{P}^n} \longrightarrow 0$$

we obtain

$$0 \longrightarrow \iota^*\Omega_{\mathbb{P}^n}^1 \longrightarrow \mathscr{O}_X(-1)^{n+1} \longrightarrow \mathscr{O}_X \longrightarrow 0 \tag{3.7}$$

which stays exact since all sheaves in the Euler sequence are locally free. Moreover, one has the so called conormal sequence

$$0 \longrightarrow \mathscr{O}_X(-d) \longrightarrow \iota^*\Omega^1_{\mathbb{P}^n} \longrightarrow \Omega^1_X \longrightarrow 0. \tag{3.8}$$

Then the claim follows from the additivity of Euler-Poincaré characteristic with respect to (3.7) and (3.8). □

Lemma 3.4.10. *For $m \in \mathbb{Z}$, one has*

$$\chi(X, \mathscr{O}_X(m)) = \binom{n+m}{n} - \binom{n+m-d}{n}.$$

Proof. Tensoring the ideal sheaf sequence (3.5) with $\mathscr{O}_{\mathbb{P}^n}(m)$ and noticing that $\iota_*\mathscr{O}_X \otimes \mathscr{O}_{\mathbb{P}^n}(m) \cong \iota_*\mathscr{O}_X(m)$ by the projection formula, we obtain

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n}(m-d) \longrightarrow \mathscr{O}_{\mathbb{P}^n}(m) \longrightarrow \mathscr{O}_X(m) \longrightarrow 0.$$

But then

$$\begin{aligned} \chi(X, \mathscr{O}_X(m)) &= \chi(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(m)) - \chi(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(m-d)) \\ &= \binom{n+m}{n} - \binom{n+m-d}{n} \end{aligned}$$

□

Now we have the tools to prove the following:

Proposition 3.4.11. *The Hodge locus of the universal family of smooth hypersurfaces in $\mathbb{P}^5$ of degree 5 is analytically dense.*

Proof. The argument is in the same spirit as in the proof of Proposition 3.4.4. First of all notice that we can still apply Donagi's Theorem 3.4.2 to use the generic injectivity of the period map and Beauville's result [10] (Theorem 2) to conclude that the algebraic monodromy group of $\mathbb{V}$ is the full orthogonal group $\mathrm{Aut}(\mathbb{V}_s, Q_s)$.

Now, notice that for a generic $s \in U_{5,5}$ the space of *algebraic* cohomology classes in $H^4(X_s, \mathbb{Z}) \cap H^{2,2}(X_s)$, namely the image in $H^4(X_s, \mathbb{Z}) \cap H^{2,2}(X_s)$ of the group $\mathrm{CH}^2(X_s)$ of codimension 2 algebraic cycles on X_s (modulo rational equivalence), has rank 1 (see for instance Shioda [43], Theorem 2.1). This result, together with the fact that the Hodge conjecture is known for quintic fourfolds, namely the cycle class map

$$\mathrm{CH}^2(X_s) \otimes \mathbb{Q} \to H^4(X_s, \mathbb{Q}) \cap H^{2,2}(X_s)$$

is surjective (see for instance Da Silva Jr [44], Corollary 2.20), implies that

$$\mathrm{rk}(H^4(X_s, \mathbb{Z}) \cap H^{2,2}(X_s)) = \mathrm{rk}(H^4(X_s, \mathbb{Q}) \cap H^{2,2}(X_s)) = 1$$

holds generically in $s \in U_{5,5}$, with $H^4(X_s, \mathbb{Z}) \cap H^{2,2}(X_s)$ being generated by the second power of the Kähler class given by the restriction of the Fubini-Study metric. Thus we have obtained that the Hodge locus contains the subset

$$\{s \in U_{5,5} : \mathrm{rk}(H^4(X_s, \mathbb{Z}) \cap H^{2,2}(X_s)) > 1\}$$

and we can argue as in the proof of Proposition 3.4.4. Consider the Hodge sub-datum $(\mathbf{M}, D_M)$, where $\mathbf{M}$ is the fixator of a single Hodge vector. Then, D_M is cut out in the Mumford-Tate domain D by $h^{4,0} + h^{3,1}$ equations, so

$$\dim D - \dim D_M \leq h^{4,0} + h^{3,1}.$$

Looking at the long exact cohomology sequence attached to the ideal sheaf sequence (3.5) we easily obtain $h^{4,0} = 0$. Furthermore, combining Lemmas 3.4.8, 3.4.9 and 3.4.10 we can compute

$$\begin{aligned} h^{3,1} &= -1 - \chi(X, \Omega^1_X) \\ &= -1 - 6\chi(X, \mathscr{O}_X(-1)) + \chi(X, \mathscr{O}_X) + \chi(X, \mathscr{O}_X(-5)) \\ &= -1 + 6\binom{-1}{5} + 1 - \binom{-5}{5} \\ &= 120. \end{aligned}$$

On the other hand

$$\dim \Phi(U_{5,5}) = \binom{10}{5} - 1 - \dim \mathrm{PGL}_6(\mathbb{C}) = 216,$$

thus we have the inequality

$$\dim \Phi(U_{5,5}) + \dim D_M - \dim D \geq 0$$

and by Theorem 3.3.2 we can conclude. □

3.4.3 Complete intersection surfaces in $\mathbb{P}^4$

Let us now go back to the case of surfaces and consider a version of the Noether-Lefschetz locus for surfaces which are (scheme-theoretic) complete intersections in $\mathbb{P}^4$. First of all let us notice that the Noether-Lefschetz Theorem holds (see for instance Spandaw [45], section 2) also in this more general case, namely a *generic* complete intersection surface in $\mathbb{P}^n$ is smooth and has Picard number 1 except in the following cases:

1) quadric surfaces in $\mathbb{P}^3$,

2) complete intersections of two quadrics in $\mathbb{P}^4$,

3) cubic surfaces in $\mathbb{P}^3$.

Thus, if we consider the open subset $U_{d_1,\ldots,d_r} \subseteq \prod_{i=1}^r \mathbb{P}(H^0(\mathbb{P}^{r+2}, \mathscr{O}(d_i)))$ parametrizing smooth surfaces in $\mathbb{P}^{r+2}$ which are complete intersections of r hypersurfaces of degrees $d_1, \ldots, d_r$ and its associated universal family, we can define the Noether-Lefschetz locus

$$\mathrm{NL}(d_1, \ldots, d_r) = \{s \in U_{d_1,\ldots,d_r} : \rho(X_s) > 1\}.$$

We remark that, while we are not aware of a generic Torelli Theorem like Theorem 3.4.2 for complete intersections, we do have a *local* Torelli Theorem that we can use, which is due to Flenner [22] (Theorem 3.1).

Theorem 3.4.12. *The period map associated to the polarized $\mathbb{Z}$-VHS $(R^{n-r} f_* \mathbb{Z})_{\mathrm{prim}}$ on the universal family of smooth complete intersections of hypersurfaces of degree $d_1, \ldots, d_r$ in $\mathbb{P}^n$ (for $1 \leq r \leq n-2$) has injective differential except in the following cases:*

1) $n = 3$, $r = 1$, $d_1 = 3$,

2) n *even,* $r = 2$, $d_1 = d_2 = 2$.

Now let us restrict to the case of complete intersection surfaces in $\mathbb{P}^4$. In this case, some computations will allow us to deduce the density of the Noether-Lefschetz locus. It is clear that in order to do a proof analogous to the one of Proposition 3.4.4 we need to compute $h^{2,0}(X)$ for a surface X which is a smooth complete intersection of two hypersurfaces of degree d_1, d_2 in $\mathbb{P}^4$.

Lemma 3.4.13. *One has*

$$h^{2,0}(X) = \binom{d_1 + d_2 - 1}{4} - \binom{d_1 - 1}{4} - \binom{d_2 - 1}{4}.$$

Proof. Let $\mathscr{I}_X \subseteq \mathscr{O}_{\mathbb{P}^4}$ be the ideal sheaf of X and assume that X is the complete intersection of the hypersurfaces defined by the polynomials $f, g \in \mathbb{C}[x_0, \ldots, x_4]$, with $\deg(f) = d_1$ and $\deg(g) = d_2$. Since X is a smooth (scheme-theoretic) complete intersection, f and g have no common factors and generate the ideal sheaf $\mathscr{I}_X$. Thus, we have a short exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^4}(-d_1 - d_2) \longrightarrow \mathscr{O}_{\mathbb{P}^4}(-d_1) \oplus \mathscr{O}_{\mathbb{P}^4}(-d_2) \longrightarrow \mathscr{I}_X \longrightarrow 0$$

where the first map sends a local section σ of $\mathscr{O}_{\mathbb{P}^4}(-d_1 - d_2)$ to $(\sigma g, -\sigma f)$ and the second map sends a local section (φ, ψ) of $\mathscr{O}_{\mathbb{P}^4}(-d_1) \oplus \mathscr{O}_{\mathbb{P}^4}(-d_2)$ to $\varphi f + \psi g$. Taking the associated long exact cohomology sequence and remembering the cohomology of line bundles on $\mathbb{P}^4$ we obtain the exact sequence

$$0 \to H^3(\mathbb{P}^4, \mathscr{I}_X) \to H^4(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(-d_1 - d_2)) \to H^4(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(-d_1) \oplus \mathscr{O}_{\mathbb{P}^4}(-d_2)) \to H^4(\mathbb{P}^4, \mathscr{I}_X) \to 0.$$

Now, considering the long exact sequence associated with the ideal sheaf sequence

$$0 \longrightarrow \mathscr{I}_X \longrightarrow \mathscr{O}_{\mathbb{P}^4} \longrightarrow \mathscr{O}_X \longrightarrow 0$$

we obtain

$$H^2(X, \mathscr{O}_X) \cong H^3(\mathbb{P}^4, \mathscr{I}_X)$$

and

$$H^4(\mathbb{P}^4, \mathscr{I}_X) \cong H^3(X, \mathscr{O}_X) = 0$$

since X has dimension 2. Putting everything together we get

$$\begin{aligned} h^{2,0}(X) = h^{0,2}(X) &= \dim H^2(X, \mathscr{O}_X) \\ &= \dim H^3(\mathbb{P}^4, \mathscr{I}_X) \\ &= \dim H^4(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(-d_1 - d_2)) - \dim H^4(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(-d_1)) - \dim H^4(\mathbb{P}^4, \mathscr{O}_{\mathbb{P}^4}(-d_2)) \end{aligned}$$

from which the claim follows immediately. □

Proposition 3.4.14. *The Noether-Lefschetz locus* $\mathrm{NL}(d_1, d_2)$ *is analytically dense provided that* $\min(d_1, d_2) \geq 2$ *and* $(d_1, d_2) \neq (2, 2)$.

Proof. We procede exactly as in the proof of Proposition 3.4.4. As before, a result of Beauville [10] (Theorem 5) ensures that the algebraic monodromy group is the full orthogonal group. Furthermore, the local Torelli Theorem 3.4.12 implies that

$$\dim \Phi(U_{d_1,d_2}) = \dim U_{d_1,d_2} - \dim \mathrm{PGL}_5(\mathbb{C}) = \left(\binom{d_1 + 4}{4} - 1\right)\left(\binom{d_2 + 4}{4} - 1\right) - 24,$$

where Φ is the period map. As before, if $(\mathbf{M}, D_M)$ is the Hodge sub-datum corresponding to the fixator of a single Hodge vector in $\mathbb{V} = (R^2 f_* \mathbb{Z})_{\text{prim}}$, then D_M is cut out in the full Mumford-Tate domain D by $h^{2,0}$ equations, so, using Lemma 3.4.13, we obtain

$$\dim D - \dim D_M \leq h^{2,0} = \binom{d_1 + d_2 - 1}{4} - \binom{d_1 - 1}{4} - \binom{d_2 - 1}{4}.$$

Then we see that the admissibility condition

$$\dim \Phi(U_{d_1,d_2}) + \dim D_M - \dim D \geq 0$$

reads

$$\left(\binom{d_1+4}{4} - 1\right)\left(\binom{d_2+4}{4} - 1\right) - 24 - \binom{d_1 + d_2 - 1}{4} + \binom{d_1 - 1}{4} + \binom{d_2 - 1}{4} \geq 0$$

which is clearly satisfied under the assumptions of the Proposition. We can thus conclude by Theorem 3.3.2, as $\mathrm{HL}(U_{d_1,d_2}, \mathbb{V}^{\otimes}, \mathbf{M}) \subseteq \mathrm{NL}(d_1, d_2)$. □

3.4.4 Curves with non-simple Jacobian

Let $g \geq 4$ be an integer and consider the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g (see Appendix A for the construction). As explained in the Appendix, $\mathcal{A}_g$ is a Hodge variety for the Hodge datum $(\mathbf{GSp}_{2g}, \mathbb{H}_g)$, where $\mathbf{GSp}_{2g}$ denotes the general symplectic group and $\mathbb{H}_g$ the Siegel upper half-space.

To be precise, let also fix a level structure, so that we have a universal family f of principally polarized abelian varieties over $\mathcal{A}_g$ and we can consider the natural integral polarized variation of Hodge structures given by $R^1 f_* \mathbb{Z}$.

Let V be a closed Hodge generic subvariety of $\mathcal{A}_g$. The natural polarized $\mathbb{Z}$-VHS $\mathbb{V}$ defined on $\mathcal{A}_g$ clearly induces by restriction a polarized $\mathbb{Z}$-VHS on V, with period map the inclusion $V \hookrightarrow \mathcal{A}_g$. Consider the strict Hodge sub-datum $(\mathbf{GSp}_2 \times \mathbf{GSp}_{2g-2}, \mathbb{H}_1 \times \mathbb{H}_{g-1})$ of $(\mathbf{GSp}_{2g}, \mathbb{H}_g)$, which corresponds to the special subvariety $\mathcal{A}_1 \times \mathcal{A}_{g-1}$ of $\mathcal{A}_g$.

Proposition 3.4.15. *Let V be a closed Hodge generic subvariety of $\mathcal{A}_g$ of dimension $d \geq g - 1$ and let $\mathbb{V}$ be the induced polarized $\mathbb{Z}$-VHS on V. Then the Hodge locus of V for $\mathbb{V}$ is analytically dense. Moreover, if $d \geq g$, then the typical Hodge locus of V for $\mathbb{V}$ is analytically dense.*

Proof. Notice that we are in a situation where we can apply Theorem 3.3.2. Indeed, the generic Mumford-Tate group of $\mathbb{V}$ restricted to V is $\mathbf{G} = \mathbf{GSp}_{2g}$, whose derived subgroup is $\mathbf{Sp}_{2g}$, which is $\mathbb{Q}$-simple. On the other hand, since $\mathbb{V}$ is not constant, the algebraic monodromy group $\mathbf{H}$ is a non trivial connected normal subgroup of $\mathbf{Sp}_{2g}$ by Theorem 2.7.5, therefore we must have

$$\mathbf{H} = \mathbf{G}^{\text{der}} = \mathbf{Sp}_{2g}.$$

Finally, notice that

$$\dim V + \dim(\mathcal{A}_1 \times \mathcal{A}_{g-1}) - \dim \mathcal{A}_g \geq 0$$

for $\dim V \geq g - 1$, hence we can conclude by Theorem 3.3.2. Furthermore, if $\dim V \geq g$, the above inequality is strict, so the Hodge sub-datum is strictly $\mathbb{V}$-admissible, thus the density of the typical Hodge locus follows from Theorem 3.3.13. □

Now consider the moduli space $\mathcal{M}_g$ of smooth projective curves of genus g. Denote by

$$j : \mathcal{M}_g \hookrightarrow \mathcal{A}_g$$

the Torelli map, associating to the isomorphism class of a curve C the class of its Jacobian $J(C)$ in $\mathcal{A}_g$. By the classical Theorem of Torelli we have that j is injective (on geometric points). Clearly, j induces an integral polarized variation of Hodge structures on $\mathcal{M}_g$.

Definition 3.4.16. The image $j(\mathcal{M}_g) \subseteq \mathcal{A}_g$ is called open Torelli locus. Its Zariski closure in $\mathcal{A}_g$ is called Torelli locus, denoted $\mathcal{T}_g$.

Remark 3.4.17. Notice that in our setting ($g \geq 4$), the Torelli locus is strictly contained in $\mathcal{A}_g$ by dimension counting and it is Hodge generic, namely $\mathcal{T}_g$ is not contained in any strict special subvariety of $\mathcal{A}_g$. This is shown for instance in Moonen-Oort [38], Remark 4.5. The strategy to prove this fact is the following. Choose a point $t \in \mathcal{T}_g$ which is Hodge generic with respect to the restriction to the Torelli locus of the natural VHS on $\mathcal{A}_g$. The image of the monodromy representation

$$\rho : \pi_1(\mathcal{T}_g, t) \to \mathbf{GL}(\mathbb{V}_t \otimes \mathbb{Q})$$

is Zariski dense in the symplectic group $\mathbf{Sp}(\mathbb{V}_t \otimes \mathbb{Q})$, as follows from the discussion in Arbarello-Cornalba-Griffiths [2], section 15.3. But then, the algebraic monodromy group of $\mathcal{T}_g$ is $\mathbf{Sp}_{2g}$, so, by Theorem 2.7.5, the derived subgroup of the Mumford-Tate group of $\mathcal{T}_g$ contains $\mathbf{Sp}_{2g}$, so $\mathbf{MT}(\mathbb{V}_t \otimes \mathbb{Q})$ must be the biggest possible, namely $\mathbf{MT}(\mathbb{V}_t \otimes \mathbb{Q}) = \mathbf{GSp}_{2g}$. Hence, $t \in \mathcal{T}_g$ is Hodge generic with respect to the whole VHS defined on $\mathcal{A}_g$.

Proposition 3.4.18. *The (typical) Hodge locus of $\mathcal{M}_g$ for the polarized $\mathbb{Z}$-VHS induced by j is analytically dense in $\mathcal{M}_g$.*

Proof. This follows easily from Proposition 3.4.15 applied to the Torelli locus, keeping in mind that $\mathcal{M}_g$ has dimension $3g-3$, which is larger than g in our setting ($g \geq 4$). □

Notice that this application of Theorem 3.3.2 actually gives a more precise information, namely that the rational translates of $\mathcal{A}_1 \times \mathcal{A}_{g-1}$ intersect the image of the Torelli map in a dense subset. This is equivalent to the density of the family of curves in $\mathcal{M}_g$ whose Jacobian is isogenous to the product of an elliptic curve and an abelian variety of dimension $g-1$. This leads to the following question. Let $1 \leq k \leq g-1$ be an integer and consider

$$\mathcal{D}_k = \{C \in \mathcal{M}_g : J(C) \text{ contains an abelian subvariety of dimension } k\} \subseteq \mathcal{M}_g,$$

for which values of g and k is $\mathcal{D}_k$ analytically dense in $\mathcal{M}_g$?

In this direction, we can prove the following result, which was originally proven (with different techniques) by Colombo-Pirola [19], Theorem 3.

Proposition 3.4.19. *For $g \geq 4$ and $1 \leq k \leq 3$, $\mathcal{D}_k$ is analytically dense in $\mathcal{M}_g$.*

Proof. First of all notice that the condition of an abelian variety A containing a proper abelian subvariety B of dimension k is equivalent to A being isogenous to the product of B with another subvariety of A, as follows from Poincaré reducibility Theorem. In other words, $\mathcal{D}_k$ is in correspondence with the set of intersections of the open Torelli locus with rational translates of the special subvariety $\mathcal{A}_k \times \mathcal{A}_{g-k}$ of $\mathcal{A}_g$. Consider the Hodge sub-datum $(\mathbf{GSp}_{2k} \times \mathbf{GSp}_{2g-2k}, \mathbb{H}_k \times \mathbb{H}_{g-k})$ of $(\mathbf{GSp}_{2g}, \mathbb{H}_g)$, which corresponds to the special subvariety $\mathcal{A}_k \times \mathcal{A}_{g-k}$. Then the admissibility condition of this sub-datum with respect to the restriction to the Torelli locus of the natural VHS on $\mathcal{A}_g$

$$\dim \mathcal{M}_g + \dim(\mathcal{A}_k \times \mathcal{A}_{g-k}) - \dim \mathcal{A}_g \geq 0$$

gives

$$3g - 3 + \frac{k(k+1)}{2} + \frac{(g-k)(g-k+1)}{2} - \frac{g(g+1)}{2} \geq 0,$$

which simplifies to

$$(3-k)g + k^2 - 3 \geq 0$$

which is clearly satisfied under the assumptions of the Proposition. Hence, the rational translates of $\mathcal{A}_k \times \mathcal{A}_{g-k}$ intersect the Torelli locus in an analytically dense subset, from which we have the density of $\mathcal{D}_k$ in $\mathcal{M}_g$. □

Remark 3.4.20. Notice that if $k \geq 4$ the admissibility condition of the Hodge sub-datum $(\mathbf{GSp}_{2k} \times \mathbf{GSp}_{2g-2k}, \mathbb{H}_k \times \mathbb{H}_{g-k})$ only holds for genus not too high, namely for $g \leq \frac{k^2-3}{k-3}$. In this case we still have density of $\mathcal{D}_k$.

On the contrary, if $g > \frac{k^2-3}{k-3}$, then we have no unconditional result, unfortunately. However, in this case, the inequality

$$\dim \mathcal{M}_g + \dim(\mathcal{A}_k \times \mathcal{A}_{g-k}) - \dim \mathcal{A}_g < 0$$

forces all intersections of the Torelli locus with rational translates of $\mathcal{A}_k \times \mathcal{A}_{g-k}$ to be atypical, hence we expect by Conjecture 3.1.8 that $\mathcal{D}_k$ is not dense in $\mathcal{M}_g$.

Appendix A

Hodge structures of type (1,0),(0,1) and abelian varieties

In this Appendix, we will construct the period domain of polarized Hodge structures of type $(1,0),(0,1)$ and its associated Hodge variety. We will start by underlining the relationship between such Hodge structures and complex abelian varieties. For a detailed study of complex abelian varieties and of their moduli spaces we refer to Birkenhake-Lange [12].

Recall that a complex torus is a compact complex manifold of the form V/Λ, where V is a finite dimensional complex vector space and Λ is a lattice in V. A complex abelian variety is a complex torus T which admits a holomorphic embedding in the projective space, which gives by Chow's Theorem an isomorphism of T onto a smooth projective algebraic variety. An abelian variety together with such a projective embedding is called *polarized* abelian variety.

Recall now that the (co)homology in degree 1 of a complex torus $T = V/\Lambda$ is simple: indeed we have the natural identification $\Lambda \cong H_1(X, \mathbb{Z})$. Moreover, T is a compact Kähler manifold, hence we have a Hodge structure of weight 1

$$H^1(T, \mathbb{Z}) \otimes \mathbb{C} \cong H^1(T, \mathbb{C}) = H^{1,0}(T) \oplus H^{0,1}(T)$$

and $H^{1,0}(T)$ identifies with the dual V^*. Indeed the holomorphic cotangent bundle of T is trivial and globally generated by the complex linear forms on V, seen as holomorphic forms on V invariant under Λ. The key fact to notice now is that T is determined by the Hodge structure on $H^1(T, \mathbb{Z})$. Indeed, given the Hodge decomposition on $H^1(T, \mathbb{C})$, we can reconstruct the torus as

$$T = (H^{1,0})^*/H^1(T, \mathbb{Z})^*.$$

Clearly, this also shows that whenever we have an integral Hodge structure of type $(1,0),(0,1)$ on a lattice Λ, namely

$$\Lambda \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1},$$

the quotient $T = H^{1,0}/\Lambda$ is a complex torus whose associated Hodge structure on $H^1(T, \mathbb{Z})$ is dual to the given one. We thus have proven the following:

Proposition A.1. *There is an equivalence of categories between the category of Hodge structures of type* $(1,0),(0,1)$ *on a free abelian group of rank* $2g$ *and the category of complex tori of dimension* g*.*

Remark A.2. A perhaps more intrinsic point of view on this equivalence is the following. If $H_{\mathbb{Z}}$ is a free abelian group of rank $2g$, then a Hodge structure of type $(1,0),(0,1)$ on $H_{\mathbb{Z}}$ corresponds to a complex structure on the real vector space $H_{\mathbb{R}} = H_{\mathbb{Z}} \otimes \mathbb{R}$, namely to an endomorphism J of $H_{\mathbb{R}}$ such that $J^2 = -\mathrm{id}$. Indeed, such a complex structure is uniquely determined by the eigenspaces of $\pm i$ of the $\mathbb{C}$-linear extension of J to $H_{\mathbb{C}} = H_{\mathbb{R}} \otimes \mathbb{C}$, hence by a decomposition $H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$ where $H^{1,0} = \overline{H^{0,1}}$. On the other hand, a complex torus can be described exactly as a quotient $H_{\mathbb{R}}/H_{\mathbb{Z}}$, seen as a differentiable manifold, together with a complex structure on $H_{\mathbb{R}}$, which automatically gives an integrable almost complex structure on this differentiable manifold. However, one has to keep in mind that this complex structure is dual to the Hodge structure on the first cohomology group of the torus. A usual convention in the literature is thus to refer to this Hodge structure as being of type $(-1,0),(0,-1)$.

Now we need to understand what happens if the Hodge structure is polarized.

Proposition A.3. *There is an equivalence of categories between the category of polarized Hodge structures of type* $(1,0),(0,1)$ *on a free abelian group of rank* $2g$ *and the category of polarized complex abelian varieties of dimension* g*.*

Proof. In view of the previous Proposition it is enough to show that, for a free abelian group Λ, a polarization Q on the Hodge structure

$$\Lambda \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$$

is equivalent to the datum of a projective embedding of the complex torus $T = H^{1,0}/\Lambda$.

Such a polarization Q is an alternating bilinear form on Λ, hence an element of

$$\bigwedge^2 \Lambda^* = \bigwedge^2 H^1(T,\mathbb{Z}) = H^2(T,\mathbb{Z}),$$

namely the cohomology class of a 2-form ω_Q on T lying the integral cohomology of T. We claim that the defining properties of a polarization force this form ω_Q to be an integral Kähler form on T. Then ω_Q induces a projective embedding of T by one of the equivalent ways of stating Kodaira's embedding Theorem (see for instance Huybrechts [31], Proposition 5.3.1 and Corollary 5.3.3).

Recall that the decomposition $\Lambda \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$ is exactly the decomposition associated with the complex structure on the tangent space of T. The form ω_Q is constant hence clearly closed, thus we are left to check that it is positive and of type (1,1). Both these properties follows from the Hodge-Riemann bilinear relations that Q satisfies since it is a polarization. In particular, the first Hodge-Riemann relation, namely the orthogonality of Hodge decomposition with respect to the hermitian form defined by $h(v,u) = iQ(v,\overline{u})$, implies that

$$Q(v,v') = 0 \text{ for all } v, v' \text{ being both in } H^{1,0} \text{ or } H^{0,1},$$

hence the 2-form ω_Q vanishes on $H^{1,0} \times H^{1,0}$ and $H^{0,1} \times H^{0,1}$. Thus ω_Q is of type (1,1). Furthermore, the second Hodge-Riemann relation, namely the positivity of h on $H^{1,0}$, implies that ω_Q is positive, so we are done. □

We can construct now the period domain D of polarized Hodge structures of type $(1,0),(0,1)$. To do so, let us fix a free abelian group $H_{\mathbb{Z}}$, the Hodge number $g = \dim H^{1,0}$ and a polarization Q. In an appropriate basis of $H_{\mathbb{Z}}$, the matrix representing Q has form

$$\left(\begin{array}{c|c} 0 & M \\ \hline -M & 0 \end{array}\right)$$

where M is a diagonal matrix. We will work here under the assumption that M can be chosen to be the identity matrix, namely we will work with a *principal* polarization. So, let $\{\delta^j, \gamma^j\}_{1\leq j\leq g}$ be a basis for $H_{\mathbb{Z}}$ for which Q has matrix

$$\left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$$

where I denotes the identity matrix. Let $\{w_i\}_{1\leq i\leq g}$ be a basis of $H^{1,0}$ as complex vector space. Let us write the change of basis as

$$w_i = \sum_{j=1}^{g} A_{ij}\delta^j + \sum_{j=1}^{g} B_{ij}\gamma^j$$

and consider the $g \times 2g$ matrix $P = (A|B)$.

The period domain D is of course a subset of the Grassmannians of g-dimensional subspaces of $H_{\mathbb{C}}$. Let us show how the Hodge-Riemann bilinear relations for Q impose restrictions on such subspaces by describing the conditions that are induced on the matrix P.

Lemma A.4. *The $g \times g$ matrix A is non singular.*

Proof. Pick a non zero vector $v \in H^{1,0}$ and consider its coordinates in the basis $\{w_i\}_i$, namely $v = \sum_{k=1}^{g} v_k w_k$. Then, we can write

$$v = \sum_{k,j} v_k A_{kj}\delta^j + \sum_{k,j} v_k B_{kj}\gamma^j.$$

The (second) Hodge-Riemann relation gives $iQ(v,\overline{v}) > 0$, which reads in basis $\{\delta^j, \gamma^j\}$

$$i\sum_{j,k,l}(v_k\overline{v_l}A_{kj}\overline{B_{lj}} - v_k\overline{v_l}B_{kj}\overline{A_{lj}}) > 0.$$

So, if we consider the matrix M defined by

$$M_{kl} = i\sum_{j}(A_{kj}\overline{B_{lj}} - B_{kj}\overline{A_{lj}}),$$

or, equivalently,

$$M = i(AB^* - BA^*)$$

where B^* denotes the Hermitian conjugate of B, we get that M is positive definite. This allows to conclude that A is non singular. Indeed, if there exists a non zero $v \in H^{1,0}$ such that ${}^t vA = 0$, then $A^*\overline{v} = 0$, so ${}^t vM\overline{v} = 0$, contradicting the positivity of M. □

As a consequence of this Lemma we can bring the matrix P in the normalized form $P = (I|Z)$, with $Z \in M_g(\mathbb{C})$, still parametrizing the same subspace $H^{1,0}$ of $H_{\mathbb{C}}$.

Lemma A.5. *With notations as above, we have that Z is a complex symmetric matrix with positive definite imaginary part.*

Proof. We need to use the first Hodge-Riemann relation, namely the fact that the decomposition $H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$ is orthogonal with respect to the hermitian form h associated to Q. This means that for all $v \in H^{1,0}$ and $u \in H^{0,1}$ one must have

$$h(v,u) = iQ(v,\overline{u}) = 0$$

or, equivalently, for all couple of vectors $v, v' \in H^{1,0}$

$$Q(v, v') = 0.$$

Once we have changed basis in order to bring P to the normalized form $(I|Z)$ this relation is equivalent to

$$\sum_{k,l}(v_k v'_l Z_{lk} - v_k v'_l Z_{kl}) = {}^t v(Z - {}^t Z)v' = 0.$$

Hence, Z is symmetric. Then, the claim that its imaginary part is positive definite is exactly the fact that the matrix M defined in the previous proof is positive definite, once the construction of M is done with the normalized form of P, namely with $A = I$ and $B = Z$. □

Now, clearly Z determines uniquely the g-dimensional subspace $H^{1,0}$ of $H_{\mathbb{C}}$ in a way so that $H_{\mathbb{C}} = H^{1,0} \oplus \overline{H^{1,0}}$ is a Hodge structure polarized by Q. But recall that we began by choosing a basis of the lattice $H_{\mathbb{Z}}$, or equivalenty by fixing an isomorphism $\mathbb{Z}^{2g} \cong H_{\mathbb{Z}}$. This is what is usually called a *marking* of the Hodge structure. So, the above argument proves the following:

Proposition A.6. *The period domain classifying* marked *Hodge structures of type* $(1,0),(0,1)$ *on a free abelian group of rank* $2g$ *with a principal polarization* Q *is the space*

$$\mathbb{H}_g = \{Z \in M_g(\mathbb{C}) : Z \text{ symmetric with positive definite imaginary part}\}$$

which is usually called Siegel upper half-space.

Let us now give a group theoretic description of the Siegel upper half-space, giving a concrete instance of Theorem 2.6.2. To do so, notice that we have a natural action of $\mathrm{Sp}(2g, \mathbb{R})$ on $\mathbb{H}_g$.

Lemma A.7. *The group* $\mathrm{Sp}(2g, \mathbb{R})$ *acts on* $\mathbb{H}_g$ *by*

$$M(Z) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \cdot Z = (AZ + B)(CZ + D)^{-1}$$

for each matrix $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in \mathrm{Sp}(2g, \mathbb{R})$.

Proof. First of all notice that the condition

$$\left(\begin{array}{c|c} {}^tA & {}^tC \\ \hline {}^tB & {}^tD \end{array}\right)\left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$$

implies that ${}^tAC$ and ${}^tBD$ are symmetric and ${}^tAD - {}^tCB = I$. Applying these identities we obtain

$${}^t\overline{(CZ + D)}(AZ + B) - {}^t\overline{(AZ + B)}(CZ + D) = Z - \overline{Z} = 2i\mathrm{Im}(Z).$$

Since the imaginary part of $Z \in \mathbb{H}_g$ is positive definite, this implies that $CZ + D$ is invertible, so the action is well defined. It remains to check that $M(Z) \in \mathbb{H}_g$. The same relations at the beginning of the proof give

$${}^t(CZ + D)(M(Z) - {}^t(M(Z)))(CZ + D) = Z - {}^tZ = 0,$$

so $M(Z)$ is symmetric. Similarly,

$${}^t\overline{(CZ + D)}\mathrm{Im}(M(Z))(CZ + D) = \mathrm{Im}(Z),$$

thus $M(Z)$ has positive definite imaginary part. □

Proposition A.8. *One has*
$$\mathbb{H}_g \cong \mathrm{Sp}(2g,\mathbb{R})/U(g).$$

Proof. Let us start by showing that the action of $\mathrm{Sp}(2g,\mathbb{R})$ on $\mathbb{H}_g$ is transitive. Let $Z \in \mathbb{H}_g$ and write $Z = X + iY$, with Y symmetric and positive definite. Then there exists $A \in \mathrm{GL}_g(\mathbb{R})$ such that $Y = A \cdot {}^tA$. Consider the matrix

$$N = \left(\begin{array}{c|c} A & X\,{}^tA^{-1} \\ \hline 0 & {}^tA^{-1} \end{array}\right).$$

Clearly we have
$${}^tN \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right) N = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right),$$

so $N \in \mathrm{Sp}(2g,\mathbb{R})$. Moreover
$$N(iI) = X + iA\,{}^tA = Z.$$

Hence the action is transitive.

Let us now compute the stabilizer of $iI \in \mathbb{H}_g$. Let $M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in \mathrm{Sp}(2g,\mathbb{R})$. M stabilizes iI if and only if $(iA + B)(iC + D)^{-1} = iI$, hence if and only if $(A - iB)(D + iC)^{-1} = I$, which gives
$$\begin{cases} A = D \\ B = -C \end{cases}$$

Furthermore, it's easy to check that the condition

$$\left(\begin{array}{c|c} {}^tA & {}^tC \\ \hline {}^tB & {}^tD \end{array}\right) \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right) \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$$

implies that the inverse of M is given by

$$M^{-1} = \left(\begin{array}{c|c} {}^tD & -{}^tB \\ \hline -{}^tC & {}^tA \end{array}\right).$$

Therefore, the stabilizer of iI consists precisely of those matrices M in $\mathrm{Sp}(2g,\mathbb{R})$ that have form $\left(\begin{array}{c|c} A & B \\ \hline -B & A \end{array}\right)$ and this implies that

$$M^{-1} = \left(\begin{array}{c|c} {}^tA & -{}^tB \\ \hline {}^tB & {}^tA \end{array}\right) = {}^tM.$$

Hence the stabilizer of iI is $\mathrm{Sp}(2g,\mathbb{R}) \cap O(2g,\mathbb{R})$, which identifies with the unitary group $U(g)$ via the map sending a matrix $\left(\begin{array}{c|c} A & B \\ \hline -B & A \end{array}\right)$ to $A + iB$. We can conclude that $\mathbb{H}_g \cong \mathrm{Sp}(2g,\mathbb{R})/U(g)$. □

Notice that, by Proposition A.3, parametrizing *marked* Hodge structures of type $(1,0),(0,1)$ on a lattice of rank $2g$ with a *principal* polarization Q is the same as parametrizing g-dimensional *principally polarized* abelian varieties A together with a *marking* of the first cohomology group $H^1(A,\mathbb{Z})$, or, equivalently, of the first homology group. If we want a classifying space which does not keep the datum of such a marking we just need to quotient by the group which permutes the markings, namely $\mathrm{Sp}(2g,\mathbb{Z})$. We thus have obtained that the moduli space of principally polarized abelian varieties (parametrizing their isomorphism classes) is

$$\mathcal{A}_g = \mathrm{Sp}(2g,\mathbb{Z})\backslash\mathbb{H}_g = \mathrm{Sp}(2g,\mathbb{Z})\backslash\mathrm{Sp}(2g,\mathbb{R})/U(g).$$

Finally notice that, since $\mathbb{H}_g$ is open in the space of symmetric $g \times g$ matrices, we have

$$\dim \mathcal{A}_g = \dim \mathbb{H}_g = \frac{g(g+1)}{2}.$$

Remark A.9. Once again, we can give a more intrinsic point of view on the construction of $\mathbb{H}_g$ and $\mathcal{A}_g$. Indeed, as we have already shown, a Hodge structure of type $(1,0),(0,1)$ on $H_{\mathbb{Z}}$ is equivalent to the datum of a complex structure on $H_{\mathbb{R}}$. Moreover the defining properties of a polarization impose the following restrictions on the complex structure.

Lemma A.10. *Let us fix a free abelian group $H_{\mathbb{Z}}$ of rank $2g$ and a symplectic form Q on it. Then Q is a polarization for the Hodge structure associated with a complex structure J on $H_{\mathbb{R}}$ if and only if:*

1) $J^*Q = Q$, that is $Q(Jv, Ju) = Q(v,u)$ for all $v, u \in H_{\mathbb{R}}$,

2) the bilinear form defined by $g_J(v,u) = Q(Jv,u)$ is positive definite.

Proof. Assume that Q is a polarization. Let v, u be arbitrary vectors in $H_{\mathbb{R}}$ and write their decomposition with respect to $H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$ as

$$v = v^{1,0} + v^{0,1}, \quad u = u^{1,0} + u^{0,1}.$$

Recall that in the Hodge structure associated with J, $H^{1,0}$ (resp. $H^{0,1}$) is the eigenspace of the complex linear extension of J, that we will still denote J, with respect to the the eigenvalue i (resp. $-i$). Then, using the fact that Q vanishes on $H^{1,0} \times H^{1,0}$ and $H^{0,1} \times H^{0,1}$ by the first Hodge-Riemann relation, we obtain

$$Q(Jv, Ju) = Q(iv^{1,0}, -iu^{0,1}) + Q(-iv^{0,1}, iu^{1,0}) = Q(v,u).$$

Notice that this property implies that g_J is symmetric, since

$$g_J(v,u) = Q(Jv,u) = Q(-v, Ju) = Q(Ju, v) = g_J(u,v).$$

Moreover, keeping the same notations and using the second Hodge-Riemann relation, we get

$$g_J(v,v) = Q(Jv,v) = Q(iv^{1,0}, v^{0,1}) + Q(-iv^{0,1}, v^{1,0}) = 2iQ(v^{1,0}, v^{0,1}) > 0, \tag{A.1}$$

so g_J is positive definite.

Conversely, assume that conditions (1) and (2) hold. Then for each couple of vectors v, v' both lying in the same eigenspace of J we have

$$Q(v,v') = Q(Jv, Jv') = -Q(v,v')$$

which implies the first Hodge-Riemann relation for Q. The second Hodge-Riemann relation follows directly from the computation (A.1). □

In view of the previous Lemma we can equivalently define the Siegel upper half-space, parametrizing Hodge structures of type $(1,0),(0,1)$ on $H_{\mathbb{Z}}$ polarized by Q, as

$$\mathfrak{S}(H_{\mathbb{R}},Q) = \{J \in \operatorname{End}(H_{\mathbb{R}}) : J^2 = -\mathrm{id}, J^*Q = Q, g_J \text{ positive definite}\}.$$

From this point of view it also clear the group theoretic description of $\mathfrak{S}(H_{\mathbb{R}},Q)$.

Indeed, let us check that the action by conjugation of the symplectic group $\mathrm{Sp}(H_{\mathbb{R}},Q)$ on $\mathfrak{S}(H_{\mathbb{R}},Q)$ is transitive. First, notice that if $J \in \mathfrak{S}(H_{\mathbb{R}},Q)$, then $h_J(v,u) = g_J(v,u) - iQ(v,u)$ is a positive definite hermitian form on the vector space $H_{\mathbb{R}}$ together with the complex structure J, which can thus be seen as a complex vector space. Now, let $J, J' \in \mathfrak{S}(H_{\mathbb{R}},Q)$ and let $\{w_i\}_i$ (resp. $\{w_i'\}_i$) be a h_J-unitary (resp. $h_{J'}$-unitary) basis for $(H_{\mathbb{R}},J)$ (resp. $(H_{\mathbb{R}},J')$). Then, $\varphi(w_i) = w_i'$ defines a complex isometry $(H_{\mathbb{R}},J) \to (H_{\mathbb{R}},J')$. Its underlying morphism φ at the level of real vector spaces satisfies

$$\varphi^*Q = -\varphi^*(\operatorname{Im}(h_{J'})) = -\operatorname{Im}(h_J) = Q,$$

so $\varphi \in \mathrm{Sp}(H_{\mathbb{R}},Q)$. Moreover, looking at the real parts of $h_J, h_{J'}$, we see that $\varphi^*h_{J'} = h_J$ implies

$$Q(J'(\varphi(v)),\varphi(u)) = Q(Jv,u) = Q(\varphi(J(v)),\varphi(u)),$$

thus $\varphi J = J'\varphi$, proving that the action by conjugation is transitive.

Furthermore, if $\varphi \in \mathrm{Sp}(H_{\mathbb{R}},Q)$, we have

$$h_J(\varphi(v),\varphi(u)) = g_J(\varphi(v),\varphi(u)) - iQ(\varphi(v),\varphi(w)) = Q(J\varphi(v),\varphi(u)) - iQ(v,u),$$

thus $\varphi^*h_J = h_J$ if and only if $\varphi^{-1}J\varphi = J$. Therefore, the stabilizer of a given complex structure J in the Siegel upper half-space is exactly the unitary group $U((H_{\mathbb{R}},J),h_J)$. Thus we can conclude

$$\mathfrak{S}(H_{\mathbb{R}},Q) \cong \mathrm{Sp}(H_{\mathbb{R}},Q)/U((H_{\mathbb{R}},J),h_J).$$

Remark A.11. As we have already briefly mentioned, this Hodge variety $\mathrm{Sp}(2g,\mathbb{Z})\backslash\mathbb{H}_g$ represents a very peculiar case. Indeed, the quotient $D = G/K$ of a non-compact simple (real) Lie group modulo a maximal compact subgroup is a symmetric space, namely it carries a G-invariant Riemannian metric and at each point $p \in D$ there is an isometry i_p fixing p and acting as $-\mathrm{id}$ on the tangent space. This is the case for $U(g)$ inside $\mathrm{Sp}(2g,\mathbb{R})$, so the Siegel upper half-space is a Hermitian symmetric space, that is a symmetric space which has a compatible structure of complex manifold. Then, by a Theorem of Baily-Borel [4], the quotient $\mathrm{Sp}(2g,\mathbb{Z})\backslash\mathbb{H}_g$ has the structure of a quasi-projective variety.

Let us again remark that this is a very special case, as for Hodge structures of higher weights, a Hodge variety $\Gamma\backslash D$ is in general not the analytification of an algebraic variety.

Let us end this Appendix by having a look at Mumford-Tate groups associated with polarized Hodge structures of type $(1,0),(0,1)$.

Let $\rho : \mathbf{S} \to \mathbf{GL}(H_{\mathbb{R}})$ be the representation of the Deligne torus that is associated with a Hodge structure of type $(1,0),(0,1)$ polarized by an alternating bilinear form Q. Recall from the proof of Proposition 1.2.7 that, in our convention, $\rho(z)$ acts on the subspace $H^{p,q}$ by multiplication by $z^{-p}\overline{z}^{-q}$, hence in our case ρ is defined as

$$\rho(z)(v) = z^{-1}v^{1,0} + \overline{z}^{-1}v^{0,1},$$

where, as usual, $v = v^{1,0} + v^{0,1}$ is the decomposition of a vector $v \in H_{\mathbb{R}}$ with respect to the decomposition $H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$.

We claim that the Hodge group $\mathbf{Hg}(\rho)$ is contained in the symplectic group $\mathbf{Sp}_{2g}$, seen as algebraic group over $\mathbb{Q}$. To show this it is enough to check that the image of the subgroup

$\mathbf{S}^1(\mathbb{R}) = \{z \in \mathbb{C} : z\overline{z} = 1\}$ under the morphism ρ (at the level of $\mathbb{R}$-valued points) is contained in the group $\mathrm{Sp}(2g, \mathbb{R})$. So, let $z \in \mathbf{S}^1(\mathbb{R})$. For $v, u \in H_{\mathbb{R}}$ we have

$$\begin{aligned} Q(\rho(z)v, \rho(z)u) &= Q(\overline{z}v^{1,0} + zv^{0,1}, \overline{z}u^{1,0} + zu^{0,1}) \\ &= Q(\overline{z}v^{1,0}, zu^{0,1}) + Q(zv^{0,1}, \overline{z}u^{1,0}) \\ &= z\overline{z}Q(v, u) \\ &= Q(v, u) \end{aligned}$$

thus $\mathbf{Hg}(\rho)(\mathbb{R}) \subseteq \mathrm{Sp}(2g, \mathbb{R})$.

An analogous computation shows that the Mumford-Tate group of ρ is contained in the general symplectic group $\mathbf{GSp}_{2g}$, seen as algebraic group over $\mathbb{Q}$, that is the group of automorphisms of $H_{\mathbb{Q}}$ that respect the form Q up to a non zero scalar. In other words

$$\mathbf{GSp}_{2g} = \mathbf{G}_m \cdot \mathbf{Sp}_{2g},$$

from which it is clear that $\mathbf{GSp}_{2g}$ is a reductive group, by Proposition B.8.

For a generic abelian variety A, the Mumford-Tate group associated with the Hodge decomposition on the first cohomology group of A is the biggest possible, namely $\mathbf{GSp}_{2g}$ and the Siegel upper half-space $\mathbb{H}_g$ can be seen, as we have discussed, as the Mumford-Tate domain associated to $\mathbf{GSp}_{2g}$.

The main way to detect abelian varieties A with smaller Mumford-Tate group, namely those lying in the Hodge locus of $\mathcal{A}_g$, is to look at the endomorphism algebra of A. Let $\mathrm{End}(A)$ be the endomorphism ring of an abelian variety A and denote by $D(A)$ its associated endomorphism algebra, namely $D(A) = \mathrm{End}(A) \otimes \mathbb{Q}$. By proposition A.3 we have

$$D(A) = \mathrm{End}_{\mathbb{Q}-\mathrm{HS}}(H^1(A, \mathbb{Q})),$$

where the right hand side denotes the $\mathbb{Q}$-algebra of endomorphisms of the Hodge structure on $H^1(A, \mathbb{Q})$, in the sense of Definition 1.2.3. On the other hand, $\mathrm{End}_{\mathbb{Q}-\mathrm{HS}}(H^1(A, \mathbb{Q}))$ is exactly the set of (rational) Hodge classes on the rational vector space $\mathrm{End}(H^1(A, \mathbb{Q}))$ carrying the Hodge structure induced from that on $H^1(A, \mathbb{Q})$. Hence, putting together the above equality and Proposition 1.3.4 we get

$$D(A) = \mathrm{End}(H^1(A, \mathbb{Q}))^{\mathbf{MT}(A)}$$

where $\mathbf{MT}(A)$ denotes the Mumford-Tate group of the Hodge structure on $H^1(A, \mathbb{Q})$. So, it is clear that the Mumford-Tate group shrinks when the abelian variety A has *exceptional* endomorphisms.

If E is an elliptic curve the situation is simple. Indeed, there are only two possibilities: either $D(E) = \mathbb{Q}$ or it is a quadratic imaginary field, in which case E is said to have *complex multiplication.*

Proposition A.12. *Let E be a complex elliptic curve with endomorphism algebra D. If $D = \mathbb{Q}$, then $\mathbf{MT}(E) = \mathbf{GL}_2$; if D is an imaginary quadratic field k, then $\mathbf{MT}(E)$ is the torus T_k.*

Proof. Notice that $\mathbf{Sp}_2 = \mathbf{SL}_2$ and the general symplectic group $\mathbf{GSp}_2$ is just $\mathbf{GL}_2$. By Proposition 1.3.5, the Mumford-Tate group of E is a connected reductive subgroup $\mathbf{M}$ of $\mathbf{GL}_2$ containing $\mathbf{G}_m \cdot \mathrm{id}$. The only such subgroups of $\mathbf{GL}_2$ are the multiplicative group $\mathbf{G}_m \cdot \mathrm{id}$, maximal tori of $\mathbf{GL}_2$ and $\mathbf{GL}_2$ itself. In the first two cases the subspace of endomorphisms of $H^1(E, \mathbb{Q})$ that are fixed by $\mathbf{M}$ is clearly bigger that $\mathbb{Q}$, hence, keeping in mind that

$$D(E) = \mathrm{End}(H^1(E, \mathbb{Q}))^{\mathbf{M}},$$

we obtain that $D = \mathbb{Q}$ implies $\mathbf{MT}(E) = \mathbf{GL}_2$.

If D is an imaginary quadratic field k, then $H^1(A, \mathbb{Q})$ is free of rank 1 over D and $\mathbf{MT}(E) \subseteq T_k$, where we see the torus T_k as a $\mathbb{Q}$-algebraic group, namely its functor of points sends any $\mathbb{Q}$-algebra R to the group of units $(R \otimes_{\mathbb{Q}} k)^{\times}$. The possibility $\mathbf{MT}(E) = \mathbf{G}_m$ is again ruled out since in this case we would have $D = \mathrm{End}(H^1(E, \mathbb{Q}))$, which is a contradiction, as $\mathrm{End}(H^1(E, \mathbb{Q}))$ is not commutative. Hence the Mumford-Tate group of E is the torus T_k. $\square$

Appendix B

Reductive groups

In this Appendix we collect some very basic facts about reductive groups, which we have used throughout the thesis. The theory of reductive groups is extremely rich, while this collection will be limited to the little part that we have actually used in this work and has no aim of providing a comprehensive introduction to this theory. We refer to Borel [13], Humphreys [30] and Milne [36] for introductions to this theory as well as for some basic facts on algebraic groups, which we assume the reader to be already familiar with.

Recall that a group scheme is a scheme whose functor of points factors through the category of groups. If k is a field, which we will assume being of characteristic zero throughout this Appendix, an *algebraic group* over k is a group scheme which is an algebraic variety over k, namely a reduced separated scheme of finite type over $\mathrm{Spec}(k)$. For our aims, we could actually restrict our attention to *linear algebraic groups*, which are closed subgroups of the general linear group $\mathbf{GL}(V)$, for some k-vector space V.

Remark B.1. Among the notions that have to be translated from abstract group theory to the theory of algebraic groups, the one of quotient requires particular attention. Indeed, if $\mathbf{G}$ is an algebraic group over k and $\mathbf{H}$ is a normal subgroup, the functor on k-algebras which takes such an algebra R to the quotient $\mathbf{G}(R)/\mathbf{H}(R)$ taken in the category of groups, is not necessarily a sheaf. Thus, it is not trivial to define a quotient $\mathbf{G}/\mathbf{H}$ in the sense of algebraic groups.

A modern perspective to solve this problem is considering the sheafification of that functor with respect to the fppf (faithfully flat, finitely presented) topology.

A more classical point of view is enough for us. If $\mathbf{H}$ is any closed subgroup of $\mathbf{G}$, then one can show that there exists a (unique up to isomorphism) separated scheme X together with a morphism of schemes $\pi : \mathbf{G} \to X$ such that:

1) for all k-algebras R, the non empty fibers of $\pi(R) : \mathbf{G}(R) \to X(R)$ are the cosets of $\mathbf{H}(R)$ in $\mathbf{G}(R)$,

2) each element of $X(R)$ lifts to an element of $\mathbf{G}(R')$ for some faithfully flat R-algebra R'.

Such a scheme X has the natural universal property of a quotient, namey every morphism $\mathbf{G} \to X'$ which is constant on the cosets of $\mathbf{H}(R)$ in $\mathbf{G}(R)$ for all k-algebras R, factors uniquely through π.

Moreover, if $\mathbf{H}$ is a normal subgroup, then X has the structure of an algebraic group. We refer to Milne [36] (Chapter 5, section c) for details.

Once we have an appropriate notion of quotients of algebraic groups we can give the following definitions.

Definition B.2. Let $\mathbf{G}$ be a connected algebraic group over k.

1) The *radical* of $\mathbf{G}$ is the largest connected normal solvable subgroup of $\mathbf{G}$.

2) The *unipotent radical* of $\mathbf{G}$ is the subgroup of unipotent elements of the radical of $\mathbf{G}$.

Definition B.3. Let $\mathbf{G}$ be a connected algebraic group over k. We say that $\mathbf{G}$ is

1) k-simple if it does not have non-trivial normal connected algebraic subgroups,

2) semisimple if its radical is trivial,

3) reductive if its unipotent radical is trivial.

Remark B.4. Notice that in general one would need to consider the geometric (unipotent) radical, namely the (unipotent) radical of the base change of $\mathbf{G}$ to an algebraic closure of k, in order to give the previous definition. However, the formation of the (unipotent) radical commutes with base change along separable field extensions (see Milne [36], Proposition 19.1 and 19.9), so in our case, where k is assumed to have characteristic zero, it is enough to consider the (unipotent) radical over k.

Proposition B.5. *A semisimple algebraic group* $\mathbf{G}$ *is the almost direct product* $\mathbf{G} = \mathbf{G}_1 \cdots \mathbf{G}_n$ *of its* k*-simple normal algebraic subgroups, namely the multiplication map*

$$m : \mathbf{G}_1 \times \cdots \times \mathbf{G}_n \to \mathbf{G}$$

is flat, surjective and has finite kernel.

Proof. See Milne [36], Theorem 21.51. □

We have the following useful criterion for reductive groups, for which we refer to Milne [36], Proposition 19.17.

Proposition B.6. *If a connected algebraic group over* k *admits a faithful semi-simple representation, then it is reductive.*

Definition B.7. Let $\mathbf{G}$ be a connected algebraic group.

1) The *adjoint group* $\mathbf{G}^{\mathrm{ad}}$ of $\mathbf{G}$ is the quotient $\mathbf{G}/Z(\mathbf{G})$ of $\mathbf{G}$ modulo its center.

2) The *derived subgroup* $\mathbf{G}^{\mathrm{der}}$ of $\mathbf{G}$ is the intersection of the normal subgroups $\mathbf{N}$ of $\mathbf{G}$ such that the quotient $\mathbf{G}/\mathbf{N}$ is abelian.

Proposition B.8. *A connected algebraic group* $\mathbf{G}$ *over* k *is reductive if and only if it is the almost direct product of a torus and a semisimple group. In this case, these groups can be given by the identity connected component of the center* $Z(\mathbf{G})$ *of* $\mathbf{G}$ *and the derived subgroup* $\mathbf{G}^{\mathrm{der}}$*. In particular, if* $\mathbf{G}$ *is reductive, then its derived subgroup and adjoint group are semisimple.*

Proof. See Milne [36] (Proposition 21.60) and Borel [13] (section 14.2). □

Example B.9. The algebraic groups $\mathbf{GL}_n$, $\mathbf{SO}_n$, $\mathbf{Sp}_{2m}$, defined over k, are reductive.

Bibliography

[1] Yves André. Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part. *Compositio Mathematica*, 82(1):1–24, 1992.

[2] Enrico Arbarello, Maurizio Cornalba, and Phillip Griffiths. *Geometry of algebraic curves: volume II with a contribution by Joseph Daniel Harris*. Springer, 2011.

[3] James Ax. On Schanuel's conjectures. *Annals of mathematics*, 93(2):252–268, 1971.

[4] Walter L. Baily and Armand Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Annals of mathematics*, pages 442–528, 1966.

[5] Benjamin Bakker, Bruno Klingler, and Jacob Tsimerman. Tame topology of arithmetic quotients and algebraicity of Hodge loci. *Journal of the American Mathematical Society*, 33(4):917–939, 2020.

[6] Benjamin Bakker and Jacob Tsimerman. The Ax–Schanuel conjecture for variations of Hodge structures. *Inventiones mathematicae*, 217:77–94, 2019.

[7] Benjamin Bakker and Jacob Tsimerman. Lectures on the Ax–Schanuel conjecture. *Arithmetic Geometry of Logarithmic Pairs and Hyperbolicity of Moduli Spaces: Hyperbolicity in Montréal*, pages 1–68, 2020.

[8] Gregorio Baldi, Bruno Klingler, and Emmanuel Ullmo. Non-density of the exceptional components of the Noether-Lefschetz locus. *Arxiv:2312.11246*, 2023.

[9] Gregorio Baldi, Bruno Klingler, and Emmanuel Ullmo. On the distribution of the Hodge locus. *Arxiv:2107.08838*, 2023.

[10] Arnaud Beauville. Le groupe de monodromie des familles universelles d'hypersurfaces et d'intersections complètes. In *Complex Analysis and Algebraic Geometry: Proceedings of a Conference held in Göttingen, June 25–July 2, 1985*, pages 8–18. Springer, 2006.

[11] José Bertin and Chris Peters. Variations of Hodge structure, Calabi-Yau manifolds and Mirror Symmetry. *Introduction to Hodge Theory, Panoramas et synthèses. Amer. Math. Soc*, 2002.

[12] Christina Birkenhake and Herbert Lange. *Complex abelian varieties*, volume 302. Springer, 2004.

[13] Armand Borel. *Linear algebraic groups*, volume 126. Springer Science & Business Media, 2012.

[14] James Carlson. Period domains and period mappings. *Summer school on Hodge theory, ICTP*, 2010.

[15] James Carlson, Stefan Müller-Stach, and Chris Peters. *Period mappings and period domains.* Cambridge University Press, 2017.

[16] Eduardo Cattani. Introduction to variation of Hodge structure. *Summer school on Hodge theory, ICTP*, 2010.

[17] Eduardo Cattani, Pierre Deligne, and Aroldo Kaplan. On the locus of Hodge classes. *Journal of the American Mathematical Society*, 8(2):483–506, 1995.

[18] Ciro Ciliberto, Joe Harris, and Rick Miranda. General components of the Noether-Lefschetz locus and their density in the space of all surfaces. *Mathematische Annalen*, 282(4):667–680, 1988.

[19] Elisabetta Colombo and Gian Pietro Pirola. Some density results for curves with nonsimple Jacobians. *Mathematische Annalen*, 288(1):161–178, 1990.

[20] Pierre Deligne and Luc Illusie. Relèvements modulo p^2 et décomposition du complexe de de Rham. *Inventiones mathematicae*, 89:247–270, 1987.

[21] Ron Donagi. Generic Torelli for projective hypersurfaces. *Compositio Mathematica*, 50(2-3):325–353, 1983.

[22] Hubert Flenner. The infinitesimal Torelli problem for zero sets of sections of vector bundles. *Mathematische Zeitschrift*, 193:307–322, 1986.

[23] Alessandro Ghigi. On some differential-geometric aspects of the Torelli map. *Bollettino dell'Unione Matematica Italiana*, 12(1):133–144, 2019.

[24] Mark Green, Phillip Griffiths, and Matt Kerr. *Mumford-Tate groups and domains: Their geometry and arithmetic.* Princeton University Press, 2012.

[25] Phillip Griffiths. Periods of integrals on algebraic manifolds, I (construction and properties of the modular varieties). *American Journal of Mathematics*, 90(2):568–626, 1968.

[26] Phillip Griffiths. Periods of integrals on algebraic manifolds, II (local study of the period mapping). *American Journal of Mathematics*, 90(3):805–865, 1968.

[27] Phillip Griffiths, Colleen Robles, and Domingo Toledo. Quotients of non-classical flag domains are not algebraic. *ArXiv:1303.0252*, 2013.

[28] Alexander Grothendieck. On the de Rham cohomology of algebraic varieties. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 29(1):95–103, 1966.

[29] Robin Hartshorne. *Algebraic Geometry*, volume 52. Springer Science & Business Media, 2013.

[30] James E. Humphreys. *Linear algebraic groups*, volume 21. Springer Science & Business Media, 2012.

[31] Daniel Huybrechts. *Complex Geometry: an introduction*, volume 78. Springer, 2005.

[32] Nicholas M. Katz and Tadao Oda. On the differentiation of de Rham cohomology classes with respect to parameters. *Journal of Mathematics of Kyoto University*, 8(2):199–213, 1968.

[33] Nazim Khelifa and David Urbanik. Existence and density of typical Hodge loci. *ArXiv:2303.16179*, 2023.

[34] Bruno Klingler. Hodge theory, between algebraicity and transcendence. In *Proc. Int. Cong. Math*, volume 3, pages 2250–2284, 2022.

[35] Bruno Klingler and Anna Otwinowska. On the closure of the positive dimensional Hodge locus. *Arxiv:1901.10003*, 2020.

[36] James S. Milne. *Algebraic groups: the theory of group schemes of finite type over a field*, volume 170. Cambridge University Press, 2017.

[37] Ben Moonen. An introduction to Mumford-Tate groups. *unpublished lecture notes, Amsterdam*, 2004.

[38] Ben Moonen and Frans Oort. The Torelli locus and special subvarieties. *ArXiv:1112.0933*, 2011.

[39] Chris A.M. Peters and Joseph H.M. Steenbrink. *Mixed Hodge structures*, volume 52. Springer Science & Business Media, 2008.

[40] Wilfried Schmid. Variation of Hodge structure: the singularities of the period mapping. *Inventiones mathematicae*, 22(3):211–319, 1973.

[41] Edoardo Sernesi. *Deformations of algebraic schemes*, volume 334. Springer Science & Business Media, 2007.

[42] Jean-Pierre Serre. Géométrie algébrique et géométrie analytique. In *Annales de l'institut Fourier*, volume 6, pages 1–42, 1956.

[43] Tetsuji Shioda. Algebraic cycles on hypersurfaces in $\mathbb{P}^n$. 10:717–733, 1987.

[44] Genival da Silva Jr. Known cases of the Hodge conjecture. *ArXiv:2105.04695*, 2021.

[45] Jeroen Spandaw. *Noether-Lefschetz problems for degeneracy loci.* Number 764. American Mathematical Soc., 2003.

[46] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry I.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2002.

[47] Claire Voisin. *Hodge Theory and Complex Algebraic Geometry II.* Cambridge University Press, 2003.

[48] Raymond O. Wells. *Differential analysis on complex manifolds*, volume 65. Springer New York, 2008.