MIRROR SYMMETRY AND CALABI-YAU FAMILIES OVER THE THRICE PUNCTURED SPHERE

- MIRROR SYMMETRY

- $$\begin{array}{ccc} X_p & \subset & \mathcal{X} \\ \downarrow & & \downarrow \\ p & \in & \mathbb{P}^1 \end{array}$$

  X_p PROJECTIVE ALGEB. CALABI-YAU MANIFOLD WHEN $p \notin \{0,1,\infty\}$

Q: WHAT DO THESE HAVE TO DO WITH ONE ANOTHER?

ANS: SEMINAL WORK OF GREENE-PLESSER & CANDELAS-DELAOSSA-GREEN-PARKES

- QUINTIC HYPERSURFACE $\subset \mathbb{P}^4$
- GENERIC QUINTIC HAS $h^{1,1} = 1$ COMING FROM HYPERPLANE CLASS
- EXPECTATION FROM MIRROR SYMMETRY IS THAT THERE EXISTS ANOTHER MIRROR CALABI-YAU THREEFOLD WITH $h^{2,1} = 1$
- QUINTIC HAS $h^{2,1} = 101$, SO:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

QUINTIC

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 101 & & 0 & \\
1 & & 1 & & 1 & & 1 \\
 & 0 & & 101 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

QUINTIC MIRROR

GREENE – PLESSER: STARTING FROM GEPNER MODELS IN CFT PROPOSED A CONSTRUCTION OF THE MIRROR

$$\sum_{i=0}^{4} x_i^5 = 0 \Big/ \text{QUOTIENT BY ACTION OF } (\mathbb{Z}/5\mathbb{Z})^3 \text{ \& RESOLVE SINGS}$$

FERMAT QUINTIC

THAT GIVES ONE CY 3-FOLD WITH THE CORRECT HODGE NUMBERS.

THE OTHERS ARISE BY DOING THE SAME CONSTRUCTION IN A FAMILY

$$\mu \sum_{i=0}^{4} x_i^5 + 5\nu \prod_{i=0}^{4} x_i = 0 \Big/ (\mathbb{Z}/5\mathbb{Z})^3 \text{ \& RESOLVE}$$

- CONSTRUCTED IN THIS WAY A FAMILY OF SMOOTH, PROJECTIVE CALABI-YAU 3-FOLDS OVER $\mathbb{P}^1 \setminus \{0,1,\infty\}$

CANDELAS-DE LA OSSA-GREEN-PARKES:

- STUDIED INTEGRALS OF $\omega^{(3,0)}$ OVER INTEGRAL 3-CYCLES ON THE CY 3-FOLDS IN THIS FAMILY
- PERIODS SATISFY GENERALIZED HYPERGEOMETRIC ODE ANNIHILATING

$$ {}_4F_3\left(\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}; 1,1,1 \,\middle|\, z\right) $$

FOR $z \in \mathbb{P}^1$ BASE OF FAMILY

- THREE REGULAR SINGULAR POINTS

0	1	∞
MAX UNIPOTENT MONODROMY	CONIFOLD MONODROMY	QUASI-UNIPOTENT MONODROMY

KONTSEVICH'S HMS:

- AUTOMORPHISMS OF D^b_{coh}(QUINTIC) CORRESPOND TO INTEGRAL MONODROMIES ABOUT $0, 1, \infty$ IN MIRROR QUINTIC FAMILY
- SHOULD APPLY TO ANY CY 3-FOLD WITH $h^{1,1}=1$
- SIMPLEST MIRRORS SHOULD BE FAMILIES OF CALABI-YAU 3-FOLDS WITH $h^{2,1}=1$ OVER $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

0	1	∞
MAX UNIPOTENT MONODROMY	CONIFOLD MONODROMY	QUASI-UNIPOTENT MONODROMY

- MIRROR SYMMETRY OFTEN TAKES HARD PROBLEM $\longleftrightarrow$ EASY PROBLEM

D-MORGAN:

- WHAT ABOUT MIRRORING THE HARD PROBLEM OF FINDING ALL CALABI-YAU 3-FOLDS WITH $h^{1,1}=1$?
- MIRROR PROBLEM: FIND ALL CALABI-YAU 3-FOLDS WITH $h^{2,1}=1$
- SIMPLEST NONTRIVIAL CASE WOULD HAVE BASE $\mathbb{P}^1 \setminus \{0,1,\infty\}$
- TO BE COMPATIBLE WITH KONTSEVICH PREDICTION SHOULD LOOK FOR $\mathbb{Z}$-VHS OVER $\mathbb{P}^1 \setminus \{0,1,\infty\}$ WITH

0	1	∞
MAX UNIPOTENT MONODROMY	CONIFOLD MONODROMY	QUASI-UNIPOTENT MONODROMY

- IS THIS EVEN A FINITE PROBLEM?

- DELIGNE: FINITENESS OF $\mathbb{Z}$-VHS & EQUIVALENCE WITH CLASSIFICATION OF INTEGRAL SYMPLECTIC MONODROMY REPRESENTATIONS

$$\rho : \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, z_0) \to Sp(4,\mathbb{Z})$$

- D-MORGAN: CLASSIFICATION OF WEIGHT 3 HODGE TYPE (1,1,1,1) $\mathbb{Z}$-VHS OVER $\mathbb{P}^1 \setminus \{0,1,\infty\}$ WITH

0	1	∞
MAX UNIPOTENT MONODROMY	CONIFOLD MONODROMY	QUASI-UNIPOTENT MONODROMY

$\Rightarrow$ 112 POSSIBLE $\mathbb{Z}$-VHS UP TO CONJUGATION EQUIVALENCE

23 OF THESE ARE "MIRROR COMPATIBLE" IN A STRONG SENSE

14 $\mathbb{R}$-VHS

m	a	a_1, a_2, a_3, a_4	$\# L_{\mathbb{Z}}$	$\# L_{\mathrm{MC}}$	t	Geometric examples
1	4	$\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}$	1	1	1	I
1	3	$\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$	1	1	1	$\mathbb{WP}^4_{1,1,1,2,5}[10]$
2	4	$\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}$	2	2	1	$\mathbb{WP}^4_{1,1,1,1,4}[8]$
					2	II
5	5	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$	2	2	1	$\mathbb{P}^4[5]$
					5	A
1	2	$\frac{1}{6}, \frac{1}{6}, \frac{5}{6}, \frac{5}{6}$	1	1	1	$\mathbb{WP}^5_{1,1,2,2,3,3}[6,6]$
2	3	$\frac{1}{6}, \frac{1}{4}, \frac{3}{4}, \frac{5}{6}$	1	1	1	$\mathbb{WP}^5_{1,1,1,2,2,3}[4,6]$ *
3	4	$\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6}$	1	1	1	$\mathbb{WP}^4_{1,1,1,1,2}[6]$
4	5	$\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{5}{6}$	11	1	1	$\mathbb{WP}^5_{1,1,1,1,1,3}[2,6]$
4	4	$\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}$	8	3	1	$\mathbb{WP}^5_{1,1,1,1,2,2}[4,4]$
					2	B
					4	C
6	5	$\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}$	1	1	1	$\mathbb{WP}^5_{1,1,1,1,1,2}[3,4]$ *
8	6	$\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}$	14	2	1	$\mathbb{P}^5[2,4]$
					2	D
9	6	$\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}$	8	2	1	$\mathbb{P}^5[3,3]$
					3	E
12	7	$\frac{1}{3}, \frac{1}{2}, \frac{1}{2}, \frac{2}{3}$	11	1	1	$\mathbb{P}^6[2,2,3]$
16	8	$\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	50	4	1	$\mathbb{P}^7[2,2,2,2]$
					2	F
					4	G
					8	H

TABLE 1. Table of "Mirror-Consistent" $\rho\colon \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \to Sp(4)$

- QUESTION REMAINS OF GEOMETRIC REALIZATION OF EACH $\mathbb{Z}$-VHS BY AN ACTUAL FAMILY OF PROJECTIVE ALGEBRAIC CALABI-YAU 3-FOLDS WITH $h^{2,1} = 1$
- FOR 13 OF THE 14 $\mathbb{R}$-VHS, THIS WAS EASY, AS EXISTING TORIC HYPERSURFACE OR CI CONSTRUCTIONS & BATYREV-BORISOV MIRROR SYMMETRY GAVE THE FAMILY
- "14th CASE" WAS MUCH MORE SUBTLE

$${}_4F_3\left(\tfrac{1}{12}, \tfrac{5}{12}, \tfrac{7}{12}, \tfrac{11}{12}; 1,1,1 \middle| z\right)$$

- NOT DIFFICULT TO SEE THAT THERE IS A 3-PARAMETER FAMILY OF $h^{2,1} = 3$ CALABI-YAU 3-FOLDS AS THE MIRROR OF [2,12] CICY IN $\mathbb{WP}^5(1,1,2,8,12)$ WHOSE GKZ-HYPERGEOMETRIC PERIOD FUNCTIONS RESTRICT TO THE DESIRED ${}_4F_3\left(\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12}; 1,1,1 \middle| z\right)$ ALONG A 1-PARAMETER SUBFAMILY
- <u>BUT</u> : EVERYWHERE ALONG THIS 1-PARAMETER SUBFAMILY THE MIRROR CICY BECOME SINGULAR
- TO THIS DAY : NO KNOWN SMOOTH FAMILY OF PROJECTIVE ALGEBRAIC CY 3-FOLDS THAT REALIZE $\mathbb{Z}$-VHS !

CLINGHER-D.-LEWIS-NOVOSELTSEV-THOMPSON:

- EVEN AFTER CREPANT PARTIAL DESINGULARIZATION, GENERIC CY3 IN FAMILY HAS A PAIR OF $\mathbb{Q}$-FACTORIAL NODAL SINGULARITIES
- NOT EVEN BIRATIONAL TO SMOOTH CY3 REALIZING THE $\mathbb{Z}$-VHS DIRECTLY
- SHOW CAN GEOMETRICALLY REALIZE $\mathbb{Z}$-VHS, AS 3^{rd} WEIGHT GRADED PIECE OF MIXED HODGE STRUCTURE ON H^3 ADMITS A PURE $\mathbb{Z}$-VHS OF WT. 3 & HODGE TYPE (1,1,1,1) THAT REALIZES THE 14^{th} CASE $\mathbb{Z}$-VHS.
- CONSTRUCTION PROVIDES A COUNTEREXAMPLE TO A CONJECTURE OF MORRISON ON MIRROR OF DEGENS

Q: WHAT GEOMETRICALLY DO ALL OF THE "MIRROR" CY 3-FOLDS HAVE IN COMMON

A: THEY ARE MIRRORS OF ... NOT SATISFYING

BETTER ANSWER COMES FROM OBSERVING INTERNAL STRUCTURE OF THESE CY 3 FOLDS

- D-HARDER-NOVOSELTSEV-THOMPSON
  THEY ARE FIBERED BY HIGH PICARD RANK (18 OR 19) K3 SURFACES
- D.-MALMENDIER
  THEY ARE FIBERED (POSSIBLY IN MULTIPLE WAYS) BY M_n-POLARIZED K3 SURFACES = MIRRORS TO

  $\langle 2n \rangle$-POLARIZED K3 SURFACES

- D-HARDER-NOVOSELTSEV-THOMPSON
  THEY ARE FIBERED BY HIGH PICARD RANK (18 OR 19) K3 SURFACES
- <u>METHOD</u>: SEARCH FOR TORICALLY INDUCED FIBRATIONS ON CY 3-FOLD HYPERSURFACES OR CI'S & RECOGNIZE FIBERS AS K3 SURFACE HYPERSURFACES OR CI'S IN OTHER TORIC VARIETIES
- MOTIVATED A PROGRAM (SUCCESSFUL!) TO COMPLETELY CLASSIFY & CONSTRUCT ALL CY 3-FOLDS THAT ARE FIBERED BY M_n-POLARIZED K3 SURFACES

- D.-MALMENDIER
  THEY ARE FIBERED (POSSIBLY IN MULTIPLE WAYS) BY M_n-POLARIZED K3 SURFACES = MIRRORS TO $\langle 2n \rangle$-POLARIZED K3 SURFACES
- METHOD: ITERATIVELY CONSTRUCT FAMILIES OF CY n-FOLDS FROM FAMILIES OF CY $(n-1)$-FOLDS
- STARTING POINT: CY 0-FOLDS
  = PAIR OF POINTS
  = SOLUTION TO $y^2 = 1-t$
  $\{\pm y\}$
  WITH PERIOD
  $${}_1F_0\left(\tfrac{1}{2} \,\middle|\, t\right)$$

- FOR (i,j,α) SUCH THAT $1 \leq i \leq 2$, $1 \leq j \leq 2\alpha$, AND $\alpha \in \{\frac{1}{2}, 1\}$ WE CONSTRUCT TWISTED FAMILIES:

$$\tilde{y}^2 = \left(1 - \frac{c_{ij}\tilde{t}}{\tilde{x}^i(\tilde{x}+1)^j}\right)\tilde{x}^2(\tilde{x}+1)^{2\alpha}$$

OF GENUS ONE CURVES

(i,j,α)	μ	singular fibers	notation
$(1,1,1)$	$\frac{1}{2}$	$I_1^*(\tilde{t}=\infty),\ I_4(\tilde{t}=0),\ I_1(\tilde{t}=1)$	X_{141}
$(2,1,1)$	$\frac{1}{3}$	$IV^*(\tilde{t}=\infty),\ I_3(\tilde{t}=0),\ I_1(\tilde{t}=1)$	X_{431}
$(1,1,\frac{1}{2})$	$\frac{1}{4}$	$III^*(\tilde{t}=\infty),\ I_2(\tilde{t}=0),\ I_1(\tilde{t}=1)$	X_{321}
$(2,1,\frac{1}{2})$	$\frac{1}{6}$	$II^*(\tilde{t}=\infty),\ I_1(\tilde{t}=0),\ I_1(\tilde{t}=1)$	X_{211}

Table 4: Families of elliptic curves

GIVING RATIONAL ELLIPTIC MODULAR SURFACES WITH SECTION FOR $\Gamma_0(n)$, $n = 1,2,3,4$.

- AT THE NEXT STEP WE TWIST UP AGAIN

derived from		ρ	Configuration of singular fibers			$MW(\pi,\sigma)$	disc Q	Λ
Srfc	μ		$u=\infty$	$u^2+u+t/4$	$u=0,-1$			
X_{141}	$\frac{1}{2}$	19	$I_8\ (A_7)$	$2\,I_1$	$2\,I_1^*\ (2\,D_5)$	[4]	2^3	M_4
X_{431}	$\frac{1}{3}$	19	$I_6\ (A_5)$	$2\,I_1$	$2\,IV^*\ (2\,E_6)$	[3]	$2\cdot 3$	M_3
X_{321}	$\frac{1}{4}$	19	$I_4\ (A_3)$	$2\,I_1$	$2\,III^*\ (2\,E_7)$	[2]	2^2	M_2
X_{211}	$\frac{1}{6}$	19	$I_2\ (A_1)$	$2\,I_1$	$2\,II^*\ (2\,E_8)$	[1]	2	M_1

Table 7: K3 surfaces from extremal rational surfaces

YIELDING WEIERSTRASS ELLIPTIC FIBRED LATTICE-POLARIZED K3 SURFACES WITH GENERIC PICARD LATTICE OF TYPE

$$M_n := H \oplus E_8 \oplus E_8 \oplus \langle -2n \rangle \quad , \; n = 1, 2, 3, 4$$

- AGAIN THE FAMILY OF K3 SURFACES IS FIBERED OVER THE THRICE PUNCTURED SPHERE

- ONCE MORE AND WE GET ALL 60 SYMPLECTICALLY RIGID $\mathbb{Z}$-VHS CLASSIFIED BY BOGNER-REITER, INCLUDING THE 14 $\mathbb{R}$-VHS OF HYPERGEOMETRIC TYPE
- EXPLICIT WEIERSTRASS ELLIPTIC M_n-POLARIZED K3 SURFACE-FIBERED CALABI-YAU 3-FOLDS
- AGAIN ARE FAMILIES OVER THE THRICE-PUNCTURED SPHERE!
- NO NEED TO STOP THERE... IN FACT GET INFINITE CY TOWERS OVER THE THRICE PUNCTURED SPHERE

MOST FAMOUS SOURCE OF CY MIRROR MANIFOLDS IN ALL DIMN'S :

DEGREE $(n+1)$ HYPERSPCES IN $\mathbb{P}^n$: X_{n-1}

MIRROR TO X_{n-1} IS FIBERED BY MIRROR TO X_{n-2} !

<u>COMMENT</u>: THE ACTUAL PROOF OF CONSTR. OF PERIODS OVER $\mathbb{Z}$-CYCLES IN CDOGP HAS A HIDDEN REALIZ. OF "QUARTIC MIRROR FIBRATION ON QUINTIC MIRROR"