Étale sheaves

§1. The étale site

Def. Grothendieck topology on a category C:

$\forall U \in C$ a set of families of maps $\underbrace{(U_i \to U)_{i\in I}}_{\text{called coverings of } U}$ s.t.

- $\forall (U_i \to U)_i$ covering, $V \to U$: $U_i \times_U V$ exist, and $(U_i \times_U V \to V)_i$ is a covering of V
- if $(U_i \to U)_{i\in I}$ covering of U, $(V_{ij} \to U_i)_{j\in J_i}$ covering of U_i $\forall i \in I$ then $(V_{ij} \to U_i \to U)_{i,j}$ covering of U, and
- $\forall U \in C$, $(U \xrightarrow{\mathrm{id}} U)$ is a covering.

<u>site</u>: category together with a Grothendieck topology

<u>sheaf</u> on a site C: presheaf $\mathcal{F}$ on C s.t. $(\forall U \in C)\ \forall (U_i \to U)_i$ covering,

$$\begin{array}{ccccc} \mathcal{F}(U) & \longrightarrow & \prod_i \mathcal{F}(U_i) & \rightrightarrows & \prod_{i,j} \mathcal{F}(U_i \times_U U_j) \\ f & \mapsto & (f|_{U_i})_i & & \\ & & (f_i)_i & \mapsto & (f_i|_{U_i\times_U U_j})_{i,j} \\ & & & \mapsto & (f_j|_{U_i\times_U U_j})_{i,j} \end{array}$$

("sheaf condition")

is an equalizer diagram (i.e. $\mathcal{F}(U) = \{(f_i)_i \in \prod_i \mathcal{F}(U_i) \mid f_i|_{U_i\times_U U_j} = f_j|_{U_i \times_U U_j}\ \forall i,j\}$)

<u>Ex.</u> (i) X top. space. C the category w/

- objects: open subsets of X,
- morphisms: inclusion maps.

C becomes a site w/

- coverings: $(U_i \hookrightarrow U)_i$ s.t. $\bigcup_{i\in I} U_i = U$.

e.g. X scheme. $\rightsquigarrow$ X_{Zar} <u>Zariski site</u> of X.

(ii) X scheme, <u>(small) étale site</u> $X_{\acute{e}t}$ of X:

- underlying category: $Et/_X$ (i.e. w/ objects ~~are~~ the étale morphisms $U \to X$, X-morphisms)
- coverings: jointly surjective families $(U_i \xrightarrow{\varphi_i} U)_i$ (i.e. $\bigcup_i \varphi_i(U_i) = U$).

§ <u>Galois coverings</u>

<u>Def.</u> $Y \to X$ faithfully flat, G finite group acting on Y over X (on the right).

$Y \to X$ is a <u>Galois covering with group G</u> if the morphism

$$Y \times G \longrightarrow Y \times_X Y, \quad (y, g) \longmapsto (y, yg),$$

is an isomorphism.

<u>Rmk.</u> Any Galois covering is surjective, finite, and étale of degree $|G|$.

<u>Ex.</u> $k'|k$ finite Galois. Then

$$k' \otimes_k k' \xrightarrow{\cong} \prod_{\sigma \in Gal(k'|k)} k', \quad \alpha \otimes \beta \longmapsto (\alpha \sigma(\beta))_\sigma,$$

so that $\operatorname{Spec} k' \longrightarrow \operatorname{Spec} k$ is a Galois covering with group $Gal(k'|k)$.

<u>Prop.</u> $Y \to X$ Galois with group G, $\mathcal{F} \in PSh(X_{\acute{e}t})$ taking disjoint unions to products.

Then $\mathcal{F}$ ~~is a sheaf~~ satisfies the sheaf condition for the covering $Y \to X$ iff the restriction map $\mathcal{F}(X) \longrightarrow \mathcal{F}(Y)$ identifies $\mathcal{F}(X)$ with $\mathcal{F}(Y)^G$.

<u>A criterion to be a sheaf</u>

<u>Prop.</u> $\mathcal{F}$ ~~[illegible]~~ presheaf on $X_{\acute{E}t}$. Assume that $\mathcal{F}$ satisfies the sheaf condition for

(a) Zariski open coverings, and

(b) étale coverings $(V \longrightarrow U)$ with U, V affine.

Then $\mathcal{F}$ is a sheaf.

Examples of sheaves on $X_{ét}$.

(i) The structure sheaf ~~on X~~ $\mathcal{O}_{X_{ét}}$.

$U \to X$ étale $\rightsquigarrow$ define $\mathcal{O}_{X_{ét}}(U) := \Gamma(U, \mathcal{O}_U)$.

$\Rightarrow$ (a) is satisfied.

for (b), we use

Lemma. $A \to B$ faithfully flat then the sequence

$$0 \to A \to B \to B \otimes_A B$$
$$b \mapsto 1 \otimes b - b \otimes 1$$

is exact.

(ii) The sheaf induced by a qcoh. $\mathcal{O}_X$-module F

$U \xrightarrow{\varphi} X$ étale $\rightsquigarrow$ define $F^{ét}(U) := \Gamma(U, \varphi^* F)$.

Condition (a) is satisfied.

for (b), we need to show that if $A \to B$ is faithfully flat and $M \in \mathrm{Mod}_A$ then

$$0 \to M \longrightarrow B \otimes_A M \longrightarrow B \otimes_A B \otimes_A M$$
$$b \otimes m \longmapsto 1 \otimes b \otimes m - b \otimes 1 \otimes m$$

is exact.

The proof is the same as for $M = A$.

(iii) Representable sheaves. ~~on~~

$Z \in Sch/_X \rightsquigarrow h_Z : Et/_X \to Set$, $U \mapsto \mathrm{Hom}_X(U, Z)$.

(a): ✓

for (b), if $Z = \mathrm{Spec}\, C$ is affine then this follows from the Lemma.

For the general case one works locally, using that $V \to U$ étale is open, in order to reduce to the ~~case~~ Z affine case.

(iv) Sheaves on $(\mathrm{Spec}\,k)_{ét}$.

Recall that étale k-algebras are precisely the finite products of finite separable field extensions of k.

A presheaf $\mathcal{F}$ of abelian groups on $(\mathrm{Spec}\,k)_{ét}$ is the same as a covariant functor $\mathrm{Et}/k \to \mathrm{Ab}$.

$\mathcal{F}$ is a sheaf iff

- $\mathcal{F}(\prod' A_i) = \bigoplus \mathcal{F}(A_i)$ for every finite family $(A_i)_i$ of étale k-algebras, and
- $\mathcal{F}(k') \xrightarrow{\cong} \mathcal{F}(K)^{\mathrm{Gal}(K/k')}$ for every finite Galois extension $K|k'$ with $k'|k$ finite.

<u>Rmk.</u> Here we use that every finite étale map $V \to U$ is dominated by a Galois map $V' \to U$, i.e. there exists a Galois covering $V' \to U$ factoring through $V \to U$ and s.t. $V' \to V$ is surjective.

<u>Prop.</u> There is an equivalence of categories

$$\left\{ \begin{array}{c} \text{sheaves of ab. grps.} \\ \text{on } (\mathrm{Spec}\,k)_{ét} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{discrete } G\text{-} \\ \text{modules} \end{array} \right\}. \qquad (\text{where } G = \mathrm{Gal}(k^{sep}|k)).$$

<u>Sketch.</u> • $\mathcal{F}$ sheaf on $(\mathrm{Spec}\,k)_{ét}$.

$$\rightsquigarrow \quad M_{\mathcal{F}} := \varinjlim_{\substack{k' \subset k^{sep} \\ k'|k \text{ fin. Galois}}} \mathcal{F}(k').$$

• M discrete G-module; $A \in \mathrm{Et}/k$.

$$\rightsquigarrow \quad \mathcal{F}_M(A) := \mathrm{Hom}_G(F(A), M). \qquad (\text{where } F(A) = \mathrm{Hom}_{k\text{-alg}}(A, k^{sep}).)$$

<u>Def.</u> X scheme, $\mathcal{F}$ presheaf on $X_{ét}$, $\underbrace{\bar{x} \to X \text{ geom. pt.}}$ Define the <u>stalk</u> of $\mathcal{F}$ at $\bar{x}$ by

$$\mathcal{F}_{\bar{x}} := \varinjlim_{\substack{(U,u) \text{ étale} \\ \text{nbhd. of } \bar{x}}} \mathcal{F}(U).$$

<u>Ex.</u> • $\mathcal{O}_{X_{ét},\bar{x}} = \mathcal{O}_{X,\bar{x}}$, the <u>étale local ring</u> at $\bar{x}$.

• $Z \to X$ of finite type then $h_{Z,\bar{x}} = Z(\mathcal{O}_{X,\bar{x}})$. (e.g. μ_n, G_a, G_m, GL_n).

Def. X scheme, $\bar{x} \to X$ geometric point, $\overbrace{\Lambda}^{\text{abelian group}}$. For $U \to X$ étale define

$$\Lambda^{\bar{x}}(U) := \bigoplus_{\mathrm{Hom}_X(\bar{x},U)} \Lambda .$$

If the image $x \in X$ of $\bar{x} \to X$ is closed then $\Lambda^{\bar{x}}$ is called the skyscraper sheaf with value Λ at x.

Rmk. • $\Lambda^{\bar{x}}$ is a sheaf (even if x is not a closed point)

• there is a natural isom. $\mathrm{Hom}(\mathcal{F}, \Lambda^{\bar{x}}) \xrightarrow{\cong} \mathrm{Hom}(\mathcal{F}_{\bar{x}}, \Lambda)$ $\forall$ sheaf $\mathcal{F}$ on $X_{ét}$.

Def. X scheme, Λ set. For $U \to X$ étale ~~define~~ set

$$\underline{\Lambda}(U) := \Lambda^{\pi_0(U)} .$$

$\underline{\Lambda}$ is called the constant sheaf on $X_{ét}$ defined by Λ.

A sheaf $\mathcal{F}$ on $X_{ét}$ is called locally constant if there exists an étale covering $(U_i \to X)_i$ s.t. $\mathcal{F}|_{U_i}$ is constant $\forall i$.

Thm. X connected, $\bar{x} \to X$ geometric point. The functor

values in finite sets

$$\left\{ \begin{matrix} \text{finite locally constant sheaves} \\ \text{(with finite stalks) on } X_{ét} \end{matrix} \right\} \longrightarrow \{ \text{finite } \pi_1(X,\bar{x})\text{-sets} \}$$

$$\mathcal{F} \longmapsto \mathcal{F}_{\bar{x}} .$$

is an equivalence.

The category (P)Sh($X_{ét}$).

Def. Let $Sh(X_{ét})$ be the category of sheaves of abelian groups on $X_{ét}$, a full subcategory of the category $PSh(X_{ét})$ of presheaves of abelian groups.

Rmk. $PSh(X_{ét})$ is an abelian category by defining kernel & cokernel "termwise"

(i.e. if $\varphi: \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves then set $(co)ker(\varphi)(U) := (co)ker(\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U))$ for all U étale over X).

A sequence $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is exact if $\mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$ is exact $\forall U \to X$ étale

Thm. For any presheaf $\mathcal{F}$ on $X_{ét}$ there exists a sheaf $\tilde{\mathcal{F}}$ on $X_{ét}$ together with a morphism $\iota : \mathcal{F} \to \tilde{\mathcal{F}}$ s.t. for any sheaf $\mathcal{G}$ with a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ there is $\tilde{\varphi} : \tilde{\mathcal{F}} \to \mathcal{G}$ s.t. the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\iota"] \arrow[dr, "\varphi"'] & & \tilde{\mathcal{F}} \arrow[dl, dashed, "\tilde{\varphi}"] \\ & \mathcal{G} & \end{tikzcd}$$

commutes.

Def. The sheaf $\tilde{\mathcal{F}}$ is called the <u>sheafification</u> of $\mathcal{F}$.

(**Rmk.** As usual, $\tilde{\mathcal{F}}$ is unique up to unique isomorphism)

Thm. ((i) The functor $\tilde{(\cdot)} : PSh(X_{ét}) \longrightarrow Sh(X_{ét})$ is exact)

(ii) TFAE:

(a) $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact in $Sh(X_{ét})$,

(b) $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$ is exact for any U étale over X, and $\mathcal{F} \to \mathcal{F}''$ is locally surjective, i.e. ...

(c) $0 \to \mathcal{F}'_{\bar{x}} \to \mathcal{F}_{\bar{x}} \to \mathcal{F}''_{\bar{x}} \to 0$ is exact for each geometric point $\bar{x} \to X$.

(iii) $Sh(X_{ét})$ is an abelian category.

Let $n \in \mathbb{N}$.

Ex. Consider the <u>Kummer sequence</u>

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow[u \mapsto u^n]{n} \mathbb{G}_m \longrightarrow 1.$$

Clearly $1 \to \mu_n(U) \to \mathbb{G}_m(U) \to \mathbb{G}_m(U)$ is exact for all $U \to X$ étale.
Now, $\mathbb{G}_m \xrightarrow{n} \mathbb{G}_m$ is rarely ~~an epim~~ surjective in $PSh(X_{ét})$, as well as in $Sh(X_{Zar})$.

Let's see if it is surjective in $Sh(X_{ét})$: For a geometric point $\bar{x} \to X$ let $A = \mathcal{O}_{X,\bar{x}}$ be the étale local ring. Then $\frac{d}{dT}(T^n - a) = n \cdot T^{n-1}$, i.e. $T^n - \bar{a} \in k[T]$ is separable if $char(k) \nmid n$, in which case it splits completely in $A[T]$ since A is strictly Henselian. Thus if $char\, k(x) \nmid n$ $\forall x \in X$ then the sequence is exact.

Direct and inverse image.

Def. $f: Y \to X$ morphism of schemes, $\mathcal{G}$ presheaf on $Y_{\acute{e}t}$. For $U \to X$ étale define

$$(f_* \mathcal{G})(U) := \mathcal{G}(U \times_X Y).$$

$f_* \mathcal{G}$ is called the <u>direct image</u> (or <u>pushforward</u>) of $\mathcal{G}$ along f.

Rmk. ~~Lemma.~~ If $\mathcal{F}$ is a sheaf then so is $f_* \mathcal{F}$.

[**Ex.** $i_{\bar{x}}: \bar{x} \to X$ geometric point, Λ abelian group. Then $i_{\bar{x},*} \Lambda = \Lambda^{\bar{x}}$ is the skyscraper sheaf at x w/ value Λ.]

$\mathcal{G}$ presheaf on $Y_{\acute{e}t}$.

Prop. (i) $j: V \hookrightarrow X$ open immersion. (ii) $i: Z \hookrightarrow X$ closed immersion (iii) $\pi: Y \to X$ finite.

$$(j_* \mathcal{G})_{\bar{x}} = \begin{cases} \mathcal{F}_{\bar{x}}, & x \in V \\ ?, & x \notin V \end{cases} \qquad (i_* \mathcal{G})_{\bar{x}} = \begin{cases} \mathcal{G}_{\bar{x}}, & x \in Z \\ 0, & x \notin Z \end{cases} \qquad (\pi_* \mathcal{G})_{\bar{x}} = \bigoplus_{y \in \pi^{-1}(x)} \mathcal{F}_{\bar{y}}^{d(y)},$$

$d(y) = $ sep. deg. of $k(y)/k(x)$

Cor. i closed immersion $\Rightarrow$ i_* is exact.

[**Ex.** k alg. closed, $X = \mathbb{A}^1_k$. $U = X \setminus \{0\}$, $j: U \hookrightarrow X$ inclusion.
$\mathcal{F}$ the locally constant $\hat{\mathbb{Z}}$ sheaf on U corresponding to a $\pi_1(U,\bar{u})$-module F.
Then $(j_* \mathcal{F})_{\bar{0}} = F^{\pi_1(U,\bar{u})}$.]

Def. $f: Y \to X$ morphism of schemes, $\mathcal{F}$ presheaf on X. ~~Set~~ For $V \to Y$ étale set

$$(f^* \mathcal{F})^{pre}(V) := \varinjlim_{\substack{V \to U \\ \downarrow \circlearrowleft \downarrow \\ Y \to X}} \mathcal{F}(U).$$

The <u>inverse image</u> (or <u>pullback</u>) of $\mathcal{F}$ along f is the sheaf $\widetilde{(f^*\mathcal{F})^{pre}} =: f^* \mathcal{F}$.

Rmk. f^* is a left adjoint to f_*, i.e. if $\mathcal{F} \in Sh(X_{\acute{e}t})$, $\mathcal{G} \in Sh(Y_{\acute{e}t})$ then there is an isomorphism

$$\mathrm{Hom}_{Sh(Y_{\acute{e}t})}(f^* \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{Sh(X_{\acute{e}t})}(\mathcal{F}, f_* \mathcal{G})$$

bifunctorial in $\mathcal{F}$ and $\mathcal{G}$.

Ex. $\pi : U \to X$ étale. Then $\pi^* : Sh(X_{ét}) \to Sh(U_{ét})$ is just the restriction map.
(this is seen (e.g.) by showing that $Hom(\mathcal{F}|_{U_{ét}}, \mathcal{G}) \cong Hom(\mathcal{F}, \pi_* \mathcal{G})$ $\forall \mathcal{F}, \mathcal{G}$.)

Rmk. $i : \bar{x} \to X$ geometric point, $\mathcal{F}$ sheaf on $X_{ét}$ then $(i^*\mathcal{F})(\bar{x}) = \mathcal{F}_{\bar{x}}$ (by def)

$\Rightarrow$ for any morphism $f : Y \to X$ and geom pt. $i : \bar{y} \to Y$

$$(f^*\mathcal{F})_{\bar{y}} = i^*(f^*\mathcal{F})(\bar{y}) = \mathcal{F}_{\bar{x}}$$

where $\bar{x}$ is the geometric point $\bar{y} \xrightarrow{i} Y \xrightarrow{f} X$.

$\Rightarrow$ f^* exact, hence f_* preserves injectives. (formally).

Prop. X scheme. The category $Sh(X_{ét})$ has enough injectives.

Extension by zero.

Def. $j : U \hookrightarrow X$ open immersion, $\mathcal{F}$ sheaf on $U_{ét}$. The extension by zero of $\mathcal{F}$ along j is the sheafification of the étale presheaf

$$(V \xrightarrow{\varphi} X) \longmapsto \begin{cases} \mathcal{F}(V), & \varphi(V) \subseteq U \\ 0 & \text{o/w.} \end{cases}$$

Rmk. $j_!$ is a left adjoint to j^*.

Prop. $\bar{x} \to X$ geometric point with image $x \in X$ then

$$(j_!\mathcal{F})_{\bar{x}} = \begin{cases} \mathcal{F}_{\bar{x}}, & x \in U \\ 0, & x \notin U. \end{cases}$$

Cor. $j_!$ exact, and j^* preserves injectives.

Pf. of Prop. 8.12.

For each $x \in X$ choose a geometric point $i_x : \bar{x} \to X$ with image x.

Let $\mathcal{F} \in Sh(X_{\acute{e}t})$, and choose for each $x \in X$ an embedding $\underset{\mathcal{F}_{\bar{x}}}{\overset{\|}{i_x^* \mathcal{F}}} \hookrightarrow I(x)$

with $I(x)$ an injective abelian group.

Define $\mathcal{F}^* := \prod_{x \in X} i_{x,*} i_x^* \mathcal{F}$ and $\mathcal{I} := \prod_{x \in X} i_{x,*} I(x)$.

The canonical maps $\mathcal{F} \to \mathcal{F}^*$ and $\mathcal{F}^* \longrightarrow \mathcal{I}$ are ~~injective~~ monomorphisms, and $\mathcal{I}$ is injective. $\square$

Prop. X scheme, $j: U \hookrightarrow X$ open, $i: Z := X \setminus U \hookrightarrow X$, $\mathcal{F}$ sheaf on X. Then the sequence

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$

is exact.

Pf. ~~For $x \in U$~~ The sequence on stalks is

$$0 \to \mathcal{F}_{\bar{x}} \xrightarrow{id} \mathcal{F}_{\bar{x}} \to 0 \to 0 \quad , \quad x \in U,$$

$$0 \to 0 \to \mathcal{F}_{\bar{x}} \xrightarrow{id} \mathcal{F}_{\bar{x}} \to 0 \quad , \quad x \notin U. \quad \square$$