The ring of crystalline periods

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Contents

Introduction

Given a finite extension of $\mathbb{Q}_p$, or more generally a p-adic field K, we would like to study its absolute Galois group G through the latter's continuous $\mathbb{Q}_p$-representations. The category of these representations is however prohibitively large - in a sense because the topologies of $\mathbb{Q}_p$ and K are too compatible. Fontaine's formalism of *period rings* lets us single out interesting full subcategories of $\mathbb{Q}_p$-representations and relates them to categories of certain semi-linear objects whose properties mirror those of a "nice" ring with a G-action.

A particularly important period ring is the *ring of crystalline periods* B_{cris}, which defines the category of *crystalline representations*. The properties of B_{cris} are rather arduous to prove, often requiring elaborate technical arguments which treatments of B_{cris} usually omit as a result. Notably, the important property that the Frobenius of B_{cris} is injective does not appear to have a full proof anywhere in the literature, as for example previously noted in [2, Theorem 9.1.8]. The main goal of this thesis is to treat these aspects in full detail.

Familiarity with the theory of p-adic representations is not strictly required if one takes the results collected in §1.3 "on faith", although it will obviously be useful. We review the important tilting construction and its adjunction to the Witt vector functor in detail in §1.1 and §1.2.

In §2, we construct the rings A_{cris} and B^+_{cris} (which give rise to B_{cris} via localization) and investigate their algebraic properties. While we do not construct A_{cris} using the more general method from [9, §2.2], we establish that our construction is equivalent in §2.3.

In §3, we relate the ring B^+_{cris} to the so-called *Gauß norms* which feature prominently in the construction of the Fargues-Fontaine curve. The Gauß norms give rise to a family of rings B^+_ρ and B^+, constructed in §3.1, that formalize what used to be ad hoc topological arguments in older texts. In §3.2, this will result in an embedding $B^+_{\rho^p} \subset B^+_{\mathrm{cris}} \subset B^+_\rho \subset B^+_{\mathrm{dR}}$ that lets us reduce many properties of B_{cris} to properties of B_{dR} or B^+_ρ. We also study the natural filtration on B_{cris} via the *cyclotomic periods* in §3.3, leading to a slightly simpler calculation of the Frobenius fixed points of B_{cris}.

In the final chapter §4, we return to the original situation from representation theory and establish how the formalism of period rings applies to B_{cris}. It turns out that for the purposes of the period ring formalism, one can replace B^+_{cris} with the slightly nicer B^+_ρ at no loss. We finish with an overview of various results on crystalline representations in §4.2 and an application to p-divisible groups which was the historical motivation for the development of B_{cris} in §4.3.

This thesis is based on a seminar on p-adic Galois representations and a lecture on the Fargues-Fontaine curve, both held by Prof. Dr. Jan Kohlhaase during the winter semesters of 2020/2021 and 2019/2020 respectively. I would like to thank him especially; his thoroughness has been an inspiration.

1 Preliminaries

We fix a prime p throughout the entire text. Rings and algebras are always commutative. Recall that every ring homomorphism between p-adically separated and complete rings is automatically continuous with respect to the p-adic topology. We will often write $\lim_n x_n$ and $\varprojlim_j x_j$ instead of $\lim_{n\to\infty} x_n$ and $\varprojlim_{j\in\mathbb{N}_0} x_j$ for the sake of brevity when this is not ambiguous. Group actions are written $g.x$ unless specified otherwise. We mark parts of equations by an ! like in $a \overset{!}{=} b = c$ to highlight where a nearby assumption applies.

1.1 The Witt-tilting adjunction

We recall the adjunction between Witt vectors and tilting, whose counit plays a central role throughout this text. The reader is assumed to be familiar with the p-typical Witt vectors; knowledge of ramified Witt vectors is not necessary because we avoid those entirely. Recall that if A is a perfect $\mathbb{F}_p$-algebra, then every Witt vector $w \in \mathrm{W}(A)$ can be written uniquely as $\sum_{i=0}^\infty p^i[w_i]$, where $[\cdot] : A \to \mathrm{W}(A)$ is the Teichmüller lift and all $w_i \in A$. This differs from the commonly found representation $w = \sum_{i=0}^\infty p^i[w_i'^{p^{-i}}]$ whose behavior is explicitly given by the structure polynomials S_n, P_n, I_n, but since $w_i^{p^i} = w_i'$, one easily obtains the adapted formulae $I_n(w_0, w_1^p, \ldots, w_n^{p^n})^{p^{-n}}$, etc. for our representation.

Definition 1.1. Let R be a ring. The *tilt* of R is the inverse limit ring

$$R^\flat := \varprojlim_{j\in\mathbb{N}_0} R/pR = \{(\overline{r}_j)_{j\in\mathbb{N}_0} \in (R/pR)^{\mathbb{N}_0} \mid \overline{r}_{j+1}^p = \overline{r}_j \text{ for all } j \in \mathbb{N}_0\},$$

whose elements we write as $\varprojlim_j \overline{r}_j$ with $\overline{r}_j \in R/pR$. $R^\flat$ is perfect because p^n-th roots are given by $(\varprojlim_j \overline{r}_j)^{1/p^n} = \varprojlim_j \overline{r}_{j+n}$. Clearly, $R^\flat \cong (R/pR)^\flat$. We obtain a functor $\cdot^\flat$ from the category of rings to the category of perfect $\mathbb{F}_p$-algebras by sending each ring homomorphism $f : R \to S$ to the map

$$f^\flat(\varprojlim_{j\in\mathbb{N}_0}(r_j + pR)) = \varprojlim_{j\in\mathbb{N}_0}(f(r_j) + pS).$$

Proposition 1.2. *Let B be a p-adically separated and complete ring.*

(i) *There is a unique multiplicative map $\cdot^\sharp : B^\flat \to B$ that yields the canonical projection $(\varprojlim_j \overline{b}_j \mapsto \overline{b}_0) : B^\flat \to B/pB$ when composed with $B \twoheadrightarrow B/pB$. It is given by*

$$(\varprojlim_{j\in\mathbb{N}_0}(b_j + pB))^\sharp = \lim_{j\to\infty} b_j^{p^j}.$$

(ii) *If $\mathcal{R}(B) := \{(b^{(j)})_{j\in\mathbb{N}_0} \in B^{\mathbb{N}_0} \mid (b^{(j+1)})^p = b^{(j)}\}$, then $\cdot^\sharp : B^\flat \to B$ factors through the multiplicative bijection*

$$\begin{aligned} B^\flat &\leftrightarrows \mathcal{R}(B) \\ \varprojlim_{j\in\mathbb{N}_0}(b^{(j)} + pB) &\leftarrow\!\shortmid (b^{(j)})_{j\in\mathbb{N}_0} \\ b = \varprojlim_{j\in\mathbb{N}_0}(b_j + pB) &\mapsto (\lim_{k\to\infty} b_{j+k}^{p^k})_j = ((b^{1/p^j})^\sharp)_j. \end{aligned}$$

(iii) *If B is an $\mathbb{F}_p$-algebra, then $\mathcal{R}(B)$ is a ring under componentwise operations and all three maps above are ring homomorphisms. If B is perfect, then $\cdot^\sharp : B^\flat \xrightarrow{\sim} B$ is a ring isomorphism.*

Proof. (i): $\cdot^\sharp$ is well-defined because $a \equiv b \mod pB$ implies $a^{p^n} \equiv b^{p^n} \mod p^{n+1}B$ for all $a, b \in B$. This also shows why $\cdot^\sharp$ is unique and given by the above formula: Every $b = \varprojlim_j (b_j + pB) \in B^\flat$ satisfies $(b^\sharp)^{1/p^n} = (b^{1/p^n})^\sharp \equiv b_n \mod pB$ by multiplicativity, so we must have $b^\sharp \equiv b_n^{p^n} \mod p^{n+1}B$ for all $n \in \mathbb{N}$.

(ii): That the two maps are inverses is verified directly. The factorization is obvious.

(iii): The set $\mathcal{R}(B)$ is an inverse limit of rings under ring homomorphisms in this case, hence itself a ring. That the three maps are ring homomorphisms is immediate. If B is perfect, the projection onto the zeroth component $\mathcal{R}(B) \xrightarrow{\sim} B$ is a ring isomorphism, so $\cdot^\sharp$ factors into $B^\flat \xrightarrow{\sim} \mathcal{R}(B) \xrightarrow{\sim} B$. □

Proposition 1.3. *Let B be a p-adically separated and complete ring and consider*

$$\theta_B : \mathrm{W}(B^\flat) \to B$$
$$\theta_B\left(\sum_{i=0}^{\infty} p^i [w_i]\right) = \sum_{i=0}^{\infty} p^i w_i^\sharp.$$

(i) *θ_B is the only ring homomorphism $\mathrm{W}(B^\flat) \to B$ which reduces to the projection $(\varprojlim_j \bar{b}_j \mapsto \bar{b}_0) : B^\flat \to B/pB$ modulo p.*

(ii) *If the Frobenius on B/pB is surjective, then so is θ_B.*

(iii) *If a group G acts on B via ring homomorphisms, then θ_B is equivariant for the functorially induced action on $\mathrm{W}(B^\flat)$ and the action on B.*

Proof. (i): By Proposition 1.2, any homomorphism with these properties must map $[b]$ to $b^\sharp$ for any $b \in B^\flat$. Since $\mathrm{W}(B^\flat)$ and B are p-adically separated and complete, the formula for θ_B follows immediately from continuity. It is however very difficult to verify directly that this formula results in a ring homomorphism; instead, decompose θ_B into three ring homomorphisms

$$\mathrm{W}(B^\flat) \xrightarrow{\alpha} \varprojlim_{f_n} W_n(B/pB) \xrightarrow{\psi} \varprojlim_{n\in\mathbb{N}} B/p^nB \xrightarrow{\ell} B,$$

where $W_n(B/pB)$ denotes the ring of truncated Witt vectors of length n, whose elements we will write as $\langle \bar{b}_0, \ldots, \bar{b}_{n-1}\rangle := (\bar{b}_0, \ldots, \bar{b}_{n-1}, 0, \ldots) + V^n(\mathrm{W}(B/pB))$ here, and

$$f_n : W_{n+1}(B/pB) \to W_n(B/pB)$$
$$f_n(\langle \bar{b}_0, \ldots, \bar{b}_n\rangle) = \langle \bar{b}_0^p, \ldots, \bar{b}_{n-1}^p\rangle$$

is the reduced Frobenius map. Note that there may not be a representation $\sum_{i=0}^\infty p^i[x_i]$ since B/pB need not be perfect, necessitating this argument using classic Witt vector components. The three homomorphisms are

$$\alpha(w) = \varprojlim_{f_n} \left(W(\varprojlim_{j\in\mathbb{N}_0} \bar{b}_j \mapsto \bar{b}_n)(w) + V^n(\mathrm{W}(B/pB)) \right),$$
$$\psi\left(\varprojlim_{f_n} \langle v_0^{(n)} + pB, \ldots, v_{n-1}^{(n)} + pB\rangle\right) = \varprojlim_{n\in\mathbb{N}} (\Phi_n(v_0^{(n)}, \ldots, v_{n-1}^{(n)}, 0) + p^nB),$$
$$\ell(\varprojlim_{n\in\mathbb{N}} (b_n + p^nB)) = \lim_{n\to\infty} b_n,$$

where Φ_n is the n-th Witt polynomial. α and ℓ are easily seen to be well-defined ring homomorphisms; the argument for ψ is slightly more involved but fundamentally relies on the facts that the n-th ghost component $\mathrm{W}(B) \to B$ is a ring homomorphism and that varying any component of $b \in B^{n+1}$ by an element of pB changes $\Phi_n(b_0, \dots, b_n)$ by an element of $p^n B$. See chapter 5 of [17], in particular Proposition 5.3, for the full details of this construction in German. The additional assumptions made there are not used to show that these maps are ring homomorphisms.

(ii): Let $b \in B$ and set $e_0 := b$. Given $e_i \in B$ for any $i \in \mathbb{N}_0$, we can construct an $x_i \in B^\flat$ with $e_i - x_i^\sharp \in pB$ by taking the zeroth component of x_i to be $e_i + pB$ and then inductively taking preimages under the surjective Frobenius on B/pB. Letting $e_{i+1} \in B$ such that $e_i - x_i^\sharp = pe_{i+1}$, we can inductively continue this construction, obtaining

$$b - \theta_B\left(\sum_{i=0}^m p^i[x_i]\right) = e_0 - \sum_{i=0}^m p^i x_i^\sharp = p^1 e_1 - \sum_{i=1}^m p^i x_i^\sharp = \ldots = p^{m+1} e_{m+1} \in p^{m+1} B$$

for all $m \in \mathbb{N}$, so that $\theta_B(\sum_{i=0}^\infty p^i[x_i]) = b$.

(iii): The functorial action on $\mathrm{W}(B^\flat)$ is $g.\sum_{i=0}^\infty p^i[w_i] = \sum_{i=0}^\infty p^i[g.w_i]$. Since the action of G on B must be p-adically continuous, its action on $B^\flat$ commutes with $\cdot^\sharp$. The statement then follows immediately from the formula. $\square$

Proposition 1.4. *The Witt vector functor W from the category of perfect $\mathbb{F}_p$-algebras to the category of p-adically separated and complete rings is left adjoint to the tilting functor $\cdot^\flat$. If A is a perfect $\mathbb{F}_p$-algebra and B is a p-adically separated and complete ring, then the counit is the map $\theta_B : \mathrm{W}(B^\flat) \to B$ from Proposition 1.3 and the unit is the map*

$$\eta_A : A \to W(A)^\flat$$
$$\eta_A(a) = \varprojlim_{k \in \mathbb{N}_0} ([a^{1/p^k}] + p\mathrm{W}(A)).$$

Therefore, the bijection between Hom-sets can be described explicitly as follows:

$$\mathrm{Hom}(\mathrm{W}(A), B) \leftrightarrows \mathrm{Hom}(A, B^\flat)$$
$$f \mapsto f^\flat \circ \eta_A$$
$$\theta_B \circ \mathrm{W}(g) \leftarrow g.$$

Proof. The final remark on Hom-sets is a simple category theoretical fact, so we just have to show $\mathrm{id}_{\mathrm{W}(A)} = \theta_{\mathrm{W}(A)} \circ \mathrm{W}(\eta_A)$ and $\mathrm{id}_{B^\flat} = (\theta_B)^\flat \circ \eta_{B^\flat}$:

$$\begin{aligned}
\theta_{\mathrm{W}(A)}\left(\mathrm{W}(\eta_A)\left(\sum_{i=0}^\infty p^i[a_i]\right)\right) &= \theta_{\mathrm{W}(A)}\left(\sum_{i=0}^\infty p^i[\eta_A(a_i)]\right) \\
&= \theta_{\mathrm{W}(A)}\left(\sum_{i=0}^\infty p^i[\varprojlim_{k \in \mathbb{N}_0} ([a_i^{1/p^k}] + p\mathrm{W}(A))]\right) \\
&= \sum_{i=0}^\infty p^i(\varprojlim_{k \in \mathbb{N}_0} ([a_i^{1/p^k}] + p\mathrm{W}(A)))^\sharp \\
&= \sum_{i=0}^\infty p^i[a_i],
\end{aligned}$$

$$\begin{aligned}
(\theta_B)^\flat(\eta_{B^\flat}(\varprojlim_{j\in\mathbb{N}_0}(b_j+pB))) &= (\theta_B)^\flat(\varprojlim_{k\in\mathbb{N}_0}([\varprojlim_{j\in\mathbb{N}_0}(b_{j+k}+pB)]+p\mathrm{W}(B^\flat))) \\
&= \varprojlim_{k\in\mathbb{N}_0}(\theta_B([\varprojlim_{j\in\mathbb{N}_0}(b_{j+k}+pB)])+pB) \\
&= \varprojlim_{k\in\mathbb{N}_0}((\varprojlim_{j\in\mathbb{N}_0}(b_{j+k}+pB))^\sharp+pB) \\
&= \varprojlim_{k\in\mathbb{N}_0}(b_k+pB). \qquad \square
\end{aligned}$$

1.2 Tilts of perfectoid fields

The tilting construction as applied to the ring of integers of so-called *perfectoid fields* will be central throughout this text. We recall the necessary facts about this situation.

Definition 1.5. A non-archimedean field $(C, |\cdot|)$ with residue field of characteristic p is *perfectoid* if:

(i) C is complete with respect to $|\cdot|$.

(ii) The set $|C| \subset [0;\infty)$ is dense.

(iii) The Frobenius on $\mathcal{O}_C/p\mathcal{O}_C$ is surjective.

Proposition 1.6. *If C is a perfectoid field, then $|x|_\flat := |x^\sharp|$ is an absolute value on $\mathcal{O}_C^\flat$.*

Proof. $|\cdot|_\flat$ is multiplicative because $\cdot^\sharp$ is multiplicative. If $x \in \mathcal{O}_C^\flat$ satisfies $x^\sharp = 0$, then under the bijection in Proposition 1.2 (ii), x corresponds to a compatible system of p^n-th roots $(x_n)_{n\in\mathbb{N}_0} \in \mathcal{R}$ such that $0 = \lim_n x_n^{p^n} = x_0$. But since $\mathcal{O}_C$ is a domain, there is only one such system, viz. the one that corresponds to $x = 0$. For the strict triangle inequality, note that for $a = \varprojlim_k(a_k + p\mathcal{O}_C)$, $b = \varprojlim_k(b_k + p\mathcal{O}_C) \in \mathcal{O}_C^\flat$, we have

$$|a+b|_\flat = \lim_{k\to\infty}|a_k+b_k|^{p^k} \le \max\{\lim_{k\to\infty}|a_k^{p^k}|, \lim_{k\to\infty}|b_k^{p^k}|\} = \max\{|a|_\flat, |b|_\flat\},$$

using that we already know that $\lim_k|a_k^{p^k}|$ and $\lim_k|b_k^{p^k}|$ exist. $\square$

Proposition 1.7. *Let $n \in \mathbb{N}$ and $x = \varprojlim_j(x_j + p\mathcal{O}_C) \in \mathcal{O}_C^\flat$. Then $x_j \in p\mathcal{O}_C$ for all $0 \le j < n$ if and only if $|x|_\flat \le |p|^{p^{n-1}}$.*

Proof. It suffices to prove the case $n = 1$ because the others follow by repeatedly taking p-th roots. If $x_0 \in p\mathcal{O}_C$, then $x_j^{p^j} \equiv x_0 \equiv 0 \mod p\mathcal{O}_C$ for all $j \in \mathbb{N}_0$, so $|x_j^{p^j}| \le |p|$ and $|x|_\flat = |x^\sharp| = \lim_j|x_j^{p^j}| \le |p|$. On the other hand, if $|x|_\flat \le |p|$, then $|x_j^{p^j}| \le |p|$ for all but finitely many $j \in \mathbb{N}_0$ since $|\cdot|$ is non-archimedean and hence $\{y \in \mathcal{O}_C \mid |y| < |p|\}$ is closed. But if $j \in \mathbb{N}_0$ is such that $|x_j^{p^j}| \le |p|$, then $x_j^{p^j} \in p\mathcal{O}_C$, so $x_0 \equiv x_j^{p^j} \equiv 0 \mod p\mathcal{O}_C$. $\square$

Remark 1.8. Since $\mathcal{O}_C^\flat$ admits an absolute value, it is an integral domain; its fraction field is denoted by $C^\flat$ and also called the *tilt of C*. This is *not* the functor $T := \cdot^\flat$ applied to C! There is however very little ambiguity in practice:

- If $\operatorname{char} C = 0$, then $T(C) := \mathcal{R}(C/pC) = \mathcal{R}(0) = 0$, so the functorial construction is completely uninteresting.

- If $\operatorname{char} C = p$, the Frobenius on $\mathcal{O}_C/p\mathcal{O}_C = \mathcal{O}_C$ is a surjective ring homomorphism whose kernel consists of nilpotent elements, therefore is trivial as $\mathcal{O}_C$ is a domain. Hence $\mathcal{O}_C$ is perfect and so is C, which means that $C \cong \mathcal{R}(C) \cong T(C)$ by 1.2 (iii). Similarly, $C^\flat = \operatorname{Frac}(\mathcal{R}(\mathcal{O}_C/p\mathcal{O}_C)) = \operatorname{Frac}\mathcal{O}_C = C$, so the two constructions agree.

- The case $\operatorname{char} C \notin \{0, p\}$ is impossible since C has a residue field of characteristic p.

Proposition 1.9 ($\mathcal{O}_C^\flat = \mathcal{O}_{C^\flat}$)**.** *If C is a perfectoid field and $C^\flat = \operatorname{Frac}\mathcal{O}_C^\flat$, then $|\cdot|_\flat$ extends to $C^\flat$ and the ring of integers of $C^\flat$ is $\mathcal{O}_C^\flat$.*

Proof. The extension of an absolute value to a fraction field is standard. If $a \in C^\flat$ and $b \in (C^\flat)^\times$ with $|a/b|_\flat \le 1$, then for all $n \in \mathbb{N}_0$, we have $|a^{1/p^n}/b^{1/p^n}|_\flat \le 1$, hence $(a^{1/p^n})^\sharp/(b^{1/p^n})^\sharp \in \mathcal{O}_C$. The latter form a compatible system of p^n-th roots in $\mathcal{O}_C$, so the bijection in Proposition 1.2 (ii) gives an element $\varprojlim_j((a^{1/p^j})^\sharp/(b^{1/p^j})^\sharp + p\mathcal{O}_C) \in \mathcal{O}_C^\flat$ whose product with $b = \varprojlim_j((b^{1/p^j})^\sharp + p\mathcal{O}_C)$ is a. Hence $a/b \in \mathcal{O}_C^\flat$. On the other hand, for every $x = \varprojlim_j(x_j + p\mathcal{O}_C) \in \mathcal{O}_C^\flat$ we have $|x_j| \le 1$ for all $j \in \mathbb{N}_0$ and thus $|x|_\flat = \lim_j |x_j^{p^j}| \le 1$, so $x \in \mathcal{O}_{C^\flat}$. □

Lemma 1.10. *If C is a perfectoid field, then $|C^\flat|_\flat = |C|$.*

Proof. It suffices to prove $|\mathcal{O}_C^\flat|_\flat = |\mathcal{O}_C|$. First let $x \in \mathcal{O}_C$ such that $|p| < |x| < 1$ and use the surjectivity of the Frobenius on $\mathcal{O}_C/p\mathcal{O}_C$ to inductively construct an element $y = \varprojlim_j(y_j + p\mathcal{O}_C) \in \mathcal{O}_C^\flat$ with $y_0 = x$. Then $y_j^{p^j} \equiv y_0 \equiv x \mod p\mathcal{O}_C$, so that we have $|y^\sharp - x| = \lim_j |y_j^{p^j} - x| \le |p| < |x|$. Hence $|y|_\flat = |(y^\sharp - x) + x| = |x|$ by the strict triangle inequality.

Now let $x \in \mathcal{O}_C$ such that $|x| \le |p|$. By the previous part and the density of $|C|$, there exists a $y \in \mathcal{O}_C^\flat$ such that $|y^2|_\flat < |p| < |y|_\flat$. Then $|p| < |x|/|y^2|_\flat < 1$, so there is a $z \in \mathcal{O}_C^\flat$ with $|z|_\flat = |x|/|y^2|_\flat$. Hence $|x| = |zy^2|_\flat$. □

Lemma 1.11. *If C is a perfectoid field, then $C^\flat$ is complete.*

Proof. We showed in Remark 1.8 that $C^\flat = C$ if $\operatorname{char} C = p$. Otherwise, it suffices to prove that $\mathcal{O}_C^\flat$ is complete. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{O}_C^\flat$. Then for each $k \in \mathbb{N}$, there is an $n \in \mathbb{N}$ such that $|x_n - x_{n+i}|_\flat \le |p|^{p^{k-1}}$ for all $i \in \mathbb{N}$. By Proposition 1.7, this means that the first k components of $x_n, x_{n+1}, \dots$ agree, so the sequence converges componentwise. One easily checks that the resulting sequence is a limit with respect to $|\cdot|_\flat$. □

Theorem 1.12.

(i) *If C is a perfectoid field, then so is $C^\flat$.*

(ii) *If C is algebraically closed, then so is $C^\flat$.*

Proof. (i): This follows from Lemma 1.11, Lemma 1.10, and the fact that $\mathcal{O}_C^\flat$ is perfect.

(ii): The case $\operatorname{char} C > 0$ is trivial. For $\operatorname{char} C = 0$, see e.g. [6, 2.1.11] or [2, 4.3.5]. □

Proposition 1.13. *If $s \in \mathcal{O}_C^\flat$ such that $|s|_\flat = |p|$, then $\mathcal{O}_C^\flat/s\mathcal{O}_C^\flat \cong \mathcal{O}_C/p\mathcal{O}_C$.*

Proof. The projection $\mathcal{O}_C^\flat \xrightarrow{\cdot^\sharp} \mathcal{O}_C \twoheadrightarrow \mathcal{O}_C/p\mathcal{O}_C$ onto the zeroth component is a ring homomorphism and surjective by 1.5 (iii). Its kernel consists of all $x \in \mathcal{O}_C^\flat$ such that $x^\sharp \in p\mathcal{O}_C$, i.e. all $x \in \mathcal{O}_C^\flat$ with $|x|_\flat \le |p| = |s|_\flat$, which are precisely the multiples of s. □

Proposition 1.14. *If C is a perfectoid field and $\xi = \sum_{i=0}^{\infty} p^i[\xi_i] \in \ker\theta_{\mathcal{O}_C}$, then the following are equivalent:*

(i) $\ker\theta_{\mathcal{O}_C} = \xi \mathrm{W}(\mathcal{O}_C^\flat)$.

(ii) $|\xi_0|_\flat = |p|$.

(iii) $|\xi_1|_\flat = 1$.

Proof. (i) $\implies$ (ii): Since $\xi \in \ker\theta_{\mathcal{O}_C}$, we have $|\xi_0^\sharp| = |-\sum_{i=1}^{\infty} p^i \xi_i^\sharp| \leq |p|$. Let $s \in \mathcal{O}_C^\flat$ such that $|s|_\flat = |p|$, using Proposition 1.10. Since $\theta_{\mathcal{O}_C}$ is surjective, there is an element $w \in \mathrm{W}(\mathcal{O}_C^\flat)$ such that $[s] - pw \in \ker\theta_{\mathcal{O}_C} = \xi\mathrm{W}(\mathcal{O}_C^\flat)$, so let $z = \sum_{i=0}^{\infty} p^i[z_i] \in \mathrm{W}(\mathcal{O}_C^\flat)$ such that $[s] - pw = \xi z$. Then $|p| = |s|_\flat = |\xi_0 z_0|_\flat \leq |\xi_0|_\flat \leq |p|$, so $|\xi_0|_\flat = |p|$.

(ii) $\implies$ (iii): Since $\xi \in \ker\theta_{\mathcal{O}_C}$, we have $|p||\xi_1^\sharp| = |-\xi_0^\sharp - \sum_{i=2}^{\infty} p^i \xi_i^\sharp| = |\xi_0^\sharp| = |p|$ by the strict triangle inequality and $|\xi_i|_\flat \leq 1$. Hence $|\xi_1|_\flat = 1$.

(iii) $\implies$ (i): Let $x = \sum_{i=0}^{\infty} p^i[x_i] \in \ker\theta_{\mathcal{O}_C}$. Since $|p\xi_1^\sharp| = |p| > |p^2| \geq |\sum_{i=2}^{\infty} p^i \xi_i^\sharp|$, we have $|\xi_0^\sharp| = |\sum_{i=1}^{\infty} p^i \xi_i^\sharp| = |p| > |\sum_{i=2}^{\infty} p^i \xi_i^\sharp|$ and hence

$$|x_0|_\flat = |x_0^\sharp| = \left|-\sum_{i=1}^{\infty} p^i x_i^\sharp\right| \leq |p| = |-p\xi_1^\sharp| = \left|\xi_0^\sharp + \sum_{i=2}^{\infty} p^i \xi_i^\sharp\right| \overset{!}{=} |\xi_0^\sharp| = |\xi_0|_\flat,$$

which implies $x_0 = \xi_0 z$ for some $z \in \mathcal{O}_C^\flat$ by Proposition 1.9. Hence $x - [z]\xi \in p\mathrm{W}(\mathcal{O}_C^\flat)$, i.e. $\ker\theta_{\mathcal{O}_C} \subset \xi\mathrm{W}(\mathcal{O}_C^\flat) + p\mathrm{W}(\mathcal{O}_C^\flat)$.

This suffices because when $x = a_0\xi + b_0 p \in \ker\theta_{\mathcal{O}_C}$ for some $a_0, b_0 \in \mathrm{W}(\mathcal{O}_C^\flat)$, we must have $\theta_{\mathcal{O}_C}(b_0 p) = \theta_{\mathcal{O}_C}(b_0)p = 0$, so $b_0 \in \ker\theta_{\mathcal{O}_C}$ since $\mathcal{O}_C$ is a domain. Inductively, we obtain elements $a_i, b_i \in \mathrm{W}(\mathcal{O}_C^\flat)$ such that

$$x = a_0\xi + b_0 p = a_0\xi + a_1\xi p + b_1 p^2 = \ldots = p^n b_n + \sum_{i=0}^{n} a_i \xi p^i$$

for all $n \in \mathbb{N}$; then $x = \xi \sum_{i=0}^{\infty} p^i a_i$ since $\mathrm{W}(\mathcal{O}_C^\flat)$ is p-adically separated and complete. □

Proposition 1.15. *Let C be a perfectoid field.*

(i) *For every $s \in \mathcal{O}_C^\flat$ with $|s|_\flat = |p|$, there is a $w \in \mathrm{W}(\mathcal{O}_C^\flat)^\times$ such that $\ker\theta_{\mathcal{O}_C}$ is generated by $[s] + pw$.*

(ii) *If C is algebraically closed, there is an $s \in \mathcal{O}_C^\flat$ with $|s|_\flat = |p|$ such that $\ker\theta_{\mathcal{O}_C}$ is generated by $[s] - p$.*

Proof. (i): Since $|\theta_{\mathcal{O}_C}(-[s])| = |\theta_{\mathcal{O}_C}([s])| = |s^\sharp| = |s|_\flat = |p|$, we have $\theta([s]) \in p\mathcal{O}_C$. By Proposition 1.3 (ii), $\theta_{\mathcal{O}_C}$ is surjective, so there is a $w = \sum_{i=0}^{\infty} p^i[w_i] \in \mathrm{W}(\mathcal{O}_C^\flat)$ with $\theta_{\mathcal{O}_C}([s] + pw) = 0$. Proposition 1.14 then implies $|w_0|_\flat = 1$, which means $w \in \mathrm{W}(\mathcal{O}_C^\flat)^\times$.

(ii): Since C is algebraically closed, we can inductively find compatible p^n-th roots $s_n \in \mathcal{O}_C$ of $p \in \mathcal{O}_C$, which defines an $s = \varprojlim_j (s_j + p\mathcal{O}_C) \in \mathcal{O}_C^\flat$ with $s^\sharp = \lim_n s_n^{p^n} = p$. Then $\theta_{\mathcal{O}_C}([s] - p) = p - p = 0$, so $[s] - p$ is a generator of $\ker\theta_{\mathcal{O}_C}$ by Proposition 1.14. □

1.3 Period rings and p-adic representations

We quickly recall the basic ideas behind the theory of period rings and some of its applications to p-adic representations. For a more detailed overview, see [3, §1]; for a general treatment of the subject, see [2], in particular §5. The starting point is the following type of field.

Definition 1.16. A *p-adic field* is a non-archimedean complete discretely valued field K of characteristic 0 with perfect residue field k of characteristic p. We usually fix the completion $C = C_K = \widehat{\overline{K}}$ of a fixed algebraic closure $\overline{K}$ of K as well.

For the remainder of the chapter, we fix such a p-adic field K and $C = C_K$. The field C is algebraically closed by Krasner's lemma. We also fix an algebraic closure $\overline{k}$ of k. The absolute Galois group of K is denoted $G := \mathrm{Gal}(K)$; its action extends to C by continuity.

The most obvious examples of p-adic fields are the finite extensions of $\mathbb{Q}_p$, which are precisely those p-adic fields whose residue field is *finite* rather than perfect. However, whenever K is a p-adic field, so is the completion of its maximal unramified extension $\widehat{K^{\mathrm{ur}}} \subset C$, with residue field an algebraic closure of k, $C_{\widehat{K^{\mathrm{ur}}}} = C_K$, and its absolute Galois group $\mathrm{Gal}(\widehat{K^{\mathrm{ur}}})$ is canonically isomorphic to the inertia group of $\mathrm{Gal}(K)$. Hence this more general definition lets us pass to the inertia group within the same formalism.

We want to study the unwieldy category $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Q}_p}(G)$ of continuous $\mathbb{Q}_p$-representations of G by singling out suitable full subcategories that contain interesting representations, such as those that arise naturally in geometry. In analogy to how one relates $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Q}_p}(G)$ to a category of étale (Φ, Γ)-modules by producing many invariants (see e.g. [12, Theorem 4.4.2]), we use the following formalism due to Fontaine [10, §1].

Definition 1.17. Let B be a ring containing $\mathbb{Q}_p$, with a G-action by ring automorphisms. B is called *$(\mathbb{Q}_p, G)$-regular* if the following properties hold:

(i) B is a domain.

(ii) $B^G = (\mathrm{Frac}\, B)^G$. In particular, B^G is a field.

(iii) Every $b \in B \setminus 0$ with $G.b \subset \mathbb{Q}_p b$ is a unit.

Definition 1.18. If B is $(\mathbb{Q}_p, G)$-regular, we define the functor

$$D_B(V) = (B \otimes_{\mathbb{Q}_p} V)^G, \qquad\qquad D_B(f : V \to W) = \mathrm{id}_B \otimes_{\mathbb{Q}_p} f$$

from $\mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Q}_p}(G)$ to the category of B^G-vector spaces Vec_{B^G}.

Definition 1.19. If B is $(\mathbb{Q}_p, G)$-regular, a *B-semilinear representation* is a B-module W with a G-action such that $g.(w + w') = g.w + g.w'$ and $g.(b \cdot w) = g.b \cdot g.w$ for all $b \in B$, $w, w' \in W$.

Definition 1.20. Let B be $(\mathbb{Q}_p, G)$-regular. A continuous $\mathbb{Q}_p$-representation V is called *B-admissible* if the following equivalent properties hold:

(i) $\dim_{B^G} D_B(V) = \dim_{\mathbb{Q}_p} V$.

(ii) $B \otimes_{\mathbb{Q}_p} V \cong B^{\dim_{\mathbb{Q}_p} V}$ as B-semilinear representations, with the componentwise action on the right.

(iii) The canonical *comparison homomorphism* $\alpha_V : B \otimes_{B^G} D_B(V) \to B \otimes_{\mathbb{Q}_p} V$ of B-semilinear representations is an isomorphism.

Note that $\dim_{B^G} D_B(V) \leq \dim_{\mathbb{Q}_p} V$ is always true and $\alpha_{B\otimes_{\mathbb{Q}_p} V}$ is always at least injective. Property 1.17 (iii) is crucial to establish the equivalence of the three properties above and ensures that admissibility is well-behaved, see e.g. [3, 1.4.4] and [2, 5.2.1]. Notably, the full subcategory $\mathrm{Rep}^B_{\mathbb{Q}_p}(G)$ of B-admissible representations is stable under subobjects, quotients, tensor products and duals. Often, the functor D_B takes values in a more refined category than Vec_{B^G}.

Theorem 1.21. *If B is $(\mathbb{Q}_p, G)$-regular, then the functor $D_B : \mathrm{Rep}^B_{\mathbb{Q}_p}(G) \to \mathrm{Vec}_{B^G}$ is exact, faithful, and commutes with tensor products and duals.*

Proof. See [2, 5.2.1]. □

Fields are trivially $(\mathbb{Q}_p, G)$-regular, but they usually don't make for good period rings. For instance, the $\overline{K}$-admissible representations are exactly the discrete representations (see [12, 3.51]), i.e. those representations that factor through some $\mathrm{Gal}(L/K)$ for a finite Galois extension L/K, which is clearly too restrictive. The C-admissible representations are exactly the potentially unramified representations (see [12, 3.55]), i.e. those representations that become unramified after restriction to some $\mathrm{Gal}(L/K)$ for a finite Galois extension L/K. While an improvement, this still excludes the non-trivial powers of the cyclotomic character χ, which arise rather naturally and ought to be admissible.

Definition 1.22. If $\eta : G \to C^\times$ is a character, we write $C(\eta)$ for the field C with the twisted G-action $g * c := \eta(g) \cdot (g.c)$.

Theorem 1.23 (Ax-Tate-Sen)**.** *If $\eta : G \to \mathcal{O}_K^\times$ is a continuous character such that $\eta(G)$ is either finite or contains $\mathbb{Z}_p^\times$ as an open subgroup, then the continuous cohomology satisfies*

$$\dim_K H^0_{\mathrm{cont}}(G, C(\eta)) = \dim_K H^1_{\mathrm{cont}}(G, C(\eta)) = \begin{cases} 0 & \text{if } \eta(I) \text{ is infinite,} \\ 1 & \text{if } \eta(I) \text{ is finite,} \end{cases}$$

where $I < G$ is the inertia subgroup of G. In particular, $C^G = K$.

Proof. The proof of this theorem is rather involved; it is the subject of [2, §14]. □

Note that the conditions of Theorem 1.23 are satisfied for all continuous $\eta : G \to \mathbb{Z}_p^\times$.

Definition 1.24. The *ring of Hodge-Tate periods* is the ring $B_{\mathrm{HT}} := \bigoplus_{n\in\mathbb{Z}} C(\chi^n)$ with componentwise addition and G-action and the multiplication induced from the C-module structure and $C(\chi^m) \otimes_C C(\chi^n) \cong C(\chi^{m+n})$. A B_{HT}-admissible representation is called a *Hodge-Tate representation*.

The ring B_{HT} is $(\mathbb{Q}_p, G)$-regular with $B^G_{\mathrm{HT}} = K$ (see e.g. [2, 5.1.2]) and already an improvement since $\dim_K D_{\mathrm{HT}}(\chi^n) = \dim_K(\chi^n \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^G = \dim_K(\bigoplus_i C(\chi^{i+n}))^G \overset{1.23}{=} 1$. The corresponding functor D_{HT} takes values in the category of graded vector spaces and we call those $i \in \mathbb{Z}$ with $\dim_K \mathrm{gr}^i D_{\mathrm{HT}}(V) > 0$ the *Hodge-Tate weights* of V. Taking the dimension itself as the multiplicity of i, the Hodge-Tate representations are then those representations V whose weights with multiplicity add up to $\dim_{\mathbb{Q}_p} V$. For example, the one-dimensional representation $\chi^n : G \to \mathbb{Q}_p^\times$ has the unique Hodge-Tate weight $-n$.

Although the ring B_{HT} ends up being a useful tool to study other period rings by reducing their properties to properties of B_{HT}, its own notion of admissibility is unsatisfactory. Not only is the category of Hodge-Tate representations still extremely large, it also turns out that D_{HT} is not full as a functor to the category of graded vector spaces. Furthermore, if L/K is a finite Galois extension, then a $V \in \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Q}_p}(G)$ will be Hodge-Tate if and only if its restriction to $\mathrm{Gal}(L)$ is. Since this is true even when L/K is ramified, the Hodge-Tate property is very imprecise. A more detailed treatment of Hodge-Tate representations that proves all these results can be found in [2, §2].

A refinement of B_{HT} that will be very useful to us is the *field of de Rham periods* B_{dR} (giving rise to *de Rham representations*), which we will explicitly construct in a slightly more general setting in 3.30. The ring B_{dR} is the fraction field of a discretely valued and topological ring B^+_{dR}, although it is very important that the topology does not come from the valuation. This equips B_{dR} with a natural G-stable filtration, so that the corresponding functor D_{dR} takes values in the category of filtered K-vector spaces Fil_K via

$$\mathrm{Fil}^n D_{\mathrm{dR}}(V) := (\mathrm{Fil}^n B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V) \cap D_{\mathrm{dR}}(V).$$

Note that we always assume filtered vector spaces to be equipped with a separated and exhaustive filtration. For more detail on filtered vector spaces, we refer to [2, §6.2]; we will also recall some basic definitions in Remark 4.11.

We will rely on the following results in §4 without giving a proof ourselves; see e.g. [2, §4.4, §6]. Note that there is no circularity and no logical gap since our construction of B_{dR} does not require these statements and §4 is in the same concrete setting as this chapter and the cited literature.

Proposition 1.25. *There is a unique G-equivariant embedding $\overline{K} \hookrightarrow B^+_{\mathrm{dR}}$ under which K obtains its usual valuation topology as the subspace topology.*

Proposition 1.26. *B_{dR} is $(\mathbb{Q}_p, G)$-regular with $B^G_{\mathrm{dR}} = K$.*

Proposition 1.27. *We have $\mathrm{gr}\, B_{\mathrm{dR}} \cong B_{\mathrm{HT}}$ and $\mathrm{gr}\, D_{\mathrm{dR}}(V) \cong D_{\mathrm{HT}}(V)$ for any de Rham representation V. In particular, de Rham representations are Hodge-Tate.*

Proposition 1.28. *$D_{\mathrm{dR}} : \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G) \to \mathrm{Fil}_K$ is an exact functor that commutes with tensor products and duals.*

Proposition 1.29. *For every $V \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G)$, the de Rham comparison isomorphism $\alpha_{V,\mathrm{dR}} : B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ is an isomorphism in Fil_K, where $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ carries the filtration $\mathrm{Fil}^n B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$.*

Proposition 1.30. *For any $V \in \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Q}_p}(G)$ and any field extension L/K such that $L \subset C$ is discretely valued, the map $L \otimes_K D^K_{\mathrm{dR}}(V) \to D^L_{\mathrm{dR}}(V)$ is an isomorphism in Fil_K. In particular, V is de Rham as a representation of G if and only if it is de Rham as a representation of $\mathrm{Gal}(L) \subset G$.*

2 The construction of A_{cris}

For each perfectoid field C of mixed characteristic, there is a ring A_{cris}, satisfying a certain universal property (2.35), that gives rise to the ring of crystalline periods via localization (3.46). In the literature (e.g. [12, 6.1] and [2, §9.1]), one also commonly finds A_{cris} defined as the p-adic completion of the ring that arises by adjoining all $\frac{\xi^n}{n!} \in \mathrm{W}(\mathcal{O}_C^\flat)[\frac{1}{p}]$ to $\mathrm{W}(\mathcal{O}_C^\flat)$, where ξ is a generator of $\ker\theta_{\mathcal{O}_C}$; that this really results in A_{cris} is however only clear with the non-trivial result that A_{cris} is p-torsion-free.

While every $a \in A_{\mathrm{cris}}$ admits a representation $\sum_{i=0}^{\infty} a_i \frac{\xi^i}{i!}$ with $a_i \in \mathrm{W}(\mathcal{O}_C^\flat)$ such that $\lim_i a_i = 0$, such representations are unfortunately not unique. This makes it cumbersome to prove various properties of A_{cris}, and as a result such proofs tend to be sketched or omitted. For example, the Frobenius endomorphism of A_{cris} is injective (2.20), but no proof of this fact seems to exist in the literature (cf. [2, 9.1.8]).

In §2.1, we construct A_{cris} as an explicit quotient, prove its basic algebraic properties, and construct its Frobenius endomorphism, which we prove to be injective. In §2.3 we prove that our construction is equivalent to the more customary definitions found in [12, 6.1] and [9, §2.2]. The latter requires some notions from the theory of divided power structures which we quickly review in §2.2. Note that only §2.1 is logically required for the later chapters.

2.1 The ring $A_{\mathrm{cris}}(S/sS)$ and its properties

Throughout this section, we fix a perfect domain and $\mathbb{F}_p$-algebra S and a non-zero $s \in S$. The case to keep in mind is the ring of integers $\mathcal{O}_C^\flat$ of the tilt of a perfectoid field C of mixed characteristic and an element $s \in \mathcal{O}_C^\flat$ with $|s|_\flat = |p|$, but our constructions work in this slightly more general setting, sometimes under the additional assumption that $\bigcap_{n\geq 1} s^n S = 0$.

Remark 2.1. Recall that if R is a $\mathbb{Z}$-torsion-free ring and $x \in nR$ for some $n \in \mathbb{N}$, then there exists a unique element whose product with n is x. We suggestively denote this element by $\frac{x}{n}$ and note that the expected identities

$$\frac{x}{n}\cdot\frac{y}{m} = \frac{xy}{nm}, \qquad \frac{x}{n} = \frac{xm}{nm}, \qquad \frac{x}{n}+\frac{y}{m} = \frac{xm+yn}{nm}$$

hold for all $n, m \in \mathbb{N}$, $x \in nR$ and $y \in mR$. If $m, n \in \mathbb{N}$ and $x \in R$ such that $x^m \in m!R$ and $x^n \in n!R$, then

$$\frac{x^m}{m!}\cdot\frac{x^n}{n!} = \frac{x^{m+n}}{m!n!} = \frac{(m+n)!x^{m+n}}{m!n!(m+n)!} = \frac{\frac{(m+n)!}{m!n!}x^{m+n}}{(m+n)!} = \frac{\binom{m+n}{n}x^{m+n}}{(m+n)!},$$

so if $x^n \in n!R$ for *all* $n \in \mathbb{N}$, we have $\frac{x^n}{n!}\cdot\frac{x^m}{m!} = \binom{m+n}{n}\frac{x^{m+n}}{(m+n)!}$. for all $m, n \in \mathbb{N}_0$.

Definition 2.2.

(i) The *divided power polynomial algebra over* $\mathrm{W}(S)$ is the $\mathrm{W}(S)$-subalgebra

$$\mathrm{W}(S)\langle X\rangle := \{\textstyle\sum_{i=0}^{n} a_i \frac{X^i}{i!} \mid n \in \mathbb{N}_0, a_i \in \mathrm{W}(S)\} < \mathrm{W}(S)[\frac{1}{p}][X].$$

(ii) The *divided power series algebra over* $\mathrm{W}(S)$ is the $\mathrm{W}(S)$-subalgebra

$$\mathrm{W}(S)\langle\langle X\rangle\rangle := \{\textstyle\sum_{i=0}^{\infty} a_i \frac{X^i}{i!} \mid a_i \in \mathrm{W}(S), \lim_i a_i = 0\} < \mathrm{W}(S)[\frac{1}{p}][\![X]\!].$$

We will usually write $X^{[i]}$ instead of $\frac{X^i}{i!}$ to preserve vertical space. Note that $X^i = i!X^{[i]}$.

Remark 2.3. From Remark 2.1 we obtain the formula $X^{[n]}X^{[m]} = \binom{m+n}{n}X^{[m+n]}$, which shows that $\mathrm{W}(S)\langle X\rangle$ really is a subalgebra. For $\mathrm{W}(S)\langle\langle X\rangle\rangle$, first note that in $\mathrm{W}(S)[\frac{1}{p}][\![X]\!]$,

$$\left(\sum_{i=0}^{\infty} a_i X^{[i]}\right)\left(\sum_{j=0}^{\infty} b_j X^{[j]}\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{i} a_j b_{i-j} X^{[j]} X^{[i-j]} = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\binom{i}{j} a_j b_{i-j}\right) X^{[i]}.$$

Now assume that $\lim_i a_i = \lim_j b_j = 0$. Then for each $k \in \mathbb{N}$ there are $m, n \in \mathbb{N}$ such that $a_i, b_j \in p^k\mathrm{W}(S)$ for all $i \geq m$ and $j \geq n$. If $i \geq m+n$, we have $\binom{i}{j} a_j b_{i-j} \in p^k\mathrm{W}(S)$ for all $0 \leq j \leq i$ as either $a_j \in p^k\mathrm{W}(S)$ (when $j \geq m$) or $b_{i-j} \in p^k\mathrm{W}(S)$ (when $j < m$ and hence $i - j > n$). Therefore $\lim_i \sum_{j=0}^{i}\binom{i}{j} a_j b_{i-j} = 0$, which proves that $\mathrm{W}(S)\langle\langle X\rangle\rangle$ is also a $\mathrm{W}(S)$-subalgebra.

Remark 2.4. There is a unique homomorphism of $\mathrm{W}(S)$-algebras

$$\frac{\mathrm{W}(S)[X_1, X_2, \ldots]}{\langle X_i X_j - \binom{i+j}{i} X_{i+j}\rangle} \to \mathrm{W}(S)\langle X\rangle$$

that maps the class of X_i to $X^{[i]}$. This is in fact an isomorphism: The map that sends $\sum_{i=0}^{n} a_i X^{[i]}$ to the class of $\sum_{i=0}^{n} a_i X_i$ is easily seen to be an inverse. We thus obtain a convenient characterization of ring homomorphisms out of $\mathrm{W}(S)\langle X\rangle$.

Proposition 2.5. $\mathrm{W}(S)\langle\langle X\rangle\rangle$ *is the p-adic completion of* $\mathrm{W}(S)\langle X\rangle$.

Proof. Let $f : \widehat{\mathrm{W}(S)\langle X\rangle} \to \mathrm{W}(S)\langle\langle X\rangle\rangle$ be the map

$$f\left(\varprojlim_{n\in\mathbb{N}}\left(\sum_{i=0}^{k_n} a_i^{(n)} X^{[i]} + p^n\mathrm{W}(S)\langle X\rangle\right)\right) = \sum_{i=0}^{\infty}(\lim_{n\to\infty} a_i^{(n)})X^{[i]},$$

where we set $a_i^{(n)} = 0$ when $i > k_n$. f is well-defined since equivalent choices of $(a_i^{(n)})_n$ differ only by a null sequence. Furthermore, one easily sees that f is a ring homomorphism through unwinding definitions and standard properties of limits.

If $\sum_{i=0}^{\infty} a_i X^{[i]} \in \mathrm{W}(S)\langle\langle X\rangle\rangle$, then for each $n \in \mathbb{N}$ there is a $k_n \in \mathbb{N}$ with $a_i \in p^n\mathrm{W}(S)$ for all $i > k_n$. Hence $f(\varprojlim_n(\sum_{i=0}^{k_n} a_i X^{[i]} + p^n\mathrm{W}(S)\langle X\rangle)) = \sum_{i=0}^{\infty} a_i X^{[i]}$, which shows that f is surjective.

On the other hand, if $\varprojlim_n(\sum_{i=0}^{k_n} a_i^{(n)} X^{[i]} + p^n\mathrm{W}(S)\langle X\rangle) \in \ker f$, then $\lim_n a_i^{(n)} = 0$ for all $n \in \mathbb{N}_0$. Hence for all $i \in \mathbb{N}_0$ and $n \in \mathbb{N}$, there is a $k_{i,n} > n$ such that $a_i^{(k_{i,n})} \in p^n\mathrm{W}(S)$; but then $a_i^{(n)} \equiv a_i^{(k_{i,n})} \equiv 0 \mod p^n\mathrm{W}(S)$ for all $i \in \mathbb{N}_0$ and $n \in \mathbb{N}$, so f is injective. □

Definition 2.6. Let $[s]$ be the Teichmüller lift of s. Then we define the $\mathrm{W}(S)$-algebras

$$A^0_{\mathrm{cris}}(S/sS) := \mathrm{W}(S)\langle X\rangle/(X^{[1]} - [s]), \qquad A_{\mathrm{cris}}(S/sS) := \mathrm{W}(S)\langle\langle X\rangle\rangle/(X^{[1]} - [s]),$$

$$B^+_{\mathrm{cris}}(S/sS) := A_{\mathrm{cris}}(S/sS)[\tfrac{1}{p}].$$

Remark 2.7. The notation S/sS should be understood formally for our purposes, but isn't arbitrary. In [9, §2.2], Fontaine constructs a functor that assigns to each p-adically separated and complete ring B that is *semiperfect* (has a surjective Frobenius on B/pB) a certain ring $A_{\mathrm{cris}}(B)$. The ring S/sS is indeed semiperfect because S is perfect; it is p-adically separated and complete since $pS = 0$. Note that these are also the preconditions of Proposition 1.3 (ii) because the map θ_B plays an important role in this construction.

When constructing the ring A_{cris} for a perfectoid field C of mixed characteristic, one uses $A_{\mathrm{cris}}(\mathcal{O}_C)$. Since $A_{\mathrm{cris}}(\mathcal{O}_C) \cong A_{\mathrm{cris}}(\mathcal{O}_C/p\mathcal{O}_C)$ by [9, 2.2.3 b] and $\mathcal{O}_C/p\mathcal{O}_C \cong \mathcal{O}_C^\flat/s\mathcal{O}_C^\flat$ for any $s \in \mathcal{O}_C^\flat$ with $|s|_\flat = |p|$ by Proposition 1.13, we are indeed looking for the ring $A_{\mathrm{cris}}(\mathcal{O}_C^\flat/s\mathcal{O}_C^\flat)$ as our notation suggests. We prove in Theorem 2.35 that the functorial definition agrees with ours in this situation, but will not otherwise concern ourselves with the functorial point of view.

The notation $A^0_{\mathrm{cris}}(S/sS)$ does not appear anywhere in [9] and is merely an analogy to A_{cris}. It cannot be functorial in S/sS since $S/sS = 0$ whenever $s \in S^\times$.

Remark 2.8. The choice of generator of sS is irrelevant. If $s' = su$ with $u \in S^\times$, the $\mathrm{W}(S)$-algebra automorphism $\chi_u : \mathrm{W}(S)\langle X\rangle \xrightarrow{\sim} \mathrm{W}(S)\langle X\rangle$ with $\chi_u(X^{[n]}) = [u]^n X^{[n]}$ and inverse $\chi_{u^{-1}}$ maps $(X^{[1]} - [s'])$ to $[u](X^{[1]} - [s])$. We therefore have

- a canonical isomorphism $A^0_{\mathrm{cris}}(S/sS) \cong A^0_{\mathrm{cris}}(S/s'S)$, induced directly from χ_u;
- a canonical isomorphism $A_{\mathrm{cris}}(S/sS) \cong A_{\mathrm{cris}}(S/s'S)$, induced from the extension of χ_u to an automorphism of $\mathrm{W}(S)\langle\langle X\rangle\rangle$ by continuity;
- and a canonical isomorphism $B^+_{\mathrm{cris}}(S/sS) \cong B^+_{\mathrm{cris}}(S/s'S)$, obtained by localization.

It may seem strange that we quotient out $X^{[1]} - [s]$ since in the perfectoid field case, $[s] \in \mathrm{W}(\mathcal{O}_C^\flat)$ can never generate $\ker \theta_{\mathcal{O}_C}$, but Proposition 2.11 shows that working with $[s]$ makes no difference.

Lemma 2.9. *If $n \in \mathbb{N}$ has the p-adic expansion $n = \sum_{i=0}^l a_i p^i$ with $0 \le a_i < p$ and $a_l \neq 0$, then*

$$v_p(n!) = \sum_{i=0}^{l} a_i \frac{p^i - 1}{p - 1}.$$

Proof. By Legendre's formula (which is proven by a simple counting argument), we have

$$v_p(n!) \overset{!}{=} \sum_{j=1}^{l} \left\lfloor \frac{n}{p^j} \right\rfloor = \sum_{j=1}^{l}\sum_{i=j}^{l} a_i p^{i-j} = \sum_{i=1}^{l}\sum_{j=1}^{i} a_i p^{i-j} = \sum_{i=1}^{l} a_i \sum_{j=0}^{i-1} p^j = \sum_{i=1}^{l} a_i \frac{p^i-1}{p-1}. \qquad \square$$

Lemma 2.10. *For each $n \in \mathbb{N}$, we have $\frac{p^n}{n!} \in p\mathbb{Z}_{(p)}$. In particular, if R is a $\mathbb{Z}_{(p)}$-algebra, then $\frac{p^n}{n!} \in pR$ for every $n \in \mathbb{N}$.*

Proof. Let $\sum_{i=0}^l a_i p^i$ with $0 \le a_i < p$ and $a_l \neq 0$ be the p-adic expansion of n. Then

$$v_p(n!) \overset{2.9}{=} \sum_{i=0}^{l} a_i \frac{p^i - 1}{p-1} < \sum_{i=0}^{l} a_i p^i = n = v_p(p^n). \qquad \square$$

Proposition 2.11. *For all $w \in \mathrm{W}(S)$, there is a unique automorphism of $\mathrm{W}(S)$-algebras $\tau_w : \mathrm{W}(S)\langle X\rangle \xrightarrow{\sim} \mathrm{W}(S)\langle X\rangle$ with $\tau_w(X^{[n]}) = \frac{(X^{[1]}+pw)^n}{n!}$. It induces isomorphisms*

$$A^0_{\mathrm{cris}}(S/sS) = \frac{\mathrm{W}(S)\langle X\rangle}{(X^{[1]} - [s])} \cong \frac{\mathrm{W}(S)\langle X\rangle}{(X^{[1]} - ([s] - pw))},$$

$$A_{\mathrm{cris}}(S/sS) = \frac{\mathrm{W}(S)\langle\langle X\rangle\rangle}{(X^{[1]} - [s])} \cong \frac{\mathrm{W}(S)\langle\langle X\rangle\rangle}{(X^{[1]} - ([s] - pw))}$$

of $\mathrm{W}(S)$-algebras that identify the classes of $X^{[n]}$.

Proof. Uniqueness is clear. $\frac{(X^{[1]}+pw)^n}{n!}$ is really an element of $\mathrm{W}(S)\langle X\rangle$ because

$$\frac{(X^{[1]}+pw)^n}{n!} = \sum_{i=0}^{n} \frac{(X^{[1]})^i}{i!}\frac{(pw)^{n-i}}{(n-i)!} = \sum_{i=0}^{n} \frac{p^{n-i}}{(n-i)!} w^{n-i} X^{[i]} \overset{2.10}{\in} \mathrm{W}(S)\langle X\rangle.$$

By Remarks 2.4 and 2.1, the images $\tau_w(X^{[n]})$ indeed induce an endomorphism τ_w which is an automorphism because its inverse is τ_{-w}. Since $\tau_w(X^{[1]}-[s]) = (X^{[1]}-([s]-pw))$, we obtain the isomorphism for $A^0_{\mathrm{cris}}(S/sS)$. By continuity, τ_w extends to an automorphism of $\mathrm{W}(S)\langle\langle X\rangle\rangle$ which yields the isomorphism for $A_{\mathrm{cris}}(S/sS)$. $\square$

Lemma 2.12. *If $\xi \in \mathrm{W}(S)$ such that $\xi \equiv [s] \mod p\mathrm{W}(S)$, then $\xi w \in n\mathrm{W}(S)$ implies $w \in n\mathrm{W}(S)$ for all $n \in \mathbb{N}$ and $w \in \mathrm{W}(S)$.*

Proof. All prime numbers except p are units in $\mathrm{W}(S)$, so it suffices to consider $n = p^k$. If $k = 1$, this follows immediately from the facts that $p \in \mathrm{W}(S)$ is a prime element and $\xi \notin p\mathrm{W}(S)$. For $k > 1$, use the case $k = 1$ to write $w = pw'$ for some $w' \in \mathrm{W}(S)$. Then $\xi w = \xi p w' \in p^n\mathrm{W}(S)$, so $\xi w' \in p^{n-1}\mathrm{W}(S)$. The lemma then follows by induction. $\square$

Proposition 2.13. *For each $k \in \mathbb{N}$ and $R \in \{\mathrm{W}(S)\langle X\rangle,\ \mathrm{W}(S)\langle\langle X\rangle\rangle\}$, we have*

$$(X^{[1]}-[s])R \cap p^k R = p^k (X^{[1]}-[s])R.$$

In particular, $A^0_{\mathrm{cris}}(S/sS)$ and $A_{\mathrm{cris}}(S/sS)$ are $\mathbb{Z}$-torsion-free.

Proof. The inclusion $\supset$ is trivial. Identify $\mathrm{W}(S)\langle X\rangle \subset \mathrm{W}(S)\langle\langle X\rangle\rangle$ and let $\sum_{i=0}^{\infty} a_i X^{[i]}$, $\sum_{i=0}^{\infty} d_i X^{[i]} \in R$ such that

$$\sum_{i=0}^{\infty} p^k a_i X^{[i]} \overset{!}{=} (X^{[1]}-[s]) \sum_{i=0}^{\infty} d_i X^{[i]} = \sum_{i=0}^{\infty} (i d_{i-1} - d_i[s]) X^{[i]},$$

where we set $d_{-1} = 0$. Whenever $d_{i-1} \in p^k\mathrm{W}(S)$ for some $i \geq 0$, the comparison of coefficients $p^k a_i = i d_{i-1} - d_i[s] \in p^k\mathrm{W}(S)$ shows $-d_i[s] \in p^k\mathrm{W}(S)$, which implies $d_i \in p^k\mathrm{W}(S)$ by Lemma 2.12. It thus follows inductively that all $d_i \in p^k\mathrm{W}(S)$, proving the statement.

The note on $\mathbb{Z}$-torsion follows from the fact that all prime numbers other than p are already units in $\mathrm{W}(S)$ and hence in $A^0_{\mathrm{cris}}(S/sS)$ and $A_{\mathrm{cris}}(S/sS)$. $\square$

Remark 2.14. In both $A_{\mathrm{cris}}(S/sS)$ and $A^0_{\mathrm{cris}}(S/sS)$, $[s]^n$ is the image of $n!X^{[n]} \in \mathrm{W}(S)\langle X\rangle$ under the projection and hence divisible by $n!$. In the notation of Remark 2.1, we have $\frac{[s]^n}{n!} \in A^0_{\mathrm{cris}}(S/sS)$ and $\frac{[s]^n}{n!} \in A_{\mathrm{cris}}(S/sS)$ for all $n \in \mathbb{N}_0$. Similarly, both rings contain $\frac{\xi^n}{n!}$ for all $\xi \in \mathrm{W}(S)$ with $\xi \equiv [s] \mod p\mathrm{W}(S)$.

Proposition 2.15. *$A_{\mathrm{cris}}(S/sS)$ is the p-adic completion of $A^0_{\mathrm{cris}}(S/sS)$.*

Proof. We have an exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathrm{W}(S)\langle X\rangle \overset{\cdot(X^{[1]}-[s])}{\hookrightarrow} \mathrm{W}(S)\langle X\rangle \twoheadrightarrow A^0_{\mathrm{cris}}(S/sS) \to 0. \qquad (*)$$

Since $A^0_{\mathrm{cris}}(S/sS)$ is $\mathbb{Z}$-flat by 2.13 and $\widehat{\mathrm{W}(S)\langle X\rangle} = \mathrm{W}(S)\langle\langle X\rangle\rangle$ by 2.5, the sequence

$$0 \to \mathrm{W}(S)\langle\langle X\rangle\rangle \overset{\cdot(X^{[1]}-[s])}{\hookrightarrow} \mathrm{W}(S)\langle\langle X\rangle\rangle \twoheadrightarrow A^0_{\mathrm{cris}}\widehat{(S/sS)} \to 0 \qquad (**)$$

obtained by completion is also exact (cf. part (iii) of [16, 0315]). We obtain an isomorphism $\widehat{A^0_{\mathrm{cris}}(S/sS)} \cong A_{\mathrm{cris}}(S/sS)$ of $\mathbb{Z}$-modules. But since all $\mathbb{Z}$-modules and homomorphisms in $(*)$ are rings and ring homomorphisms, so are all $\mathbb{Z}$-modules and homomorphisms in $(**)$. Consequently $\widehat{A^0_{\mathrm{cris}}(S/sS)} \cong A_{\mathrm{cris}}(S/sS)$ is also a ring isomorphism. $\square$

We can now characterize equality in $A_{\mathrm{cris}}(S/sS)$ and $A^0_{\mathrm{cris}}(S/sS)$. Note that the argument in the proof of Proposition 2.13 is an instance of this.

Proposition 2.16. *Let $\xi \in \mathrm{W}(S)$ such that $\xi \equiv [s] \mod p\mathrm{W}(S)$. Every $a \in A_{\mathrm{cris}}(S/sS)$ can be written (in many ways) as a p-adically convergent series $a = \sum_{i=0}^{\infty} a_i \frac{\xi^i}{i!}$, where $a_i \in \mathrm{W}(S)$ for all $i \in \mathbb{N}_0$ and $\lim_i a_i = 0 \in \mathrm{W}(S)$. Furthermore, $a = 0$ if and only if there exist $d_i \in \mathrm{W}(S)$ for all $i \in \mathbb{N}_0$ such that*

$$\lim_{i\to\infty} d_i = 0 \in \mathrm{W}(S), \qquad a_0 = -d_0\xi, \qquad a_i = id_{i-1} - d_i\xi \quad \text{for all } i \in \mathbb{N}.$$

Proof. a is the image of some $\sum_{i=0}^{\infty} a_i X^{[i]} \in \mathrm{W}(S)\langle\langle X\rangle\rangle$ under the canonical projection $\pi : \mathrm{W}(S)\langle\langle X\rangle\rangle \twoheadrightarrow A_{\mathrm{cris}}(S/sS)$ from 2.11, which is p-adically continuous and thus satisfies

$$a = \pi\left(\sum_{i=0}^{\infty} a_i X^{[i]}\right) = \sum_{i=0}^{\infty} a_i \pi(X^{[i]}) = \sum_{i=0}^{\infty} a_i \frac{\xi^i}{i!}.$$

Clearly $a = 0$ if and only if $\sum_{i=0}^{\infty} a_i X^{[i]} \in \ker \pi$, which is to say that there is some $d = \sum_{i=0}^{\infty} d_i X^{[i]} \in \mathrm{W}(S)\langle\langle X\rangle\rangle$ such that

$$\sum_{i=0}^{\infty} a_i X^{[i]} \overset{!}{=} (X^{[1]} - \xi)d = \sum_{i=1}^{\infty} (id_{i-1} - d_i\xi)X^{[i]} - d_0\xi.$$

The statement follows by comparing coefficients. $\square$

Proposition 2.17. *Let $\xi \in \mathrm{W}(S)$ such that $\xi \equiv [s] \mod p\mathrm{W}(S)$. Every $a \in A^0_{\mathrm{cris}}(S/sS)$ can be written (in many ways) as $\sum_{i=0}^{n} a_i \frac{\xi^i}{i!}$, where $n \in \mathbb{N}$ and $a_i \in \mathrm{W}(S)$ for all $0 \leq i \leq n$. We have $a = 0$ if and only if there exist $d_i \in \mathrm{W}(S)$ for all $i \in \mathbb{N}_0$ such that $d_i = 0$ for all but finitely many $i \in \mathbb{N}_0$ and*

$$a_0 = -d_0\xi, \qquad a_i = id_{i-1} - d_i\xi \text{ for all } 0 < i \leq n, \qquad id_{i-1} = d_i\xi \text{ for all } i > n.$$

Proof. This is proven completely analogously to Proposition 2.16. $\square$

Proposition 2.18. *Consider the canonical maps $\mathrm{W}(S) \overset{\iota_0}{\to} A^0_{\mathrm{cris}}(S/sS) \overset{\iota_1}{\to} A_{\mathrm{cris}}(S/sS)$.*

(i) *ι_0 is injective.*

(ii) *ι_1 is injective if and only if $\iota := \iota_1 \circ \iota_0$ is injective.*

(iii) *ι is injective if and only if $\bigcap_{n\geq 1} s^n S = 0$.*

Proof. (i): Let $w = \sum_{i=0}^{\infty} p^i[w_i] \in \mathrm{W}(S)$ such that $\iota_0(w) = 0$, i.e. that there are $d_i \in \mathrm{W}(S)$ such that $w = -d_0[s]$, $id_{i-1} = d_i[s]$ for all $i > 0$, and $d_i = 0$ for all but finitely many $i \in \mathbb{N}_0$. Since $A^0_{\mathrm{cris}}(S/sS)$ is $\mathbb{Z}$-torsion-free, $id_{i-1} = d_i[s]$ shows that $d_{i-1} = 0$ whenever $d_i = 0$ for some $i > 0$; but that inductively means $d_0 = 0$, hence $w = -d_0[s] = 0$.

(ii): If ι_1 is injective, then so is $\iota = \iota_1 \circ \iota_0$ by part (i). On the other hand, assume that ι is injective and let $a = \sum_{i=0}^{n} a_i \frac{[s]^i}{i!} \in \ker \iota_1$. Then $n!a \in \mathrm{W}(S) \subset A^0_{\mathrm{cris}}(S/sS)$ and $\iota(n!a) = 0$, so $n!a = 0$. But $A^0_{\mathrm{cris}}(S/sS)$ is $\mathbb{Z}$-torsion-free, so $a = 0$.

(iii): We identify $\mathrm{W}(S) \subset \mathrm{W}(\operatorname{Frac} S)$. Since the Teichmüller lift for $\mathrm{W}(S)$ is the only multiplicative section of $\mathrm{W}(S) \twoheadrightarrow S$, the lift for $\mathrm{W}(\operatorname{Frac} S)$ restricts to the lift for $\mathrm{W}(S)$ and we can unambiguously denote both by $[\cdot]$.

Let $w = \sum_{i=0}^{\infty} p^i [w_i] \in \ker \iota$. Then by Proposition 2.16 there are $d_i \in \mathrm{W}(S)$ such that

$$\lim_{i\to\infty} d_i = 0 \in \mathrm{W}(S), \qquad w = -d_0[s], \qquad i d_{i-1} = d_i[s] \quad \text{for all } i \in \mathbb{N}.$$

It inductively follows that $-w[s^{-1}]^{i+1} i! = d_i \in \mathrm{W}(S)$ for all $i \in \mathbb{N}_0$. In particular, we have $w[s^{-n}] \in \mathrm{W}(S)[\frac{1}{p}]$ for all $n \in \mathbb{N}$, so that for each $n \in \mathbb{N}$ there exists a $k_n \in \mathbb{N}_0$ such that

$$p^{k_n} w[s^{-n}] = \sum_{i=0}^{\infty} p^{i+k_n} [w_i s^{-n}] \in \mathrm{W}(S) \cap \mathrm{W}(\operatorname{Frac} S).$$

Due to the uniqueness of the Teichmüller expansion in $\mathrm{W}(\operatorname{Frac} S)$ and its compatibility with $\mathrm{W}(S)$, we see that $w_i s^{-n} \in S$, i.e. $w_i \in s^n S$, for all $i, n \in \mathbb{N}_0$.

If $\bigcap_{n\geq 1} s^n S = 0$, this means $w_i = 0$ for all $i \in \mathbb{N}_0$, i.e. $w = 0$. On the other hand, if $x \in \bigcap_{n \geq 1} s^n S \setminus 0$, then all $xs^{-n} \in S$ and the elements $d_i := -[xs^{-(i+1)}] i! \in \mathrm{W}(S)$ for $i \in \mathbb{N}_0$ show that $[x] \in \ker \iota$ by Proposition 2.16. □

Remark 2.19 (Hom-set characterization). Let $\xi \in \mathrm{W}(S)$ such that $\xi \equiv [s] \mod p\mathrm{W}(S)$.

(i) By Remark 2.4 and Proposition 2.11, ring homomorphisms $A^0_{\mathrm{cris}}(S/sS) \to R$ correspond to pairs $(f, (r_i)_{i\in\mathbb{N}})$, where $f : \mathrm{W}(S) \to R$ is a ring homomorphism and the $r_i \in R$ form a sequence with $r_1 = f(\xi)$ and $r_i r_j = \binom{i+j}{i} r_{i+j}$ for all $i, j \in \mathbb{N}$. Of course the r_i correspond to the images of $\frac{\xi^i}{i!} \in A^0_{\mathrm{cris}}(S/sS)$.

(ii) If $\bigcap_{n\geq 1} s^n S = 0$, then the ring homomorphisms $A_{\mathrm{cris}}(S/sS) \to B$ with p-adically separated and complete B correspond to ring homomorphisms $A^0_{\mathrm{cris}}(S/sS) \to B$ by Proposition 2.18 and hence pairs $(f, (b_i)_{i\in\mathbb{N}})$ as in (i). If $\bigcap_{n\geq 1} s^n S \neq 0$, one can still induce every ring homomorphism $A_{\mathrm{cris}}(S/sS) \to B$ this way, but distinct pairs need not induce distinct homomorphisms.

(iii) If B is p-adically separated and complete as well as $\mathbb{Z}$-torsion-free, then any ring homomorphism $f : \mathrm{W}(S) \to B$ admits at most *one* extension $f' : A_{\mathrm{cris}}(S/sS) \to B$ since $n! f'(\frac{\xi^n}{n!}) = f'(n! \frac{\xi^n}{n!}) = f'(\xi^n) = f(\xi)^n$ uniquely determines all $f'(\frac{\xi^n}{n!})$. Such an extension exists if and only if $f(\xi)^n \in n!B$ for all $n \in \mathbb{N}$ since the compatibility condition of (i) is automatically satisfied due to Remark 2.1.

Theorem 2.20.

(i) *There is a unique ring homomorphism $\varphi_{\mathrm{cris}} : A_{\mathrm{cris}}(S/sS) \to A_{\mathrm{cris}}(S/sS)$ satisfying $\varphi_{\mathrm{cris}}(A^0_{\mathrm{cris}}(S/sS)) \subset A^0_{\mathrm{cris}}(S/sS)$ that extends the Frobenius $\varphi : \mathrm{W}(S) \xrightarrow{\sim} \mathrm{W}(S)$.*

(ii) *$\varphi_{\mathrm{cris}} : A_{\mathrm{cris}}(S/sS) \to A_{\mathrm{cris}}(S/sS)$ is injective.*

(iii) *The localization of φ_{cris} at p is an injective endomorphism of $B^+_{\mathrm{cris}}(S/sS)$. We likewise denote it by φ_{cris}.*

Proof. (i): By Remark 2.19, it suffices to show $\varphi([s])^n \in n!A^0_{\mathrm{cris}}(S/sS)$. Indeed we have

$$\varphi([s])^n = [s]^{np} = (np)!\frac{[s]^{np}}{(np)!} \in (np)!A^0_{\mathrm{cris}}(S/sS) \subset n!A^0_{\mathrm{cris}}(S/sS).$$

(ii): Let $a = \sum_{i=0}^{\infty} a_i \frac{[s]^i}{i!} \in A_{\mathrm{cris}}(S/sS)$ such that

$$\varphi_{\mathrm{cris}}(a) = \sum_{j=0}^{\infty} \varphi(a_j)\frac{(jp)!}{j!}\frac{[s]^{jp}}{(jp)!} = 0.$$

By 2.16, this means that there are Witt vectors $g_i \in \mathrm{W}(S)$ such that for all $i \in \mathbb{N}$,

$$\lim_{i\to\infty} g_i = 0 \in \mathrm{W}(S), \qquad \varphi(a_0) = -g_0[s], \qquad ig_{i-1} - g_i[s] = \begin{cases} 0 & \text{if } i \notin p\mathbb{Z} \\ \varphi(a_{i/p})\frac{i!}{(i/p)!} & \text{if } i \in p\mathbb{Z}. \end{cases}$$

If we set $g_{-1} := 0$, we have, for all $j \in \mathbb{N}_0$ and $0 < r < p$,

$$g_{jp+r}[s] = (jp+r)g_{jp+(r-1)}, \qquad\qquad g_{jp}[s] = (jp)g_{jp-1} - \frac{(jp)!}{j!}\varphi(a_j). \tag{$*$}$$

This gives us the following implications for all $j \in \mathbb{N}_0$ and $0 < r < p$:

$$g_{jp+(r-1)} \in \tfrac{(jp+r-1)!}{j!}\mathrm{W}(S) \overset{(*)}{\Longrightarrow} g_{jp+r}[s] \in \tfrac{(jp+r)!}{j!}\mathrm{W}(S) \overset{2.12}{\Longrightarrow} g_{jp+r} \in \tfrac{(jp+r)!}{j!}\mathrm{W}(S),$$

$$g_{jp+(p-1)} \in \tfrac{(jp+(p-1))!}{j!}\mathrm{W}(S) \overset{(*)}{\Longrightarrow} g_{(j+1)p}[s] \in \tfrac{((j+1)p)!}{(j+1)!}\mathrm{W}(S) \overset{2.12}{\Longrightarrow} g_{(j+1)p} \in \tfrac{((j+1)p)!}{(j+1)!}\mathrm{W}(S).$$

Since trivially $g_0 \in \frac{(0p)!}{0!}\mathrm{W}(S)$, we inductively get $g_{jp+(p-1)} \in \frac{(jp+(p-1))!}{j!}\mathrm{W}(S)$ for all $j \in \mathbb{N}_0$, which lets us define

$$c_j := \frac{j!g_{jp+(p-1)}}{(jp+(p-1))!} \in \mathrm{W}(S), \qquad\qquad d_j := \varphi^{-1}(c_j) \in \mathrm{W}(S).$$

Then

$$c_j[s]^p = \frac{j!g_{jp+(p-1)}[s]^p}{(jp+(p-1))!} \overset{(*)}{=} \frac{j!g_{jp+(p-2)}[s]^{p-1}}{(jp+(p-2))!} \overset{(*)}{=} \dots \overset{(*)}{=} \frac{j!g_{jp}[s]}{(jp)!}$$

for all $j \in \mathbb{N}_0$; hence $-d_0[s] = \varphi^{-1}(-c_0[s]^p) = \varphi^{-1}(-g_0[s]) = a_0$ and

$$\begin{aligned} jd_{j-1} - d_j[s] &= \varphi^{-1}(jc_{j-1} - c_j[s]^p) \\ &= \varphi^{-1}\left(\frac{j(j-1)!g_{jp-1}}{(jp-1)!} - \frac{j!g_{jp}}{(jp)!}[s]\right) \\ &= \varphi^{-1}\left(\frac{j!(jp\, g_{jp-1} - g_{jp}[s])}{(jp)!}\right) \\ &\overset{(*)}{=} \varphi^{-1}\left(\frac{j!\varphi(a_j)\frac{(jp)!}{j!}}{(jp)!}\right) = a_j \end{aligned} \tag{$**$}$$

for all $j \in \mathbb{N}$. It now only remains to show $\lim_i d_i = 0$, since the sequence $(d_i)_{i\in\mathbb{N}_0}$ then satisfies the conditions of 2.16 and shows that $a = 0$.

We have shown in $(**)$ that $d_{m+j}[s] = (m+j)d_{m+j-1} - a_{m+j}$ for all $j, m \in \mathbb{N}$, so we inductively get $d_{m+j}[s]^j \in (\frac{(m+j)!}{m!}d_m, a_{m+1}, \dots, a_{m+j}) \lhd \mathrm{W}(S)$ for all $j, m \in \mathbb{N}$. Now let

$k \in \mathbb{N}$ be arbitrary and choose $n \in \mathbb{N}$ large enough so that $a_{(p^n-1)+j} \in p^k \mathrm{W}(S)$ for all $j \in \mathbb{N}$ and $v_p(\frac{p^n!}{(p^n-1)!}) = v_p(p^n) = n \geq k$. Then

$$d_{(p^n-1)+j}[s]^j \in (\tfrac{(p^n-1+j)!}{(p^n-1)!} d_{p^n-1}, a_{p^n}, \ldots, a_{p^n+j-1}) \subset p^k \mathrm{W}(S)$$

for all $j \in \mathbb{N}$, i.e. $d_{(p^n-1)+j} \in p^k \mathrm{W}(S)$ for all $j \in \mathbb{N}$ by Lemma 2.12, showing $\lim_i d_i = 0$.

(iii): Since $\varphi_{\text{cris}}(p) = p \in B^+_{\text{cris}}(S/sS)^\times$, this is a consequence of Proposition 2.13. □

We close with a negative result that illustrates one of the reasons the ring $A_{\text{cris}}(S/sS)$ is not very pleasant to deal with. This is essentially Exercise 9.4.1 in [2].

Lemma 2.21. *There is an isomorphism of* $\mathrm{W}(S)$*-algebras*

$$\mathrm{W}(S)\langle X\rangle \cong \frac{\mathrm{W}(S)[Y_0, Y_1, \ldots]}{(Y_j^p - c_j Y_{j+1}; j \in \mathbb{N}_0)},$$

where $c_j := p^{j+1}!(p^j!)^{-p} \in p\mathbb{Z}_{(p)}$.

Proof. Let $I := (Y_j^p - c_j Y_{j+1}; j \in \mathbb{N}_0)$. First note that we really have $c_j \in p\mathbb{Z}_{(p)}$ because

$$v_p(p^{j+1}!) = \frac{p^{j+1}-1}{p-1} > \frac{p^{j+1}-p}{p-1} = p v_p(p^j!) = v_p((p^j!)^p).$$

Define a $\mathrm{W}(S)$-algebra homomorphism $f' : \mathrm{W}(S)[Y_0, Y_1, \ldots] \to \mathrm{W}(S)\langle X\rangle$ via the images $f'(Y_j) := X^{[p^j]}$, which satisfies $f'(Y_j^p - c_j Y_{j+1}) = 0$ and thus induces a homomorphism $f : \mathrm{W}(S)[Y_0, Y_1, \ldots]/I \to \mathrm{W}(S)\langle X\rangle$ that we claim to be bijective. Let $n \in \mathbb{N}$ have the p-adic expansion $n = \sum_{i=0}^l a_i p^i$; set $n_k := \sum_{i=0}^k a_i p^i$ for $0 \leq k \leq l$ and $n_{-1} := 0$. Then

$$X^{[n]} = \prod_{i=0}^l X^{[a_i p^i]} \binom{n_k}{n_{k-1}}^{-1} = \prod_{i=0}^l \left(X^{[p^i]}\right)^{a_i} \frac{(p^i!)^{a_i}}{(a_i p^i)!} \binom{n_i}{n_{i-1}}^{-1}.$$

This proves that f is surjective because $v_p\left(\frac{(p^i!)^{a_i}}{(a_i p^i)!}\binom{n_i}{n_{i-1}}^{-1}\right)$ is

$$a_i \cdot \frac{p^i-1}{p-1} - a_i \frac{p^i-1}{p-1} - \left(\sum_{j=0}^{i} a_j \frac{p^j-1}{p-1} - \sum_{j=0}^{i-1} a_j \frac{p^j-1}{p-1} - a_i \frac{p^i-1}{p-1}\right) = 0.$$

Let $x = \sum_{k=0}^n a_k \prod_{i=0}^m Y_i^{e_{k,i}} \in \ker f'$. After adding appropriate elements of I, we may assume $e_{k,i} < p$ for all k, i. Set $e_k := \sum_{i=0}^m e_{k_i} p^i \in \mathbb{N}_0$. After combining summands, we may assume that all e_k are distinct. But then

$$0 = f'(x) = f'\left(\sum_{k=0}^n a_k \prod_{i=0}^m Y_i^{e_{k,i}}\right) = \sum_{k=0}^n a_k z_k X^{[e_k]}$$

for non-zero $z_k \in \mathbb{Z}_{(p)} \subset \mathrm{W}(S)$ and pairwise distinct $X^{[e_k]}$, so all $a_k = 0$, i.e. $x = 0$. Since we added elements of I in the reduction step, we have shown $x \in I$, so f is injective. □

Proposition 2.22. *Let* $\bigcap_{n\geq 1} s^n S = 0$. *Neither* $A_{\text{cris}}(S/sS)$ *nor* $A^0_{\text{cris}}(S/sS)$ *is noetherian.*

Proof. Since $[s]^p \in pA^0_{\mathrm{cris}}(S/sS)$, Lemma 2.21 shows that

$$\begin{aligned} A^0_{\mathrm{cris}}(S/sS)/pA^0_{\mathrm{cris}}(S/sS) &\cong \frac{\mathrm{W}(S)[Y_0, Y_1, \ldots]}{(p, Y_0 - [s], Y_j^p - c_j Y_{j+1}; j \in \mathbb{N}_0)} \\ &\cong \frac{\mathrm{W}(S)[Y_0, Y_1, \ldots]}{(p, Y_0 - [s], Y_j^p; j \in \mathbb{N}_0)} \\ &\cong \frac{\mathrm{W}(S)[Y_1, \ldots]}{(p, [s]^p, Y_j^p; j \in \mathbb{N})} \\ &\cong \frac{(S/s^pS)[Y_1, \ldots]}{(Y_j^p; j \in \mathbb{N})}. \end{aligned}$$

We must have $s \notin S^\times$ because $\bigcap_{n \geq 1} s^n S = 0$. Therefore $S/s^pS \neq 0$, so the ring above is not noetherian. But if either $A_{\mathrm{cris}}(S/sS)$ or $A^0_{\mathrm{cris}}(S/sS)$ was noetherian, so would be all their quotients, in particular $A_{\mathrm{cris}}(S/sS)/pA_{\mathrm{cris}}(S/sS) \overset{2.15}{\cong} A^0_{\mathrm{cris}}(S/sS)/pA^0_{\mathrm{cris}}(S/sS)$. $\square$

2.2 Notes on divided power structures

The theory of divided power structures is usually intended to work around $\mathbb{Z}$-torsion. In the $\mathbb{Z}$-torsion-free case it is almost trivial; we effectively used it throughout §2.1 already. We will however have to consider general rings for Definition 2.32 and Theorem 2.35, necessitating this short excursion into the general theory. A classic reference is [1, §3]; 09PD and 07H7 of [16] might also be of interest.

Definition 2.23. Let R be a ring and $I \lhd R$ an ideal. A *divided power structure on I (in R)* is a family $\gamma = (\gamma_i)_{i \in \mathbb{N}}$ of maps $\gamma_i : I \to I$ satisfying the following properties. We write $\gamma_0(x) := 1$.

(i) For all $x \in I$, $\gamma_1(x) = x$.

(ii) For all $x \in I$, $r \in R$ and $i \in \mathbb{N}_0$, $\gamma_i(rx) = r^i \gamma_i(x)$.

(iii) For all $x, y \in I$ and $i \in \mathbb{N}_0$, $\gamma_i(x+y) = \sum_{j=0}^{i} \gamma_j(x)\gamma_{i-j}(y)$.

(iv) For all $x \in I$ and $i, j \in \mathbb{N}_0$, $\gamma_i(x)\gamma_j(x) = \frac{(i+j)!}{i!j!}\gamma_{i+j}(x) = \binom{i+j}{i}\gamma_{i+j}(x)$.

(v) For all $x \in I$ and $i, j \in \mathbb{N}_0$, $\gamma_i(\gamma_j(x)) = C_{i,j}\gamma_{ij}(x)$, where $C_{i,j} = (ij)!/(i!(j!)^i) \in \mathbb{Z}$.

We also call (R, I, γ) a *divided power algebra* or (I, γ) a *divided power ideal.*

If (R, I, γ) and (S, J, δ) are divided power algebras, a ring homomorphism $f : R \to S$ is a *divided power homomorphism* if $f(I) \subset J$ and $\delta_i(f(x)) = f(\gamma_i(x))$ for all $i \in \mathbb{N}$ and $x \in I$.

Note that $C_{i,j} \in \mathbb{Z}$ since it is the number of ways to partition a set of ij elements into i sets of j elements. In older texts like [1], one also finds the terms *PD-structure* and *PD-morphism*, based on the French *puissances divisées.* The composition of two divided power homomorphisms is clearly itself a divided power homomorphism, so divided power algebras with divided power homomorphisms form a category.

Example 2.24. It follows inductively from properties (i) and (iv) that $n!\gamma_n(x) = x^n$ for all $n \in \mathbb{N}$ and $x \in I$, so if R is a $\mathbb{Q}$-algebra, every ideal admits exactly one divided power structure, given by $\gamma_n(x) = \frac{x^n}{n!}$, from whose properties (i)-(v) are derived in the first place. In this case, the theory is completely trivial.

Proposition 2.25. *Let R be a $\mathbb{Z}$-torsion-free ring. If γ and δ are divided power structures on ideals $I \triangleleft R$ and $J \triangleleft R$ respectively, then $\gamma_n(x) = \delta_n(x)$ for all $n \in \mathbb{N}$ and $x \in I \cap J$. In particular, there is at most one divided power structure on any given ideal.*

Proof. Note that $n!\gamma_n(x) = x^n = n!\delta_n(x)$ for $x \in I \cap J$ and cancel $n!$. □

Therefore we are only concerned with existence in the $\mathbb{Z}$-torsion-free case. The next proposition saves us from a laborious verification of all five properties.

Proposition 2.26. *If R is a $\mathbb{Z}$-torsion-free ring and $I \triangleleft R$ is an ideal, then the following are equivalent:*

(i) *I admits a divided power structure.*

(ii) *For all $x \in I$ and all $n \in \mathbb{N}$, we have $x^n \in n!I$.*

(iii) *There is a generating set S of I such that $x^n \in n!I$ for all $x \in S$ and $n \in \mathbb{N}$.*

Proof. The implications (i) $\implies$ (ii) $\implies$ (iii) are trivial and don't even require that R is $\mathbb{Z}$-torsion-free. For (iii) $\implies$ (ii), it suffices to show that $(x+y)^n \in n!R$ for all $n \in \mathbb{N}$ whenever $x^n, y^n \in n!R$ for all $n \in \mathbb{N}$. Indeed,

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} = \sum_{i=0}^{n} \binom{n}{i} \frac{i!x^i}{i!} \frac{(n-i)!y^{n-i}}{(n-i)!} = \sum_{i=0}^{n} n! \frac{x^i}{i!} \frac{y^{n-i}}{(n-i)!} \in n!I.$$

(ii) $\implies$ (i) follows by taking $\gamma_n(x)$ to be $\frac{x^n}{n}$. To verify that this is a divided power structure, multiply the equations in 2.23 (i)-(v) by $1!, i!, (i!)^2, i!j!$ or $i!j!^i$ respectively and note that doing so produces an equivalence because R is $\mathbb{Z}$-torsion-free. □

Example 2.27. The ideal $p\mathbb{Z}_p \triangleleft \mathbb{Z}_p$ admits a unique divided power structure because $p^n \in n!p\mathbb{Z}_p \subset n!\mathbb{Z}_p$ by Lemma 2.10.

Proposition 2.28. *If (A, aA, γ) with $a \in A$ is a divided power algebra and $f : A \to B$ is a ring homomorphism, then $\delta_n(f(a)b) := b^n f(\gamma_n(a))$ for $b \in B$ is the unique divided power structure on $f(a)B \triangleleft B$ that makes f a divided power homomorphism.*

Proof. Uniqueness is clear by 2.23 (ii). δ_n is well-defined because whenever $b, b' \in B$ satisfy $f(a)b = f(a)b'$, we have

$$f(\gamma_n(a))b^n - f(\gamma_n(a))(b')^n = f(\gamma_n(a))(b^n - (b')^n) = f(\gamma_n(a))(b - b') \sum_{i=0}^{n-1} b^i (b')^{n-1-i} = 0$$

since $f(\gamma_n(a)) \in f(aA) \subset f(a)B$ is a multiple of $f(a)$. We verify the five properties of 2.23 directly. (i) and (ii) are obvious. For (iii), let $b, b' \in B$ and $n \in \mathbb{N}$ and note that

$$\begin{aligned}
\sum_{i=0}^{n} \delta_i(bf(a))\delta_{n-i}(b'f(a)) &= \sum_{i=0}^{n} b^i (b')^{n-i} f(\gamma_i(a)\gamma_{n-i}(a)) \\
&= \sum_{i=0}^{n} \binom{n}{i} b^i (b')^{n-i} f(\gamma_n(a)) \\
&= (b+b')^n f(\gamma_n(a)) \\
&= \delta_n((b+b')f(a))
\end{aligned}$$

since (iv) already holds for γ. Similarly for any $b \in B$ and $i, j \in \mathbb{N}_0$,

$$\begin{aligned}
\delta_i(bf(a))\delta_j(bf(a)) &= b^{i+j} f(\gamma_i(a)\gamma_j(a)) \\
&= b^{i+j} f\left(\binom{i+j}{i}\gamma_{i+j}(a)\right) \\
&= \binom{i+j}{i} b^{i+j}\delta_{i+j}(f(a)) \\
&= \binom{i+j}{i}\delta_{i+j}(bf(a)).
\end{aligned}$$

For each $n \in \mathbb{N}$, let $a_n \in A$ such that $\gamma_n(a) = a_n a$, i.e. $\delta_n(f(a)) = f(\gamma_n(a)) = f(a_n)f(a)$. Then $a_j^i a_i a = C_{i,j} a_{ij} a$ for all $i, j \in \mathbb{N}_0$ by 2.23 (v), so

$$\begin{aligned}
\delta_i(\delta_j(f(a)b)) &\overset{\text{(ii)}}{=} b^{ij}\delta_i(\delta_j(f(a))) = b^{ij}\delta_i(f(a_j)f(a)) \overset{\text{(ii)}}{=} b^{ij} f(a_j^i)\delta_i(f(a)) \\
&= b^{ij} f(a_j^i) f(a_i) f(a) = b^{ij} C_{i,j} f(a_{ij}) f(a) = C_{i,j}\delta_{ij}(f(a)b).
\end{aligned}$$ □

Corollary 2.29. *Every p-adically separated and complete ring B admits a unique divided power structure γ on pB such that each $\gamma_n(p) \in pB$ is the image of $\frac{p^n}{n!} \in p\mathbb{Z}_p$ inside B.*

Proposition 2.30. *If (A, I, γ) and (B, J, δ) are two divided power algebras, $f : A \to B$ is a ring homomorphism, and S is a generating set of I, then f is a divided power homomorphism if and only if $f(I) \subset J$ and $f(\gamma_n(s)) = \delta_n(f(s))$ for all $n \in \mathbb{N}$ and $s \in S$.*

Proof. The condition is trivially necessary. Let $x_1, x_2 \in I$ such that $f(\gamma_n(x_i)) = \delta_n(f(x_i))$ for all $n \in \mathbb{N}$ and $i \in \{1, 2\}$. It then suffices to show that for all $a_1, a_2 \in A$ and $n \in \mathbb{N}$, we also have $f(\gamma_n(a_1x_1 + a_2x_2)) = \delta_n(f(a_1x_1 + a_2x_2))$:

$$\begin{aligned}
f(\gamma_n(a_1x_1 + a_2x_2)) &= \sum_{i=0}^{n} f(\gamma_i(a_1x_1)) f(\gamma_{n-i}(a_2x_2)) \\
&= \sum_{i=0}^{n} f(a_1^i) f(\gamma_i(x_1)) f(a_2^{n-i}) f(\gamma_{n-i}(x_2)) \\
&= \sum_{i=0}^{n} f(a_1^i)\delta_i(f(x_1)) f(a_2^{n-i})\delta_{n-i}(f(x_2)) \\
&= \sum_{i=0}^{n} \delta_i(f(a_1x_1))\delta_{n-i}(f(a_2x_2)) \\
&= \delta_n(f(a_1x_1 + a_2x_2)).
\end{aligned}$$ □

2.3 Alternative constructions of A_{cris}

We investigate how our construction of A_{cris} compares to two more common alternatives. Throughout this section, we fix a perfectoid field of mixed characteristic $(C, |\cdot|)$ and apply the constructions of §2.1 to $S := \mathcal{O}_C^\flat$ and some fixed $s \in S$ such that $|s|_\flat = |p|$. Since $\mathcal{O}_C^\flat$ is the ring of integers of the valued field $C^\flat$, all such s are associated and the rings $A_{\text{cris}}^0 := A_{\text{cris}}^0(\mathcal{O}_C^\flat / s\mathcal{O}_C^\flat)$, $A_{\text{cris}} := A_{\text{cris}}(\mathcal{O}_C^\flat / s\mathcal{O}_C^\flat)$ and $B_{\text{cris}}^+ := B_{\text{cris}}^+(\mathcal{O}_C^\flat / s\mathcal{O}_C^\flat)$ are uniquely determined up to canonical isomorphism. Note also that our occasional extra assumption $\bigcap_{n \geq 1} s^n \mathcal{O}_C^\flat = 0$ holds. We additionally fix a $\xi \in \mathrm{W}(\mathcal{O}_C^\flat)$ with $\xi \equiv [s] \mod p\mathrm{W}(\mathcal{O}_C^\flat)$ which generates $\ker \theta_{\mathcal{O}_C}$; this is guaranteed to exist by Proposition 1.15.

The first alternative construction we consider is the one given in [12, Definition 6.1], where A_{cris} is defined as the p-adic completion of $\mathrm{W}(\mathcal{O}_C^\flat)[\frac{\xi^n}{n!} \mid n \in \mathbb{N}]$.

Theorem 2.31. *There is a unique isomorphism of* $\mathrm{W}(\mathcal{O}_C^\flat)$*-algebras*

$$A_{\text{cris}}^0 \cong A := \{\sum_{i=0}^n a_i \frac{\xi^i}{i!} \in \mathrm{W}(\mathcal{O}_C^\flat)[\tfrac{1}{p}] \,|\, n \in \mathbb{N}_0,\ a_i \in \mathrm{W}(\mathcal{O}_C^\flat)\} < \mathrm{W}(\mathcal{O}_C^\flat)[\tfrac{1}{p}].$$

Proof. A is $\mathbb{Z}$-torsion-free, so it suffices to show that the $\mathrm{W}(\mathcal{O}_C^\flat)$-algebra homomorphism $\pi : \mathrm{W}(\mathcal{O}_C^\flat)\langle X\rangle \twoheadrightarrow A$ with $\pi(X^{[n]}) = \frac{\xi^n}{n!}$ satisfies $\ker \pi \subset (X^{[1]} - \xi)$ (surjectivity and the other inclusion are trivial). Let $a = \sum_{i=0}^n a_i X^{[i]} \in \ker \pi$. If $n = 0$, then clearly $a = a_0 = 0$. Otherwise, let $d_m := \sum_{i=1}^{n-m} \frac{m!}{(m+i)!} a_{m+i}\xi^{i-1} \in \mathrm{W}(\mathcal{O}_C^\flat)[\frac{1}{p}]$ for all $0 \le m < n$.

We claim that all $d_m \in \mathrm{W}(\mathcal{O}_C^\flat)$. In fact it suffices to show $d_m\xi \in \mathrm{W}(\mathcal{O}_C^\flat)$ because if $p^n d_m \in \mathrm{W}(\mathcal{O}_C^\flat)$ for some $n \in \mathbb{N}_0$, then $(p^n d_m)\xi = p^n(d_m\xi) \in p^n\mathrm{W}(\mathcal{O}_C^\flat)$. Lemma 2.12 then implies $p^n d_m \in p^n\mathrm{W}(\mathcal{O}_C^\flat)$, hence $d_m \in \mathrm{W}(\mathcal{O}_C^\flat)$.

For $m = 0$, we indeed have

$$a_0 = -[s]\sum_{i=1}^n a_i \frac{\xi^{i-1}}{i!} = -\xi d_0 \in \mathrm{W}(\mathcal{O}_C^\flat)$$

and thus $d_0 \in \mathrm{W}(\mathcal{O}_C^\flat)$. Now note that

$$\begin{aligned} d_m\xi + a_m &= \sum_{i=1}^{n-m} \frac{m!}{(m+i)!} a_{m+i}\xi^i + \frac{m!}{m!} a_m \xi^{m-m} \\ &= \sum_{i=0}^{n-m} \frac{m!}{(m+i)!} a_{m+i}\xi^i \\ &= m \cdot \sum_{i=1}^{n-(m-1)} \frac{(m-1)!}{((m-1)+i)!} a_{(m-1)+i}\xi^{i-1} \\ &= m d_{m-1} \end{aligned}$$

for all $0 < m < n$, so inductively $d_m\xi \in \mathrm{W}(\mathcal{O}_C^\flat)$ and consequently $d_m \in \mathrm{W}(\mathcal{O}_C^\flat)$. Now by construction, we have

$$a = \sum_{i=1}^n a_i X^{[i]} + a_0 = \sum_{i=1}^n (i d_{i-1} - d_i\xi) - \xi d_0 = (X^{[1]} - \xi)\sum_{i=0}^{n-1} d_i X^{[i]}. \qquad \square$$

Finally, we show that our construction of A_{cris} results in the *universal p-adic formal divided power thickening* of $\mathcal{O}_C$ and therefore agrees with the more general universal construction in [9, §2.1].

Definition 2.32. Let R be a ring. A *p-adic formal divided power thickening of R* is a p-adically separated and complete ring B together with a surjective ring homomorphism $\rho : B \twoheadrightarrow R$ and a divided power structure δ on $\ker \rho$ such that $\delta_n(b) = \gamma_n(b)$ for all $n \in \mathbb{N}$ and $b \in \ker \rho \cap pB$, where γ is the natural divided power structure on pB from 2.29.

Lemma 2.33. *If (B, ρ, δ) is a p-adic formal divided power thickening of a ring R, then $\rho^\flat : B^\flat \to R^\flat$ is an isomorphism.*

Proof. The key observation is that if $k \in \ker\rho$, then $k^p = p!\delta_p(k) \in pB$.

For injectivity, note that if $\varprojlim_j (b_j + B/pB) \in \ker\rho^\flat$, then there are $k_j \in \ker\rho$ with $b_j \equiv k_j \mod pB$ for all $j \in \mathbb{N}_0$, which shows $b_j \equiv b_{j+1}^p \equiv k_{j+1}^p \equiv 0 \mod pB$.

For surjectivity, let $r = \varprojlim_j (r_j + R/pR) \in R^\flat$. Since ρ is surjective, there exist elements $b_j \in B$ such that $\rho(b_j) = r_{j+1}$ for all $j \in \mathbb{N}_0$. These elements necessarily satisfy $\rho(b_{j+1}^p) \equiv r_{j+2}^p \equiv r_{j+1} \equiv \rho(b_j) \mod pR$, so $b_{j+1}^p \equiv b_j + k_j \mod pB$ for some $k_j \in \ker\rho$. Hence $(b_{j+1}^p)^p \equiv b_j^p + k_j^p \equiv b_j^p \mod pB$. But then $b := \varprojlim_j (b_j^p + pB) \in B^\flat$ satisfies

$$\rho^\flat(b) = \varprojlim_{j\in\mathbb{N}_0} (\rho(b_j^p) + pR) = \varprojlim_{j\in\mathbb{N}_0} (r_{j+1}^p + pR) = \varprojlim_{j\in\mathbb{N}_0} (r_j + pR) = r. \qquad \square$$

Proposition 2.34. *$\theta_{\mathcal{O}_C}$ extends to a unique ring homomorphism $\theta_{\mathrm{cris}} : A_{\mathrm{cris}} \twoheadrightarrow \mathcal{O}_C$. Its kernel admits divided powers, turning $(A_{\mathrm{cris}}, \theta_{\mathrm{cris}})$ into a p-adic formal divided power thickening of $\mathcal{O}_C$.*

Proof. Since $\theta_{\mathcal{O}_C}$ is already surjective, so are all its extensions. The compatibility between divided power structures follows from 2.25 and 2.13. Note that

$$\left|\frac{\theta_{\mathcal{O}_C}([s])^n}{n!}\right| = \frac{|(s^\sharp)^n|}{|n!|} = \frac{|s^n|_\flat}{|n!|} = \left|\frac{p^n}{n!}\right| \overset{2.27}{\leq} 1,$$

so we have $\theta_{\mathcal{O}_C}([s])^n \in n!\mathcal{O}_C$. By Remark 2.19 (iii), this suffices to construct the unique extension $\theta_{\mathrm{cris}} : A_{\mathrm{cris}} \twoheadrightarrow \mathcal{O}_C$ as $\mathcal{O}_C$ is $\mathbb{Z}$-torsion-free and p-adically separated and complete.

Since $\mathcal{O}_C$ is $\mathbb{Z}$-torsion-free, all $\frac{\xi^n}{n!} \in A_{\mathrm{cris}}$ lie in $\ker\theta_{\mathrm{cris}}$. To construct the divided power structure on $\ker\theta_{\mathrm{cris}}$, first note the following:

- If $w = \xi v \in \ker\theta_{\mathcal{O}_C}$ with some $v \in \mathrm{W}(\mathcal{O}_C^\flat)$, then $w^n = \xi^n v^n = n! v^n \frac{\xi^n}{n!} \in n!\ker\theta_{\mathrm{cris}}$.
- For all $i, n \in \mathbb{N}$, we have $(\frac{\xi^i}{i!})^n = \frac{(in!)}{(i!)^n}\frac{\xi^{in}}{(in)!} = n!C_{n,i}\frac{\xi^{in}}{(in)!} \in n!\ker\theta_{\mathrm{cris}}$, where $C_{n,i}$ is the integer $\frac{(in)!}{n!(i!)^n}$ from Definition 2.23.
- If $x \in \ker\theta_{\mathrm{cris}}$, then $(px)^n = n!\frac{p^n}{n!}x^n \in n!\ker\theta_{\mathrm{cris}}$.

Now write any $x \in \ker\theta_{\mathrm{cris}}$ as $x = a_0 + \sum_{i=1}^m a_i \frac{\xi^i}{i!} + p\sum_{i=m+1}^\infty \frac{a_i}{p}\frac{\xi^i}{i!}$, where all $a_i \in \mathrm{W}(\mathcal{O}_C^\flat)$ such that $\lim_i a_i = 0$, and $m \in \mathbb{N}$ such that $a_i \in p\mathrm{W}(\mathcal{O}_C^\flat)$ for all $i > m$. Then x is a finite sum of elements of the three types discussed above, so the existence of the divided power structure follows from 2.26 (iii). $\square$

Theorem 2.35 (Universality). *If (B, ρ, δ) is a p-adic formal divided power thickening of $\mathcal{O}_C$, then there is a unique divided power ring homomorphism*

$$\alpha : (A_{\mathrm{cris}}, \ker\theta_{\mathrm{cris}}, \gamma) \to (B, \ker\rho, \delta)$$

such that $\theta_{\mathrm{cris}} = \rho \circ \alpha$.

Proof. The map $\rho \circ - : \mathrm{Hom}(\mathrm{W}(\mathcal{O}_C^\flat), B) \to \mathrm{Hom}(\mathrm{W}(\mathcal{O}_C^\flat), \mathcal{O}_C)$ factors into the bijections (cf. 1.4, 2.33) between homomorphism sets of $\mathbb{Z}_p$-algebras (equivalently, rings)

$$\mathrm{Hom}(\mathrm{W}(\mathcal{O}_C^\flat), B) \underset{-^\flat\circ\eta_{\mathcal{O}_C^\flat}}{\overset{\theta_B\circ\mathrm{W}(-)}{\leftrightarrows}} \mathrm{Hom}(\mathcal{O}_C^\flat, B^\flat) \underset{\rho^\flat\circ-}{\overset{(\rho^\flat)^{-1}\circ-}{\leftrightarrows}} \mathrm{Hom}(\mathcal{O}_C^\flat, \mathcal{O}_C^\flat) \underset{\theta_{\mathcal{O}_C}\circ\mathrm{W}(-)}{\overset{-^\flat\circ\eta_{\mathcal{O}_C}}{\leftrightarrows}} \mathrm{Hom}(\mathrm{W}(\mathcal{O}_C^\flat), \mathcal{O}_C)$$

because for each homomorphism $f : \mathrm{W}(\mathcal{O}_C^\flat) \to B$, we have

$$\theta_{\mathcal{O}_C} \circ \mathrm{W}(\rho^\flat \circ f^\flat \circ \eta_{\mathcal{O}_C^\flat}) = \theta_{\mathcal{O}_C} \circ \mathrm{W}((\rho \circ f)^\flat \circ \eta_{\mathcal{O}_C^\flat}) \overset{1.4}{=} \rho \circ f.$$

Hence there is a unique homomorphism $\alpha' : \mathrm{W}(\mathcal{O}_C^\flat) \to B$ with $\rho \circ \alpha' \overset{!}{=} \theta_{\mathcal{O}_C} = \theta_{\mathrm{cris}|\mathrm{W}(\mathcal{O}_C^\flat)}$. We define $\alpha : A_{\mathrm{cris}} \to B$ via the images $\alpha(\frac{\xi^n}{n!}) := \delta_n(\alpha'(\xi))$, which satisfy the compatibility conditions of Remark 2.19 (i) because of 2.23 (i) and 2.23 (iv). By continuity and the construction of α', we can check $\theta_{\mathrm{cris}} = \rho \circ \alpha$ on all elements $\frac{\xi^n}{n!}$:

$$\rho\left(\alpha\left(\frac{\xi^n}{n!}\right)\right) = \rho(\delta_n(\alpha'(\xi))) = \frac{n!\rho(\delta_n(\alpha'(\xi)))}{n!} = \frac{\rho(\alpha'(\xi)^n)}{n!} = \frac{\theta_{\mathcal{O}_C}(\xi)^n}{n!} = \theta_{\mathrm{cris}}\left(\frac{\xi^n}{n!}\right).$$

It only remains to show that α commutes with the divided power structures. By construction we have $\alpha(\gamma_n(\xi)) = \delta_n(\alpha(\xi))$ for all $n \in \mathbb{N}$, so by the decomposition into sums from the proof of 2.34 and Proposition 2.30, it suffices to check compatibility on $p \ker \theta_{\mathrm{cris}}$. But if $x \in \ker \theta_{\mathrm{cris}}$, then $\alpha(\gamma_n(px)) = \alpha\left(\frac{p^n}{n!}x^n\right) = \frac{p^n}{n!}\alpha(x)^n = \delta_n(p\alpha(x)) = \delta_n(\alpha(px))$. □

3 The rings of Gauß norm completions and B_{cris}

If C is a perfectoid field of mixed characteristic, the ring $B^+_{\mathrm{cris}} := B^+_{\mathrm{cris}}(\mathcal{O}^\flat_C/s\mathcal{O}^\flat_C)$ can be considered a subring of the field of de Rham periods B_{dR} (cf. §1.3 and 3.30). This embedding is conveniently studied through completions of $\mathrm{W}(\mathcal{O}^\flat_C)[\frac{1}{p}]$ with respect to *Gauß norms*, which play an important role in the construction of the Fargues-Fontaine curve and can often be used as a replacement for B^+_{cris}. An overview of how these rings fit into the general theory can be found in the beginning of [7]; a detailed and more general treatment can be found in [6, §1].

Our approach is not fundamentally different from the classical one. Many of our arguments in §3.2 and §3.3 can be found in their original, more ad hoc form in [11, §4]. Another overview of this approach to $B^+_{\mathrm{cris}} \subset B_{\mathrm{dR}}$ via Gauß norms can be found in [3, §3]; note however that Caruso fixes a logarithmic valuation, so that his B^+_μ corresponds to our B^+_ρ for $\rho = |p|^\mu$. Furthermore, Caruso's B_μ and Fontaine's $B_{\mathfrak{a}}$ in [11] do not correspond to Fontaine's B_ρ in [6], but instead $B^+_\rho[\frac{1}{t}]$, which is the notation we will use.

3.1 Bounded Laurent series in mixed characteristic

Throughout this section, we fix a perfectoid field F of characteristic $p > 0$ with absolute value $|\cdot|$, which notably means that F is perfect. The case to keep in mind is $F = C^\flat$ for a perfectoid field C of mixed characteristic.

Definition 3.1. We write $B^{b,+} := \mathrm{W}(\mathcal{O}_F)[\frac{1}{p}]$.

Note that every $x \in B^{b,+}$ can be written as $x = \sum_{i\in\mathbb{Z}} p^i[x_i]$, where all $x_i \in \mathcal{O}_F$ are uniquely determined by x and almost all $x_i = 0$ for $i < 0$. Therefore, whenever some $x \in B^{b,+}$ is given, we will simply write x_i for these coefficients when this is unambiguous.

The $+$ in $B^{b,+}$ refers to the fact that the *Newton polygon* of each $x \in B^{b,+}$ (the convex hull of all $(i, -\log_p(|x_i|)) \in \mathbb{R}^2$ and the "points at infinity" $(0,\infty)$ and $(\infty, 0)$) lies above the x-axis. For $x \in B^{b,+}$, this simply corresponds to $|x_i| \le 1$, but for the other rings we construct its definition and meaning become less elementary. Like its classical counterpart, the Newton polygon contains divisibility information; we will only use it indirectly through the function ψ_x from Proposition 3.10, to which it is related via the Legendre transform. See [6, §1.5, §1.6.3] for details.

The b in $B^{b,+}$ is to be understood as "bounded" since $B^{b,+}$ is a subring of the ring $B^b := \{x = \sum_i p^i[x_i] \in \mathrm{W}(F)[\frac{1}{p}] \mid \sup_i |x_i| < \infty\}$, which can be thought of as a mixed characteristic analogue of the ring of bounded formal Laurent series, in p instead of a formal variable. The rings we construct in this section can be similarly considered rings of functions. We do not need this point of view, but will shortly sketch it here since it motivates various constructions and terminology.

A $w \in \mathrm{W}(\mathcal{O}_F)$ is *primitive of degree 1* when $w_0 \neq 0$ and $w_1 \in \mathcal{O}_F^\times$. For each such element, one has a projection $\mathrm{W}(\mathcal{O}_F) \twoheadrightarrow \mathrm{W}(\mathcal{O}_F)/w\mathrm{W}(\mathcal{O}_F)$. The ring $\mathrm{W}(\mathcal{O}_F)/w\mathrm{W}(\mathcal{O}_F)$ admits the absolute value $|x + w\mathrm{W}(\mathcal{O}_F)|_w := \inf_{z\in\mathrm{W}(\mathcal{O}_F)} \sup_{i\in\mathbb{N}_0} |(x+wz)_i|$ and therefore is a domain; it turns out to be the ring of integers of a perfectoid field C of characteristic zero with an isometric isomorphism $C^\flat \cong F$ and $|p|_w = |w_0|$. Under this isomorphism, the projection is identified with $\theta_{\mathcal{O}_C}$ and its localization $\theta : B^{b,+} \to C$ can be considered an evaluation homomorphism at a point of magnitude $|p|_w$. For more details, see [6, §2.2].

The analogy between $B^{b,+}$ and Laurent series makes the following definition natural.

Definition 3.2. Let $\rho \in (0;1)$ and $x = \sum_i p^i[x_i] \in B^{b,+}$. The *$\rho$-Gauß norm of x* is

$$|x|_\rho := \sup_{i\in\mathbb{Z}} \rho^i |x_i|.$$

Note that this supremum is actually a maximum since $x_i = 0$ for almost all $i < 0$ and $\lim_i \rho^i |x_i| \leq \lim_i \rho^i = 0$. This also shows that the maximum is attained for only finitely many $i \in \mathbb{Z}$ if $x \neq 0$; we will develop bounds on possible i in Proposition 3.10. We already called $|\cdot|_\rho$ a norm, but this is non-trivial and requires a technical argument.

Lemma 3.3. *For each $n \in \mathbb{N}_0$, the Witt structure polynomial*

$$\tilde{S}_n = S_n(X_0, X_1^p, \dots, X_n^{p^n}, Y_0, Y_1^p, \dots, Y_n^{p^n}) \in \mathbb{Z}[X_0, \dots, X_n, Y_0, \dots, Y_n]$$

adapted to the representation $\sum_{i=0}^\infty p^i[x_i]$ is homogeneous of degree p^n.

Proof. The case $n = 0$ is trivial. Let Φ_m be the m-th Witt polynomial and note that $\Phi_m(X_0, X_1^p, \dots, X_m^{p^m}) = \sum_{i=0}^m p^i X_i^{p^m}$ is always a homogeneous polynomial of degree p^m. Let all $\tilde{S}_i$ for $i < n+1$ be homogeneous polynomials of degree p^i. Then the polynomial $\Phi_n(\tilde{S}_0^p, \tilde{S}_1^p, \dots, \tilde{S}_n^p) = \sum_{i=0}^n p^i \tilde{S}_i^{p^{n+1-i}}$ is homogeneous of degree p^{n+1}. Now note that

$$\begin{aligned} p^{n+1}\tilde{S}_{n+1} + \Phi_n(\tilde{S}_0^p, \dots, \tilde{S}_n^p) &= \Phi_{n+1}(\tilde{S}_0, \dots, \tilde{S}_{n+1}) \\ &= \Phi_{n+1}(X_0, X_1^p, \dots, X_{n+1}^{p^{n+1}}) + \Phi_{n+1}(Y_0, Y_1^p, \dots, Y_{n+1}^{p^{n+1}}) \end{aligned}$$

by the defining property of S_{n+1}, which shows that $\tilde{S}_{n+1}$ is a sum of homogeneous polynomials of degree p^{n+1} as claimed. $\square$

Proposition 3.4. *$|\cdot|_\rho$ is a non-archimedean absolute value on $B^{b,+}$.*

Proof. First note that if $|x|_\rho = \sup_i \rho^i |x_i| = 0$, then all $|x_i| = 0$, i.e. $x = 0$. It suffices to show the triangle inequality and multiplicativity on $\mathrm{W}(\mathcal{O}_F) \subset B^{b,+}$ because for all $x, y \in B^{b,+}$, there is an $m \in \mathbb{N}$ such that $p^m x, p^m y \in \mathrm{W}(\mathcal{O}_F)$; since it is clear from the definition of $|\cdot|_\rho$ that $|p^n x|_\rho = \rho^n |x|_\rho$ for all $n \in \mathbb{Z}$, we then have

$$|x+y|_\rho = \rho^{-m}|p^m x + p^m y|_\rho \overset{!}{\leq} \rho^{-m}\max\{|p^m x|_\rho, |p^m y|_\rho\} = \max\{|x|_\rho, |y|_\rho\},$$

$$|xy|_\rho = \rho^{-2m}|p^m x p^m y|_\rho \overset{!}{=} \rho^{-2m}|p^m x|_\rho |p^m y|_\rho = |x|_\rho |y|_\rho.$$

Hence let $x, y \in \mathrm{W}(\mathcal{O}_F)$. The strict triangle inequality follows from

$$\begin{aligned} |x+y|_\rho &= \sup_{i\geq 0} \rho^i |S_i(x_0, x_1^p, \dots, x_i^{p^i}, y_0, \dots, y_i^{p^i})^{p^{-i}}| \\ &\overset{3.3}{\leq} \sup_{i\geq 0} \max_{0\leq j\leq i}\{\rho^i |x_j|,\ \rho^i |y_j|\} \\ &\leq \sup_{i\geq 0} \max_{0\leq j\leq i}\{\rho^j |x_j|,\ \rho^j |y_j|\} \\ &= \max\{\sup_{i\geq 0} \rho^i |x_i|,\ \sup_{i\geq 0} \rho^i |y_i|\} \\ &= \max\{|x|_\rho,\ |y|_\rho\}. \end{aligned}$$

For multiplicativity, assume without loss of generality that $x, y, xy \neq 0$ since $B^{b,+}$ is a domain and let $n \in \mathbb{N}$ be large enough such that $|p|_\rho^{n+1} < |x|_\rho, |y|_\rho, |xy|_\rho$. Then we have $xy \equiv \sum_{i=0}^n p^i \sum_{j=0}^i [x_j y_{i-j}] \mod p^{n+1}\mathrm{W}(\mathcal{O}_F)$, so the strict triangle inequality implies

$$|xy|_\rho \stackrel{!}{=} \left| \sum_{i=0}^n p^i \sum_{j=0}^i [x_j y_{i-j}] \right|_\rho \stackrel{!}{\leq} \max_{0\leq i\leq n} \rho^i \max_{0\leq j\leq i} |x_j||y_{i-j}| \leq (\max_{0\leq i\leq n} \rho^i |x_i|)(\max_{0\leq i\leq n} \rho^i |y_i|) \leq |x|_\rho |y|_\rho.$$

Finally, let $i_x, i_y \in \mathbb{Z}$ be minimal such that $|x|_\rho = \rho^{i_x}|x_{i_x}|$ and $|y|_\rho = \rho^{i_y}|y_{i_y}|$, and let $x' = \sum_{i=i_x}^\infty p^i[x_i]$, $y' = \sum_{i=i_y}^\infty p^i[y_i]$. Then $x'y' = \sum_{i=i_x+i_y}^\infty p^i[w_i]$ for some $w_i \in \mathrm{W}(\mathcal{O}_F)$, where notably $w_{i_x+i_y} = x_{i_x} y_{i_y}$. Hence $|x|_\rho |y|_\rho = \rho^{i_x+i_y}|x_{i_x} y_{i_y}| \leq |x'y'|_\rho$. By construction, $|x - x'|_\rho < |x|_\rho$ and $|y - y'|_\rho < |y|_\rho$, which shows

$$\begin{aligned} |xy - x'y'|_\rho &= |(x - x')y - x'(y' - y)|_\rho \\ &\leq \max\{|(x - x')y|_\rho,\ |x'(y' - y)|_\rho\} \\ &\leq \max\{|x - x'|_\rho |y|_\rho,\ |x'|_\rho |y' - y|_\rho\} \\ &< |x|_\rho |y|_\rho = |x'y'|_\rho. \end{aligned}$$

Therefore $|x|_\rho |y|_\rho \leq |x'y'|_\rho = |xy|_\rho$ as well by the strict triangle inequality. □

Remark 3.5. Since $|x| = 1$ for all $x \in \mathbb{F}_p^\times \subset \mathcal{O}_F^\times$, the Gauß norms restrict to the p-adic absolute value on $\mathbb{Q}_p = \mathrm{W}(\mathbb{F}_p)[\frac{1}{p}] \subset B^{b,+}$, normalized to $|p| = \rho$. The topology on $\mathrm{W}(\mathcal{O}_F)$ induced by $|\cdot|_\rho$ is however *not* the p-adic topology but the coarser *weak topology*, which arises as the product topology from the identification

$$(\sum_{i=0}^\infty p^i[w_i] \mapsto (w_i)_i) : \mathrm{W}(\mathcal{O}_F) \leftrightarrows \prod_{i=0}^\infty \mathcal{O}_F,$$

where each $\mathcal{O}_F$ is given its usual valuation topology. Contrast this with how the p-adic topology arises by equipping $\mathcal{O}_F$ with the discrete topology instead. Clearly, $\mathrm{W}(\mathcal{O}_F)$ is Hausdorff and complete with respect to the weak topology.

Proposition 3.6. *If $\rho \in (0;1)$, then $|\cdot|_\rho$ induces the weak topology on $\mathrm{W}(\mathcal{O}_F)$.*

Proof. Note that for all $x \in \mathrm{W}(\mathcal{O}_F)$ and $n \in \mathbb{N}_0$, the function $f_{x,n} : \mathrm{W}(\mathcal{O}_F) \to \mathcal{O}_F$ with

$$f_{x,n}(y) := S_n(x_0, \ldots, x_n^{p^n}, y_0, \ldots, y_n^{p^n})^{1/p^n}$$

is continuous with respect to the weak topology on $\mathrm{W}(\mathcal{O}_F)$ because S_n is a polynomial. In particular, translation is a homeomorphism with respect to both topologies on $\mathrm{W}(\mathcal{O}_F)$ and it suffices to consider neighborhoods of zero.

Let $U = \{y \in \mathrm{W}(\mathcal{O}_F) \mid |y|_\rho < \varepsilon\}$ for some $\varepsilon > 0$. Then there is an $n \in \mathbb{N}_0$ such that $\varepsilon\rho^{-n} > 1$, so that

$$\begin{aligned} \{y \in \mathrm{W}(\mathcal{O}_F) \mid |y_i| < \varepsilon\rho^{-i} \text{ for all } 0 \leq i \leq n\} &\stackrel{!}{=} \{y \in \mathrm{W}(\mathcal{O}_F) \mid |y_i| < \varepsilon\rho^{-i} \text{ for all } i \geq 0\} \\ &= \{y \in \mathrm{W}(\mathcal{O}_F) \mid |y|_\rho < \varepsilon\} \end{aligned}$$

is a weakly open neighborhood of zero contained in U.

On the other hand, if $V = \{y \in \mathrm{W}(\mathcal{O}_F) \mid |y_i| < \varepsilon_i \text{ for all } 0 \leq i \leq n\}$ for some $n \in \mathbb{N}_0$ and $\varepsilon_0, \ldots, \varepsilon_n > 0$, then for $\varepsilon := \min\{\varepsilon_0, \varepsilon_1\rho^{-1}, \ldots, \varepsilon_n\rho^{-n}\}$ the set

$$\begin{aligned} \{y \in \mathrm{W}(\mathcal{O}_F) \mid |y|_\rho < \varepsilon\} &= \{y \in \mathrm{W}(\mathcal{O}_F) \mid |y_i| < \varepsilon\rho^{-i} \text{ for all } i \geq 0\} \\ &\subset \{y \in \mathrm{W}(\mathcal{O}_F) \mid |y_i| < \varepsilon\rho^{-i} \text{ for all } 0 \leq i \leq n\} \end{aligned}$$

is a $|\cdot|_\rho$-neighborhood of zero contained in V. □

In particular, we see that all Gauß norms on $\mathrm{W}(\mathcal{O}_C^\flat)$ are equivalent and that $\mathrm{W}(\mathcal{O}_C^\flat)$ is complete with respect to them. This situation is very different from the one for $B^{b,+}$, where completion and varying the Gauß norm are important tools.

Definition 3.7. Let $\rho \in (0;1)$.

(i) We write B_ρ^+ for the completion of $B^{b,+}$ with respect to $|\cdot|_\rho$. The absolute value $|\cdot|_\rho$ extends to an absolute value on B_ρ^+ which we likewise denote by $|\cdot|_\rho$.

(ii) The *ring of power-bounded elements of* B_ρ^+ is $\mathring{B}_\rho^+ := \{x \in B_\rho^+ \mid |x|_\rho \leq 1\}$.

The ring B_ρ^+ too has an interpretation as a ring of functions, specifically those that converge on the closed annulus with radius between ρ and 1. We can make this more precise using Proposition 3.25. For the ring $\mathring{B}_\rho^+$, the following compatibility statement regarding the p-adic and $|\cdot|_\rho$ topologies holds.

Proposition 3.8. *Let* $\rho \in (0;1)$.

(i) *We have* $p^n \mathring{B}_\rho^+ = \{x \in \mathring{B}_\rho^+ \mid |x|_\rho \leq \rho^n\}$ *for all* $n \in \mathbb{N}_0$. *In particular, the* p*-adic and* $|\cdot|_\rho$*-norm topologies coincide on* $\mathring{B}_\rho^+$*, which is thus* p*-adically separated and complete.*

(ii) *The ring* $\mathring{B}_\rho^+$ *is the* p*-adic completion of* $S_\rho := B^{b,+} \cap \mathring{B}_\rho^+$.

(iii) *The natural homomorphism* $\mathring{B}_\rho^+[\frac{1}{p}] \to B_\rho^+$ *is an isomorphism.*

Proof. (i): If $x \in \mathring{B}_\rho^+$ and $n \in \mathbb{N}_0$, then $|p^n x|_\rho = |p^n|_\rho |x|_\rho \leq \rho^n$. On the other hand, if $x \in \mathring{B}_\rho^+$ satisfies $|x|_\rho \leq \rho^n$, then $|p^{-n}x|_\rho \leq 1$, so $p^{-n}x \in \mathring{B}_\rho^+$ and $x = p^n(p^{-n}x) \in p^n \mathring{B}_\rho^+$.

(ii): Let $x \in \mathring{B}_\rho^+$ and $x^{(n)} \in B^{b,+}$ such that $x = \lim_n x^{(n)}$. Since $|\cdot|_\rho$ is non-archimedean, either $x = 0$ and $|x^{(n)}|_\rho < 1$ for almost all $n \in \mathbb{N}$, or $|x^{(n)}|_\rho = |x| \leq 1$ for almost all $n \in \mathbb{N}$; either way $x^{(n)} \in S_\rho$ for almost all $n \in \mathbb{N}$, so $S_\rho \subset \mathring{B}_\rho^+$ is dense. The result now follows immediately from (i) since $\mathring{B}_\rho^+ \subset B_\rho^+$ is closed.

(iii): The homomorphism is injective because we localized the injective $\mathring{B}_\rho^+ \hookrightarrow B_\rho^+$. It is surjective because any $x \in B_\rho^+$ can be written as $p^{-n}(p^n x)$ for some $n \in \mathbb{N}$ such that $p^n x \in \mathring{B}_\rho^+$. □

Lemma 3.9. *If* $0 < \sigma \leq \rho < 1$, *then* $|x|_\rho \leq |x|_\sigma^{\log_\sigma(\rho)}$ *for all* $x \in B^{b,+}$.

Proof. Write $x = \sum_{i \in \mathbb{Z}} p^i [x_i]$ and let $n \in \mathbb{Z}$ such that $|x|_\rho = \rho^n |x_n|$. Then

$$|x|_\rho = \rho^n |x_n| = \sigma^{n \log_\sigma(\rho)} |x_n| \leq \sigma^{n \log_\sigma(\rho)} |x_n|^{\log_\sigma(\rho)} = (\sigma^n |x_n|)^{\log_\sigma(\rho)} \leq |x|_\sigma^{\log_\sigma(\rho)},$$

where the first inequality uses $|x_n| \leq 1$ and $0 < \log_\sigma(\rho) \leq 1$. □

Proposition 3.10. *For* $b \in B^{b,+} \setminus 0$, *let* $\psi_b : (0;\infty) \to \mathbb{R}$ *be* $\psi_b(t) = \log_p(|b|_{p^{-t}})$.

(i) ψ_b *is convex (hence continuous) and piecewise affine linear with integral slopes.*

(ii) *Let* $t \in (0;\infty)$. *Then*

$$d^-\psi_b(t) := \lim_{\substack{h \to 0 \\ h < 0}} \frac{\psi_b(t+h) - \psi_b(t)}{h} = \min\{i \in \mathbb{Z} \mid |b|_{p^{-t}} = p^{-ti}|x_i|\},$$

$$d^+\psi_b(t) := \lim_{\substack{h \to 0 \\ h > 0}} \frac{\psi_b(t+h) - \psi_b(t)}{h} = \max\{i \in \mathbb{Z} \mid |b|_{p^{-t}} = p^{-ti}|x_i|\}.$$

Proof. Let $0 < r < s$ and consider the restriction of ψ_b to $[r;s]$. Write $\sigma = p^{-s} < p^{-r} = \rho$ and set $C = \min\{|b|_\rho^{\log_\rho(\sigma)}, |b|_\rho\}$ ($\log_\rho$, not $\log_p$!). Then for all $t \in [r;s]$,

$$|b|_{p^{-t}} \overset{3.9}{\geq} |b|_\rho^{\log_\rho(p^{-t})} \geq \begin{cases} |b|_\rho^{\log_\rho(\rho)} = |b|_\rho & \text{if } |b|_\rho \geq 1, \\ |b|_\rho^{\log_\rho(\sigma)} & \text{if } |b|_\rho < 1, \end{cases}$$

i.e. $|b|_{p^{-t}} \geq C > 0$. Therefore, if $n \in \mathbb{N}$ is large enough so that $p^{-n} < C$ and $b_i = 0$ for all $i < -n$, we have, for all $t \in [r;s]$,

$$\psi_b(t) = \log_p\left(\left|\sum_{i=-n}^{n} p^i[b_i]\right|_{p^{-t}}\right) = \max_{-n\leq i\leq n}(\log_p(|b_i|) - it). \qquad (*)$$

(i): By elementary analysis, being convex and piecewise affine linear with integral slopes is stable under finite maxima and can be checked on compact subintervals of $(0;\infty)$, so this follows directly from $(*)$ since each $\log_p(|b_i|) - it$ is of this form.

(ii): Let $0 < r < t < s$ and choose an $n \in \mathbb{N}$ as before so that $(*)$ holds. Since all $(t' \mapsto \log_p(|b_i|) - it')$ are continuous, there is a $\delta > 0$ such that for all $h \in (-\delta;\delta)$,

$$\begin{aligned}
\psi_b(t+h) &= \max\{\log_p(|b_i|) - i(t+h) \mid -n \leq i \leq n\} \\
&\overset{!}{=} \max\{\log_p(|b_i|) - i(t+h) \mid -n \leq i \leq n,\ \psi_b(t) = \log_p(|b_i|) - it\} \\
&= \max\{\log_p(|b_i|) - i(t+h) \mid -n \leq i \leq n,\ |b|_{p^{-t}} = p^{-it}|b_i|\} \\
&\leq \max\{\log_p(|b_i|) - i(t+h) \mid i \in \mathbb{Z},\ |b|_{p^{-t}} = p^{-it}|b_i|\} \\
&\leq \max\{\log_p(|b_i|) - i(t+h) \mid i \in \mathbb{Z}\} \\
&= \psi_b(t+h).
\end{aligned}$$

Since $\{\log_p(|b_i|) - it \mid i \in \mathbb{Z}, |b|_{p^{-t}} = p^{-it}|b_i|\}$ is a singleton set, it follows that

$$\begin{aligned}
\frac{\psi_b(t+h) - \psi_b(t)}{h} &= \frac{\max\{\log_p(|b_i|) - i(t+h) - (\log_p(|b_i|) - it) \mid i \in \mathbb{Z},\ |b|_{p^{-t}} = p^{-it}|b_i|\}}{h} \\
&= \frac{\max\{ih \mid i \in \mathbb{Z},\ |b|_{p^{-t}} = p^{-it}|b_i|\}}{h} \\
&= \begin{cases} \min\{i \in \mathbb{Z} \mid |b|_{p^{-t}} = p^{-ti}|x_i|\} & \text{if } h < 0, \\ \max\{i \in \mathbb{Z} \mid |b|_{p^{-t}} = p^{-ti}|x_i|\} & \text{if } h > 0 \end{cases}
\end{aligned}$$

for all $h \in (-\delta;\delta) \setminus 0$. □

Corollary 3.11. *For all $b \in B^{b,+} \setminus 0$, the function $(\rho \mapsto |b|_\rho) : (0;1) \to \mathbb{R}$ is continuous.*

Proof. The function factors into the continuous functions $\exp_p \circ \psi_b \circ (-\log_p)$. □

If $0 < \sigma \leq \rho < 1$, the composition $(B^{b,+}, |\cdot|_\sigma) \xrightarrow{\text{id}} (B^{b,+}, |\cdot|_\rho) \hookrightarrow B_\rho^+$ is continuous by Lemma 3.9 and extends to a continuous ring homomorphism $\iota_{\sigma,\rho} : B_\sigma^+ \to B_\rho^+$. Since $\iota_{\sigma,\rho}$ simply maps $\lim_n x_n \in B_\sigma^+$ with $x_n \in B^{b,+}$ to $\lim_n x_n \in B_\rho^+$, the maps $\iota_{\sigma,\rho}$ form a directed system. They turn out to be injective, but this is surprisingly non-trivial.

Lemma 3.12. *If $0 < \sigma \leq \rho < 1$, then $|\iota_{\sigma,\rho}(x)|_\rho \leq |x|_\sigma^{\log_\sigma(\rho)}$ for all $x \in B_\sigma^+$.*

Proof. Write $x = \lim_n x_n \in B_\sigma^+$, where all $x_n \in B^{b,+}$. By the continuity of $\iota_{\sigma,\rho}$,

$$|\iota_{\sigma,\rho}(x)|_\rho = \lim_{n\to\infty} |x_n|_\rho \overset{3.9}{\leq} \lim_{n\to\infty} |x_n|_\sigma^{\log_\sigma(\rho)} = |x|_\sigma^{\log_\sigma(\rho)}. \qquad □$$

Lemma 3.13. *Let $0 < \sigma < \tau < \rho < 1$, $\lambda := \log_{\sigma/\rho}(\tau/\rho) \in (0;1)$, and $x \in B^+_\sigma$. Then*

$$|\iota_{\sigma,\tau}(x)|_\tau \leq |x|^\lambda_\sigma |\iota_{\sigma,\rho}(x)|^{(1-\lambda)}_\rho.$$

Proof. For $b \in B^{b,+}$, this is a multiplicative rewording of the convexity statement in 3.10, although it can also be proven directly by a straightforward manipulation. We obtain the lemma by taking limits as in 3.12. □

Proposition 3.14. *If $0 < \sigma < \rho < 1$, then $\iota_{\sigma,\rho} : B^+_\sigma \to B^+_\rho$ is injective.*

Proof. Assume to the contrary that there is an $x \in B^+_\sigma \setminus 0$ such that $\iota_{\sigma,\rho}(x) = 0$. Then $|\iota_{\sigma,\tau}(x)|_\tau \leq 0$ for all $\tau \in (\sigma;\rho]$ by Lemma 3.13. For each $\varepsilon \in (0;1)$, there is a $y \in B^{b,+}$ such that $|x-y|_\sigma < \min\{\varepsilon^{\log_\rho(\sigma)}, |x|_\sigma\}$ (and notably, $|x|_\sigma = |y|_\sigma$) since $B^{b,+} \subset B^+_\sigma$ is dense and $x \neq 0$. Furthermore there is a $\tau \in (\sigma;\rho)$ such that $||y|_\sigma - |y|_\tau| < \varepsilon$ by continuity (3.11), and hence $|y|_\sigma \leq ||y|_\sigma - |y|_\tau| + |y|_\tau < \varepsilon + |y|_\tau$. Therefore, using $|x-y|_\sigma < 1$,

$$|x|_\sigma = |y|_\sigma < \varepsilon + |y|_\tau = \varepsilon + |\iota_{\sigma,\tau}(x-y)|_\tau \overset{3.12}{\leq} \varepsilon + |x-y|^{\log_\sigma(\tau)}_\sigma \overset{!}{\leq} \varepsilon + |x-y|^{\log_\sigma(\rho)}_\sigma < 2\varepsilon.$$

This holds for all $\varepsilon \in (0;1)$, so absurdly $|x|_\sigma = 0$. □

Note how Proposition 3.14 requires convexity, continuity and Lemma 3.12, which is completely specific to Gauß norms. In general, maps induced from the identity in this manner need not be injective. The following result shows that the choice of ρ does matter.

Proposition 3.15. *The map $\iota_{\sigma,\rho} : B^+_\sigma \hookrightarrow B^+_\rho$ is not surjective when $0 < \sigma < \rho < 1$.*

Proof. Call Cauchy/null sequences with respect to $|\cdot|_\rho$ ρ-Cauchy/ρ-null for short. Since F is perfectoid, there is an $x \in \mathcal{O}_F$ with $\sigma < |x| < \rho$; then the sequence $(p^{-n}[x]^n)_n$ is ρ-null but not σ-null because

$$\lim_{n\to\infty} |p^{-n}[x]^n|_\rho = \lim_{n\to\infty} (\rho^{-1}|x|)^n = 0, \qquad \lim_{n\to\infty} |p^{-n}[x]^n|_\sigma = \lim_{n\to\infty} (\sigma^{-1}|x|)^n = \infty.$$

Therefore the sequence $x_n := \sum_{i=0}^n p^{-i}[x^i] \in B^{b,+}$ is ρ-Cauchy but not σ-Cauchy. Now recall the construction of the completion as classes of Cauchy sequences. If the class of $(x_n)_n$ in B^+_ρ was in $\iota_{\sigma,\rho}(B^+_\sigma)$, there would be a ρ-null sequence $(e_n)_n$ in $B^{b,+}$ such that $(x_n + e_n)_n$ is σ-Cauchy; but by Proposition 3.14, ρ-null and σ-Cauchy together imply σ-null, so it absurdly follows that $(x_n)_n$ is σ-Cauchy. □

Definition 3.16. We write $B^+ := \varprojlim_{0<\rho<1} B^+_\rho$, where the transition maps of the inverse limit are the maps $\iota_{\sigma,\rho} : B^+_\sigma \to B^+_\rho$ for each pair $0 < \sigma \leq \rho < 1$.

Due to Proposition 3.14, we may identify the various B^+_ρ as subrings of each other and view B^+ as the intersection of all B^+_ρ, which gives it an interpretation as functions that converge on the unit disk. The ring B^+ can frequently be substituted for B^+_{cris} and tends to have nicer properties.

Remark 3.17. The natural choice of topology on B^+ is the inverse limit topology, i.e. the coarsest topology such that all inclusions $\iota_\rho : B^+ \hookrightarrow B^+_\rho$ are continuous. Equivalently, one uses $B^+ \cong \varprojlim_{n\geq 2} B^+_{1/n}$ and equips B^+ with the coarsest topology such that all $B^+ \overset{\iota_{1/n}}{\hookrightarrow} B^+_{1/n}$ for $n \geq 2$ are continuous, since $B^+ \overset{\iota_\rho}{\hookrightarrow} B^+_\rho$ factors into $B^+ \overset{\iota_{1/n}}{\hookrightarrow} B^+_{1/n} \overset{\iota_{1/n,\rho}}{\hookrightarrow} B^+_\rho$ for $n = \lceil 1/\rho \rceil$. This shows that the topology of B^+ is metrizable via

$$d(x,y) := \sup_{n\geq 2} 2^{-n} \frac{|x-y|_{1/n}}{1+|x-y|_{1/n}}$$

(the restriction of the product metric for $B^+ \cong \varprojlim_{n\geq 2} B^+_{1/n}$), so a sequence converges in B^+ if and only if its image in every $B^+_{1/n}$ (or equivalently, in every B^+_ρ) converges. It follows immediately that B^+ is the completion of $B^{b,+}$ with respect to this metric.

Proposition 3.18. $\{x \in B^+ \mid |x|_\rho \leq 1$ *for all* $0 < \rho < 1\} = \bigcap_{0<\rho<1} \mathring{B}^+_\rho = \mathrm{W}(\mathcal{O}_F)$.

Proof. Clearly $\mathrm{W}(\mathcal{O}_F) \subset \bigcap_\rho \mathring{B}^+_\rho$. On the other hand, let $x \in B^+ \setminus 0$ with $|x|_\rho \leq 1$ for all $\rho \in (0;1)$ and let $x^{(1)}, x^{(2)}, \ldots \in B^{b,+} \setminus 0$ be a sequence that converges to x with respect to all Gauß norms. Set $y^{(n)} := \sum_{i=0}^\infty p^i[x_i^{(n)}] \in \mathrm{W}(\mathcal{O}_F)$. We claim that $\lim_n |x^{(n)} - y^{(n)}|_\rho = 0$ for all ρ, which implies that x is a $|\cdot|_\rho$-limit of elements of $\mathrm{W}(\mathcal{O}_F)$ and hence itself in $\mathrm{W}(\mathcal{O}_F)$ by Remark 3.5 and Proposition 3.6.

Let $\rho, \varepsilon \in (0;1)$. Since $\lim_n x^{(n)} = x$ in $B^+_{\varepsilon\rho}$, there is an $n \in \mathbb{N}$ with $|x^{(j)}|_{\varepsilon\rho} = |x|_{\varepsilon\rho} \leq 1$ for all $j > n$. Hence $|x_i^{(j)}|(\varepsilon\rho)^i \leq 1$ for all $j > n$ and $i \in \mathbb{Z}$, which in particular means

$$|x^{(j)} - y^{(j)}|_\rho = \sup_{i<0} |x_i^{(j)}|\rho^i \leq \sup_{i<0} \varepsilon^{-i} = \varepsilon. \qquad \square$$

Proposition 3.19. *Let* $\rho \in (0;1)$ *and* $\varphi : B^{b,+} \xrightarrow{\sim} B^{b,+}$ *the Frobenius automorphism.*

(i) *There exists a unique continuous isomorphism* $\varphi_\rho : B^+_\rho \xrightarrow{\sim} B^+_{\rho^p}$ *that restricts to* φ.

(ii) φ_ρ *restricts to a continuous isomorphism* $\mathring{\varphi}_\rho : \mathring{B}^+_\rho \xrightarrow{\sim} \mathring{B}^+_{\rho^p}$.

(iii) *If* $0 < \sigma < \rho < 1$, *then* $\varphi_\sigma = \varphi_\rho \circ \iota_{\sigma,\rho}$.

Proof. (i): Uniqueness follows from the density of $B^{b,+} \subset B^+_\rho$. If $x \in B^{b,+}$, then

$$|\varphi(x)|_{\rho^p} = \sup_{i\in\mathbb{Z}} \rho^{pi}|x_i|^p = \sup_{i\in\mathbb{Z}} (\rho^i|x|_\rho)^p = |x|_\rho^p, \qquad (*)$$

so φ maps Cauchy sequences relative to $|\cdot|_\rho^p$ (i.e. $|\cdot|_\rho$) to Cauchy sequences relative to $|\cdot|_{\rho^p}$. Hence φ extends to a continuous homomorphism $\varphi_\rho : B^+_\rho \to B^+_{\rho^p}$. This is in fact an isomorphism: φ^{-1} similarly extends to a continuous homomorphism $B^+_{\rho^p} \to B^+_\rho$, so since either composition of the two is continuous and the identity on the dense subset $B^{b,+}$, it is itself the identity.

(ii): Note that $(*)$ still holds after taking limits and that every $x \in \mathring{B}^+_\rho$ can be written as a limit of elements of $\mathring{B}^+_\rho \cap B^{b,+}$ by Proposition 3.8.

(iii): This follows immediately from continuity. $\square$

Corollary 3.20. *For any* $\rho \in (0;1)$, *the ring* $B^+ = \bigcap_{n\geq 0} B^+_{\rho^{p^n}} = \bigcap_{n\geq 0} \varphi_\rho^n(B^+_\rho)$ *is the largest subring of* B^+_ρ *where* φ_ρ *is bijective. The restriction of* φ_ρ *to* B^+ *is continuous and denoted by* φ_B.

Proposition 3.21. *Let* $\mathfrak{m}_F = \{x \in \mathcal{O}_F \mid |x| < 1\}$. *For all* $x \in 1 + \mathfrak{m}_F$, *the series*

$$\log([x]) := \sum_{i=1}^\infty (-1)^{i+1} \frac{([x]-1)^i}{i}$$

converges in B^+. *The resulting map* $\log([\cdot]) : 1 + \mathfrak{m}_F \to B^+$ *is a continuous group homomorphism with respect to the additive structure on* B^+.

Proof. Let $\rho \in (0;1)$ and $w \in \mathrm{W}(\mathcal{O}_F)$ such that $[x]-1 = [x-1]+pw$. Then

$$|[x]-1|_\rho \le \max\{|x-1|,\ |pw|_\rho\} \le \max\{|x-1|,\ \rho\} < 1.$$

The series converges in B^+_ρ because if $\lambda := \log_\rho(|[x]-1|_\rho) \in (0;\infty)$, then

$$\lim_{n\to\infty}\left|\frac{([x]-1)^n}{n}\right|_\rho = \lim_{n\to\infty}\rho^{\lambda n - v_p(n)} \le \lim_{n\to\infty}\rho^{\lambda n-\log_p(n)} = 0.$$

Since $[\cdot] : \mathcal{O}_F \to \mathrm{W}(\mathcal{O}_F)$ is an isometry for $|\cdot|_\rho$ and satisfies $[xy]=[x][y]$ for all $x, y \in \mathcal{O}_F$, one obtains a continuous group homomorphism $1+\mathfrak{m}_F \to B^+_\rho$ for every $\rho \in (0;1)$ through the same formal arguments as for the ordinary logarithm. These homomorphisms are evidently compatible with the inclusions $\iota_{\sigma,\rho}$ for $\sigma \le \rho$, so we obtain a continuous group homomorphism $\log([\cdot]) : 1+\mathfrak{m}_F \to B^+$. □

Proposition 3.22. *If $x \in 1+\mathfrak{m}_F$, then $\varphi_B(\log([x])) = p\log([x])$.*

Proof. Since φ_B is continuous, we have

$$\varphi_B(\log([x])) = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{([x^p]-1)^i}{i} = \log([x^p]) = p\log([x]). \qquad \square$$

Remark 3.23. One can in fact prove that every $b \in B^+$ with $\varphi_B(b) = pb$ is of the form $b = \log([x])$ for some $x \in 1+\mathfrak{m}_F$, see [6, 4.4.7].

3.2 The chain of inclusions $B^+_{|p|^p} \hookrightarrow B^+_{\mathrm{cris}} \hookrightarrow B^+_{|p|} \hookrightarrow B^+_{\mathrm{dR}}$

Throughout this section, we fix a perfectoid field C of mixed characteristic. Then $F := C^\flat$ is perfectoid of characteristic p by Theorem 1.12 (i), so we have the rings $B^{b,+}, B^+_\rho, \mathring{B}^+_\rho, B^+$ from §3.1. Furthermore, we fix an arbitrary $s \in \mathcal{O}_C^\flat$ with $|s|_\flat = |p|$ and obtain the rings $A_{\mathrm{cris}} := A_{\mathrm{cris}}(\mathcal{O}_C^\flat/s\mathcal{O}_C^\flat)$ and $B^+_{\mathrm{cris}} := B^+_{\mathrm{cris}}(\mathcal{O}_C^\flat/s\mathcal{O}_C^\flat)$ from 2.6. Note that $\bigcap_{n\ge 1} s^n\mathcal{O}_C^\flat = 0$ by simple absolute value considerations, so all conditional results of §2.1 apply.

Our goal is to realize B^+_{cris} as an intermediate ring between $B^+_{|p|^p}$ and $B^+_{|p|}$ and to embed it into the *field of de Rham periods* B_{dR}, which will be an important tool for our representation-theoretic arguments in §4.

Definition 3.24. We write $\theta : B^{b,+} \twoheadrightarrow C$ for the localization of $\theta_{\mathcal{O}_C} : \mathrm{W}(\mathcal{O}_C^\flat) \twoheadrightarrow \mathcal{O}_C$ at p.

Since $B^{b,+}$ is a domain and $C = \mathcal{O}_C[\frac{1}{p}]$, the map θ is surjective. As explained in the beginning of §3.1, the map θ can be thought of as an evaluation homomorphism. It is central for all constructions in this chapter.

Proposition 3.25. *For all $\rho \in (0;|p|]$, there is a continuous and surjective ring homomorphism $\theta_\rho : B^+_\rho \twoheadrightarrow C$ which restricts to θ. It satisfies $|\theta_\rho(x)| \le |x|_\rho$ for all $x \in B^+_\rho$ if and only if $\rho = |p|$.*

Proof. We first show $|\theta(x)| \le |x|_\rho$ for $x \in B^{b,+}$ and $\rho = |p|$. The case $x \in \ker\theta$ is trivial; otherwise choose $n \in \mathbb{N}$ such that $\rho^n < |\theta(x)|$ and $x_i = 0$ for all $i < -n$. Since $|x_i|_\flat \le 1$ for all $i \in \mathbb{Z}$ and $\rho = |p|$, we have

$$|\theta(x)| = \left|\sum_{i\in\mathbb{Z}} p^i x_i^\sharp\right| = \left|\sum_{-n\le i\le n} p^i x_i^\sharp\right| \le \sup_{-n\le i\le n}|p|^i|x_i^\sharp| \overset{!}{\le} \sup_{i\in\mathbb{Z}}\rho^i|x_i|_\flat = |x|_\rho.$$

Norm-decreasing maps are uniformly continuous; since θ is already surjective, so is the unique continuous extension θ_ρ. The norm inequality for θ_ρ follows from the one for θ by taking limits. For $\rho < |p|$, take $\theta_\rho := \theta_{|p|} \circ \iota_{\rho,|p|}$, which is clearly surjective and continuous. It isn't norm-decreasing because $|\theta(p)| = |p| > \rho = |p|_\rho$. $\square$

This too can be considered an evaluation homomorphism, which makes more precise how B^+_ρ is a ring of functions that converge on the annulus with radius between ρ and 1. Given an $f \in B^+_\sigma$ for some $\sigma \in (0;1)$, one "evaluates" f at a primitive element of degree 1 corresponding to a perfectoid field C of mixed characteristic with $\rho = |p| \geq \sigma$ via $\theta_\sigma(f) = \theta_\rho(\iota_{\sigma,\rho}(f)) \in C$, which is indeed $\lim_n \theta(f_n) \in C$ for $f_n \in B^{b,+}$ with $f = \lim_n f_n$.

The map θ cannot extend to B^+_ρ when $\rho > |p|$ because $\lim_n |p^{-n}(s^\sharp)^n| = 1$ even though $\lim_n |p^{-n}[s]^n|_\rho = 0$. The restriction to $(0;|p|]$ is therefore fundamental and the reason why $B^+_{|p|}$ is called $B^+_{\max}$ by some authors.

Proposition 3.26. *If $\rho \in (0;|p|]$, then $\ker\theta_\rho$ is the topological closure of $\ker\theta$ in B^+_ρ. In particular, whenever $\ker\theta = \xi B^{b,+}$ for some $\xi \in B^{b,+}$, then $\ker\theta_\rho = \xi B^+_\rho$ as well.*

Proof. Let $x = \lim_n x^{(n)} \in \ker\theta_\rho \subset B^+_\rho$, where $x^{(n)} \in B^{b,+}$. Since θ_ρ is continuous, we have $\lim_n \theta(x^{(n)}) = \theta_\rho(x) = 0$ and may pass to a subsequence such that $\theta(x^{(n)}) \in p^n\mathcal{O}_C$ holds for all $n \in \mathbb{N}$. Now let

$$v_n := \sum_{i<n} p^i[x^{(n)}_i] \in B^{b,+}, \qquad\qquad w_n := \sum_{i=0}^{\infty} p^i[x^{(n)}_{n+i}] \in \mathrm{W}(\mathcal{O}_C^\flat)$$

for each $n \in \mathbb{N}$. Then by construction, $x^{(n)} = v_n + p^n w_n$, so that

$$\theta(v_n) = \theta(x^{(n)}) - p^n\theta_{\mathcal{O}_C}(w_n) \in p^n\mathcal{O}_C.$$

Since $\theta_{\mathcal{O}_C}$ is surjective, there is a $u_n \in \mathrm{W}(\mathcal{O}_C^\flat)$ such that $\theta(v_n) = p^n\theta_{\mathcal{O}_C}(u_n)$. Setting $y^{(n)} := v_n - p^n u_n \in \ker\theta$, we then have

$$|x^{(n)} - y^{(n)}|_\rho = |p^n w_n + p^n u_n|_\rho = |p^n|_\rho |w_n + u_n|_\rho \leq \rho^n.$$

This shows that $\lim_n y^{(n)} = x$, so $\ker\theta_\rho$ lies in the closure of $\ker\theta$; the other inclusion is trivial since $\ker\theta_\rho = \theta_\rho^{-1}(\{0\})$ is closed.

Finally, if $\ker\theta = \xi B^{b,+}$ for some $\xi \in B^{b,+}$, there are $z^{(n)} \in B^{b,+}$ such that $y^{(n)} = \xi z^{(n)}$ for each $n \in \mathbb{N}$. These form a Cauchy sequence since $|z^{(n)} - z^{(n')}|_\rho = |\xi|_\rho |y^{(n)} - y^{(n')}|_\rho$ for all $n, n' \in \mathbb{N}$; hence $x = \xi \lim_n z^{(n)} \in \xi B^+_\rho$. $\square$

Proposition 3.27. *If $\rho \in (0;|p|]$ and $\xi \in B^{b,+}$ is a generator of $\ker\theta$, then for each $n \in \mathbb{N}$, the map $B^{b,+}/\xi^n B^{b,+} \to B^+_\rho/\xi^n B^+_\rho$ induced by $B^{b,+} \hookrightarrow B^+_\rho$ is an isomorphism.*

Proof. The equality

$$B^+_\rho \overset{3.25}{=} \theta_\rho^{-1}(C) = \theta_\rho^{-1}(\theta_\rho(B^{b,+})) = B^{b,+} + \ker\theta_\rho \overset{3.26}{=} B^{b,+} + \xi B^+_\rho$$

inductively extends to $B^+_\rho = B^{b,+} + \xi^n B^+_\rho$ for $n \geq 1$, showing surjectivity.

Injectivity also follows by induction over n. The base case is $\ker\theta_\rho \cap B^{b,+} = \ker\theta$, which was the subject of 3.26. Then $\xi^n B^+_\rho \cap B^{b,+} \subset \ker\theta_\rho \cap B^{b,+} = \ker\theta = \xi B^{b,+}$ shows

$$\xi^n B^+_\rho \cap B^{b,+} \overset{!}{\subset} \xi^n B^+_\rho \cap \xi B^{b,+} = \xi(\xi^{n-1}B^+_\rho \cap B^{b,+}) \overset{\text{ind.}}{=} \xi(\xi^{n-1}B^{b,+}) = \xi^n B^{b,+}$$

for all $n > 1$. Note that $\xi^n B^+_\rho \cap \xi B^{b,+} = \xi(\xi^{n-1}B^+_\rho \cap B^{b,+})$ uses that B^+_ρ is a domain. $\square$

Lemma 3.28. *If $\xi \in \mathrm{W}(\mathcal{O}_C^\flat)$ is a generator of $\ker\theta_{\mathcal{O}_C}$, then*

$$\psi_\xi(t) = \log_p(|\xi|_{p^{-t}}) = \begin{cases} -t & \text{if } t < -\log_p(|p|), \\ \log_p(|p|) & \text{if } t \geq -\log_p(|p|). \end{cases}$$

for all $t \in (0;\infty)$, where ψ_ξ is the function from Proposition 3.10. In particular,

$$d^+\psi_\xi(-\log_p(|p|)) - d^-\psi_\xi(-\log_p(|p|)) = 1.$$

Proof. Proposition 1.14 implies $|\xi_0|_\flat = |p|$ and $|\xi_1|_\flat = 1$, so that

$$\begin{aligned} \log_p(|\xi|_{p^{-t}}) &= \sup_{i\geq 0}\,(\log_p(|\xi_i|_\flat) - ti) \\ &= \max\{\log_p(|\xi_0|_\flat),\ \log_p(|\xi_1|_\flat) - t,\ \sup_{i\geq 2}\,(\log_p(|\xi_i|_\flat) - ti)\} \\ &= \max\{\log_p(|p|),\ -t,\ \sup_{i\geq 2}\,(\log_p(|\xi_i|_\flat) - ti)\} \\ &= \max\{\log_p(|p|),\ -t\} \\ &= \begin{cases} -t & \text{if } t < -\log_p(|p|), \\ \log_p(|p|) & \text{if } t \geq -\log_p(|p|) \end{cases} \end{aligned}$$

for all $t \in (0;\infty)$ since all $|\xi_i|_\flat \leq 1$. The remark on derivatives follows immediately. □

Proposition 3.29. *If $\rho \in (0;|p|]$, then B_ρ^+ is $\ker\theta_\rho$-adically separated.*

Proof. We first consider the case $\rho = |p|$. Let $\xi \in \mathrm{W}(\mathcal{O}_C^\flat)$ be a generator of $\ker\theta_{\mathcal{O}_C}$ (hence of $\ker\theta_\rho$) and $f \in \bigcap_{n\geq 0}\xi^n B_\rho^+ \setminus 0$. Since $B^{b,+} \subset B_\rho^+$ is dense, there is an $f' \in B^{b,+}$ with $|f - f'|_\rho < |f|_\rho$. Let $\lambda = -\log_p(\rho)$ and $N = d^+\psi_{f'}(\lambda) - d^-\psi_{f'}(\lambda) \overset{3.10}{\in} \mathbb{N}_0$. By assumption there exists a $g \in B_\rho^+ \setminus 0$ such that $f = \xi^N g$, for which there is again a $g' \in B^{b,+}$ such that $|g - g'|_\rho < |g|_\rho$. Therefore

$$|\xi^N g' - f'|_\rho = |\xi^N(g' - g) + (f - f')|_\rho \leq \max\{|\xi^N(g' - g)|_\rho,\ |f - f'|_\rho\} \overset{!}{<} |f|_\rho = |f'|_\rho,$$

so since $|\cdot|_\rho$ is continuous in ρ (3.11), we have $\psi_{f'} = \psi_{\xi^N g'} = N\psi_\xi + \psi_{g'}$ near λ. Hence

$$\begin{aligned} N &= d^+\psi_{f'}(\lambda) - d^-\psi_{f'}(\lambda) \\ &\overset{!}{=} N(d^+\psi_\xi(\lambda) - d^-\psi_\xi(\lambda)) + d^+\psi_{g'}(\lambda) - d^-\psi_{g'}(\lambda) \\ &\overset{3.28}{=} N + d^+\psi_{g'}(\lambda) - d^-\psi_{g'}(\lambda), \end{aligned}$$

which shows $d^+\psi_{g'}(\lambda) - d^-\psi_{g'}(\lambda) = 0$. By Proposition 3.10 (ii), this means that there is a unique $m \in \mathbb{Z}$ with $|g'|_\rho = \rho^m|g'_m|_\flat$, so $|g' - p^m[g'_m]|_\rho < |g'|_\rho$, so

$$|\theta(g' - p^m[g'_m])| \overset{3.25}{\leq} |g' - p^m[g'_m]|_\rho \overset{!}{<} |g'|_\rho = |p^m[g'_m]|_\rho = |p^m g'^\sharp_m| = |\theta(p^m[g'_m])|.$$

This shows $|\theta(g')| = |\theta(p^m[g'_m])|$, and thus

$$|\theta_\rho(g - g')| \overset{3.25}{\leq} |g - g'|_\rho < |g|_\rho = |g'|_\rho = |\theta(p^m[g'_m])| \overset{!}{=} |\theta(g')|,$$

whence we finally conclude

$$|\theta_\rho(g)| \overset{!}{=} |\theta(g')| = |\theta(p^m[g'_m])| = \rho^m|g'_m|_\flat = |g'|_\rho = |g|_\rho.$$

But since $g \neq 0$, this means $g \notin \ker\theta_\rho$, which absurdly implies $f \notin \xi^{N+1}B_\rho^+$.

In the general case, let $\sigma \in (0; |p|]$. Then we have an injection

$$B_\sigma^+ \hookrightarrow B_\rho^+ \hookrightarrow \varprojlim_{n\in\mathbb{N}} B_\rho^+/\xi^n B_\rho^+ \overset{3.27}{\cong} \varprojlim_{n\in\mathbb{N}} B^{b,+}/\xi^n B^{b,+} \overset{3.27}{\cong} \varprojlim_{n\in\mathbb{N}} B_\sigma^+/\xi^n B_\sigma^+. \qquad \square$$

This result isn't too surprising; it essentially states that "functions" $f \in B_\rho^+$ with zeros of infinite order vanish, just like in the classical situation.

Definition 3.30. Let $\rho \in (0; |p|]$.

(i) The *field of de Rham periods* is the fraction field (cf. 3.31) $B_{\mathrm{dR}} := \operatorname{Frac} B_{\mathrm{dR}}^+$, where

$$B_{\mathrm{dR}}^+ := \varprojlim_{n\in\mathbb{N}} B^{b,+}/(\ker\theta)^n \overset{3.27}{\cong} \varprojlim_{n\in\mathbb{N}} B_\rho^+/(\ker\theta_\rho)^n.$$

The ring B_{dR}^+ is equipped with the inverse limit topology relative to the quotient topologies induced by the norm $|\cdot|_\rho$. The choice of ρ or $B^{b,+}$ vs. B_ρ^+ is irrelevant since $B^{b,+}/(\ker\theta)^n \cong B_\rho^+/(\ker\theta_\rho)^n$ is a homeomorphism by definition. The map $B_\rho^+ \hookrightarrow B_{\mathrm{dR}}^+$ is continuous by definition and injective by Proposition 3.29.

(ii) We write $\theta_{\mathrm{dR}} : B_{\mathrm{dR}}^+ \twoheadrightarrow C$ for the composition $B_{\mathrm{dR}}^+ \twoheadrightarrow B^{b,+}/(\ker\theta) \xrightarrow{\theta} C$, or equivalently, $B_{\mathrm{dR}}^+ \twoheadrightarrow B_\rho^+/(\ker\theta_\rho) \xrightarrow{\theta_\rho} C$ for any $\rho \in (0; |p|]$. This map is by definition continuous and open.

Proposition 3.31. *B_{dR}^+ is a complete discrete valuation ring with residue field C and maximal ideal $\ker\theta_{\mathrm{dR}}$. Every generator $\xi \in \mathrm{W}(\mathcal{O}_C^\flat)$ of $\ker\theta_{\mathcal{O}_C}$ is a uniformizer of B_{dR}^+.*

Proof. [17, 5.10] Since $C = B^{b,+}/\ker\theta = B_{\mathrm{dR}}^+/\ker\theta_{\mathrm{dR}}$ is a field, it follows that $\ker\theta_{\mathrm{dR}}$ is maximal and $\xi \in B^{b,+}$ is a prime element. The maximal ideals of $B^{b,+}/\xi^n B^{b,+}$ correspond to maximal ideals of $B^{b,+}$ that contain ξ^n; the only such ideal is $\xi B^{b,+}$, so all $B^{b,+}/\xi^n B^{b,+}$ are local with maximal ideal $\xi B^{b,+}/\xi^n B^{b,+}$.

Let $x = \varprojlim_n (x_n + \xi^n B^{b,+}) \in B_{\mathrm{dR}}^+ \setminus \ker\theta_{\mathrm{dR}}$. Then $x_1 \notin \xi B^{b,+}$ by definition of θ_{dR}, hence $x_n \notin \xi B^{b,+}$, i.e. $x_n + \xi^n B^{b,+} \in (B^{b,+}/\xi^n B^{b,+})^\times$, for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $x_n' \in B^{b,+}$ such that $x_n x_n' \equiv 1 \mod \xi^n B^{b,+}$. Then

$$(x_{n+1}' - x_n')x_n x_{n+1} \equiv x_{n+1}' x_n x_{n+1} - x_n' x_n x_{n+1} \equiv x_n - x_{n+1} \equiv 0 \mod \xi^n B^{b,+};$$

it follows that $x_{n+1}' - x_n' \in \xi^n B^{b,+}$ because $x_n, x_{n+1} \notin \xi B^{b,+}$, $\xi \in B^{b,+}$ is a prime element, and $B^{b,+}$ is a domain (cf. proof of 2.12). Hence $\varprojlim_n (x_n' + \xi^n B^{b,+}) = x^{-1} \in B_{\mathrm{dR}}^+$, showing that B_{dR}^+ is local with maximal ideal $\ker\theta_{\mathrm{dR}}$.

Now let $y = \varprojlim_n (y_n + \xi^n B^{b,+}) \in \ker\theta_{\mathrm{dR}}$. Then all $y_n \in \ker\theta$, so for each $n \in \mathbb{N}$ there is a $y_n' \in B^{b,+}$ such that $y_n = \xi y_n'$. Note that y_n' is unique modulo $\xi^{n-1}B^{b,+}$. We have $\xi(y_{n+1}' - y_n') = y_{n+1} - y_n \in \xi^n B^{b,+}$, so $y_{n+2}' - y_{n+1}' \in \xi^n B^{b,+}$ for all $n \in \mathbb{N}$. Then the element $y' := \varprojlim_n (y_{n+1}' + \xi^n B^{b,+})$ satisfies $y = \xi y'$ because $\xi y_{n+1}' \equiv y_{n+1} \equiv y_n \mod \xi^n B^{b,+}$. This proves that $\ker\theta_{\mathrm{dR}} = \xi B_{\mathrm{dR}}^+$.

Note that in the above context, for any $z := \varprojlim_n (z_n + \xi^n B^{b,+}) \in B_{\mathrm{dR}}^+$ that satisfies $y = \xi z$, we must have $\xi z_n \equiv y_n \equiv y_{n+1} \mod \xi^n B^{b,+}$ and hence $z_n \equiv y_{n+1}' \mod \xi^n B^{b,+}$ by the uniqueness statement. Therefore the decomposition $y = \xi z$ is unique. Since B_{dR}^+ is ξ-adically separated by construction, we can repeat this decomposition until we obtain $y = \xi^n u$ with unique $n \in \mathbb{N}$ and $u \in B_{\mathrm{dR}}^+ \setminus \ker\theta_{\mathrm{dR}} = (B_{\mathrm{dR}}^+)^\times$. Since $\xi \in B^{b,+}$ is not nilpotent, this proves the statement. $\square$

Note that the valuation topology is very different from the inverse limit topology; $\ker \theta_{\mathrm{dR}} \subset B_{\mathrm{dR}}^+$ is open with respect to the former, so the induced topology on the residue field C would be discrete rather than the normal valuation topology.

Remark 3.32. Unlike for B_{cris}^+, B_ρ^+ and B^+, there is no Frobenius lift for B_{dR}^+ because $\varphi : \mathrm{W}(\mathcal{O}_C^\flat) \xrightarrow{\sim} \mathrm{W}(\mathcal{O}_C^\flat)$ doesn't preserve $\ker \theta_{\mathcal{O}_C}$. Using 1.15, let $w \in \mathrm{W}(\mathcal{O}_C^\flat)^\times$ such that $\xi = [s] + pw$ is a generator of $\ker \theta_{\mathcal{O}_C}$. Then $\theta_{\mathcal{O}_C}(\varphi(w)) \in \mathcal{O}_C^\times$ and hence

$$\theta_{\mathcal{O}_C}(\varphi(\xi)) = (s^\sharp)^p + p\theta_{\mathcal{O}_C}(\varphi(w)) \neq 0$$

since $|(s^\sharp)^p| = |p|^p < |p| = |p\theta_{\mathcal{O}_C}(\varphi(w))|$. This is a major defect of B_{dR}^+.

Proposition 3.33. *Let $\rho = |p|$ and recall the sets $S_\rho, S_{\rho^p} \subset B^{b,+}$ from Proposition 3.8.*

(i) *There exist (unique) continuous homomorphisms of $\mathrm{W}(\mathcal{O}_C^\flat)$-algebras*

$$S_{\rho^p} \hookrightarrow A_{\mathrm{cris}}^0 \hookrightarrow S_\rho, \qquad \mathring{B}_{\rho^p}^+ \to A_{\mathrm{cris}} \to \mathring{B}_\rho^+, \qquad B_{\rho^p}^+ \to B_{\mathrm{cris}}^+ \to B_\rho^+.$$

The composition $S_{\rho^p} \hookrightarrow A_{\mathrm{cris}}^0 \hookrightarrow S_\rho$ is the canonical inclusion $S_{\rho^p} \hookrightarrow S_\rho$.

(ii) *The maps from (i) make the following diagrams commute:*

$$\begin{tikzcd} A_{\mathrm{cris}} \arrow[r, hook, "\varphi_{\mathrm{cris}}"] \arrow[d] & A_{\mathrm{cris}} \\ \mathring{B}_\rho^+ \arrow[r, "\mathring{\varphi}_\rho"] & \mathring{B}_{\rho^p}^+ \arrow[u] \end{tikzcd} \quad \begin{tikzcd} B_{\mathrm{cris}}^+ \arrow[r, hook, "\varphi_{\mathrm{cris}}"] \arrow[d] & B_{\mathrm{cris}}^+ \\ B_\rho^+ \arrow[r, "\varphi_\rho"] & B_{\rho^p}^+ \arrow[u] \end{tikzcd} \quad \begin{tikzcd} \mathring{B}_{\rho^p}^+ \arrow[r, hook, "\iota_{\rho^p,\rho}"] \arrow[d] & \mathring{B}_\rho^+ \\ A_{\mathrm{cris}} \arrow[ur] & \end{tikzcd} \quad \begin{tikzcd} B_{\rho^p}^+ \arrow[r, hook, "\iota_{\rho^p,\rho}"] \arrow[d] & B_\rho^+ \\ B_{\mathrm{cris}}^+ \arrow[ur] & \end{tikzcd}$$

In particular, we have continuous injections $B_{\rho^p}^+ \hookrightarrow B_{\mathrm{cris}}^+ \hookrightarrow B_\rho^+$ such that φ_{cris} is the restriction of φ_ρ and $B_{\rho^p}^+ \hookrightarrow B_{\mathrm{cris}}^+ \hookrightarrow B_\rho^+$ is the canonical inclusion $B_{\rho^p}^+ \hookrightarrow B_\rho^+$.

Proof. (i): It suffices to construct $S_{\rho^p} \hookrightarrow A_{\mathrm{cris}}^0$ and $A_{\mathrm{cris}}^0 \hookrightarrow S_\rho$; the other maps arise through completeness and localization. Continuity is trivial for them by Proposition 3.8; uniqueness follows because all maps in question are determined on $\mathrm{W}(\mathcal{O}_C^\flat)$. We define $A_{\mathrm{cris}}^0 \hookrightarrow S_\rho$ via $\mathrm{W}(\mathcal{O}_C^\flat) \hookrightarrow S_\rho$ and the images $\frac{[s]^n}{n!}$, which lie in S_ρ because

$$\left|\frac{[s]^n}{n!}\right|_\rho = \frac{\rho^n}{|n!|_\rho} = \frac{|p^n|_\rho}{|n!|_\rho} = \left|\frac{p^n}{n!}\right|_\rho \overset{2.10}{\leq} 1$$

for all $n \in \mathbb{N}$. For $S_{\rho^p} \hookrightarrow A_{\mathrm{cris}}^0$, identify $A_{\mathrm{cris}}^0 \subset S_\rho \subset B^{b,+}$ and let $x \in S_{\rho^p}$. Then we have $\sum_{i=0}^\infty p^i[x_i] \in \mathrm{W}(\mathcal{O}_C^\flat) \subset A_{\mathrm{cris}}^0$. Furthermore, for each $n \in \mathbb{N}$, we have $p^{-n}[x_{-n}] \in S_{\rho^p}$ by the definition of $|\cdot|_{\rho^p}$ as a maximum; hence

$$|x_{-n}|_\flat = \rho^{np}|p^{-n}[x_{-n}]|_{\rho^p} \leq \rho^{np} \cdot 1 = \rho^{np}|p^{-n}[s^{np}]|_{\rho^p} = |s^{np}|_\flat,$$

which shows $x_{-n}s^{-np} \in \mathcal{O}_C^\flat$. Therefore every

$$p^{-n}[x_{-n}] = [x_{-n}s^{-np}] \cdot p^{-n}[s^p]^n = [x_{-n}s^{-np}] \cdot \left((p-1)!\frac{[s^p]}{p!}\right)^n \in A_{\mathrm{cris}}^0,$$

which proves that $x \in A_{\mathrm{cris}}^0$. The resulting homomorphism $S_{\rho^p} \to A_{\mathrm{cris}}^0$ is injective because its composition with $A_{\mathrm{cris}}^0 \to S_\rho$ is just the injection $S_{\rho^p} \hookrightarrow S_\rho$; this can be verified on $\mathrm{W}(\mathcal{O}_C^\flat)$, where it is trivial.

(ii): It suffices to verify the first and third diagram; the others follow by eliminating denominators. For the first diagram, this can be done on $\mathrm{W}(\mathcal{O}_C^\flat)$ by Remark 2.19 (iii), where commutativity follows from the fact that the maps in (i) are $\mathrm{W}(\mathcal{O}_C^\flat)$-algebra homomorphisms and the fact that $\mathring{\varphi}_\rho$ and φ_{cris} both restrict to $\varphi : B^{b,+} \xrightarrow{\sim} B^{b,+}$. For the third, we can verify commutativity on S_{ρ^p} by continuity and density; this was done in (i). □

Note that the injectivity of the maps $A_{\text{cris}} \hookrightarrow \mathring{B}_\rho^+$ and $B_{\text{cris}}^+ \hookrightarrow B_\rho^+$ fundamentally relies on the injectivity of φ_{cris}. From the inclusions $A_{\text{cris}} \hookrightarrow B_{\text{cris}}^+ \hookrightarrow B_\rho^+ \hookrightarrow B_{\text{dR}}^+$, we conclude:

Corollary 3.34. *There is a canonical continuous inclusion $B_{\text{cris}}^+ \hookrightarrow B_{\text{dR}}^+$.*

Corollary 3.35. *A_{cris} and B_{cris}^+ are domains.*

Corollary 3.36. *B^+ is the largest subring of B_{cris}^+ where φ_{cris} is bijective.*

Proof. If $\rho = |p|$, then $B^+ \overset{3.20}{=} \bigcap_n \varphi_{\rho^p}^n(B_{\rho^p}^+) \overset{3.33}{\subset} \bigcap_n \varphi_{\text{cris}}^n(B_{\text{cris}}^+) \overset{3.33}{\subset} \bigcap_n \varphi_\rho^n(B_\rho^+) \overset{3.20}{=} B^+$. $\square$

3.3 The cyclotomic periods t_ε and the induced filtration

We continue in the setting of §3.2, but now additionally assume that C contains all p^n-th roots of unity. This is a rather natural assumption in the context of perfectoid fields due to the Fontaine-Wintenberger theorem and no restriction in practice as we will only end up considering algebraically closed C in §4 anyway.

In this case there exists a distinguished set of uniformizers t_ε of B_{dR}^+, the so-called *cyclotomic periods*, that can be thought of as a p-adic analogue of $2\pi i$ from the classic complex setting. These elements will allow us to finally define the period ring B_{cris} and a family of closely related period rings that are "almost as good" in the sense that they can be used as a replacement at no loss in many situations. It will turn out in §4 that the Galois group acts on the cyclotomic periods via the cyclotomic character χ (hence the name), which will ensure that the naturally occuring representations χ^n for $n \in \mathbb{Z}$ are admissible.

Definition 3.37. We write

$$\mathcal{U} := \{ \varprojlim_{n\in\mathbb{N}_0} (\varepsilon_n + p^n\mathcal{O}_C) \in \mathcal{O}_C^\flat \mid \varepsilon_0 = 1,\ \varepsilon_1 \neq 1,\ \varepsilon_{n+1}^p = \varepsilon_n \}$$

for the image of compatible systems of primitive p^n-th roots of unity in $\mathcal{O}_C$ under the bijection in Proposition 1.2 (ii). Given $\varepsilon \in \mathcal{U}$, we always let $\varepsilon_n \in \mathcal{O}_C$ denote the uniquely corresponding p^n-th primitive root of unity.

Lemma 3.38. *Every primitive p^n-th root of unity $\zeta_{p^n} \in C$ satisfies $|\zeta_{p^n} - 1| = |p|^{\frac{1}{p^{n-1}(p-1)}}$.*

Proof. Note that $\zeta_{p^n} - 1$ is a root of the polynomial $F(X) = \sum_{i=0}^{p-1}(X+1)^{ip^{n-1}} \in \mathbb{Z}[X]$, which is Eisenstein with respect to p because the constant term is equal to p and

$$\sum_{i=0}^{p-1}(X+1)^{ip^{n-1}} \equiv \sum_{i=0}^{p-1}(X^{p^{n-1}}+1)^i \mod p,$$

$$\sum_{i=0}^{p-1}(X^{p^{n-1}}+1)^i = \sum_{i=0}^{p-1}\sum_{j=0}^{i}\binom{i}{j}X^{jp^{n-1}} = \sum_{j=0}^{p-1}X^{jp^{n-1}}\sum_{i=j}^{p-1}\binom{i}{j} = \sum_{j=0}^{p-1}X^{jp^{n-1}}\binom{p}{j+1}.$$

For the last equality, perform a case distinction on the lowest chosen element. All roots of $F(X)$, in particular $\zeta_{p^n} - 1$, then have absolute value $|p|^{\frac{1}{(p-1)p^{n-1}}}$ by the usual Newton polygon argument for Eisenstein polynomials. $\square$

Proposition 3.39. *If $\varepsilon \in \mathcal{U}$, then $|\varepsilon^{p^n} - 1|_\flat = |p|^{p^{n+1}/(p-1)}$ for any $n \in \mathbb{Z}$. In particular, $\varepsilon \in 1 + \mathfrak{m}_{C^\flat}$.*

Proof. By a direct calculation,

$$|\varepsilon^n - 1|_\flat = |\lim_{k\to\infty}(\varepsilon_k^{p^n} - 1)^{p^k}| = \lim_{\substack{k\to\infty\\ k\geq n}} |\varepsilon_{k-n} - 1|^{p^k} \overset{3.38}{=} \lim_{\substack{k\to\infty\\ k\geq n}} |p|^{\frac{p^k}{p^{k-n-1}(p-1)}} = |p|^{p^{n+1}/(p-1)}. \quad \square$$

Definition 3.40. For each $\varepsilon \in \mathcal{U}$, the corresponding *cyclotomic period* is

$$t_\varepsilon := \log([\varepsilon]) \overset{3.21}{=} \sum_{i=1}^{\infty} (-1)^{i+1} \frac{([\varepsilon]-1)^i}{i} \in B^+.$$

The following proposition shows that in all rings we consider, the choice of $\varepsilon \in \mathcal{U}$ is basically irrelevant. We will therefore often write t instead of t_ε when the ε is not used elsewhere, notably for localizations $A[\frac{1}{t}]$ of $\mathbb{Z}_p$-algebras A that contain all t_ε.

Lemma 3.41. *For all $\varepsilon, \varepsilon' \in \mathcal{U}$, there is a unique $a \in \mathbb{Z}_p^\times$ such that*

$$\varepsilon^a := \varprojlim_{n\in\mathbb{N}_0} (\varepsilon_n^{a+p^n\mathbb{Z}} + p\mathcal{O}_C) = \varepsilon'.$$

Proof. For all $n \in \mathbb{N}$, there is a unique $\overline{a_n} = a_n + p^n\mathbb{Z}$ with $\varepsilon_n^{a_n} = \varepsilon'_n$ and $\gcd(a_n, p^n) = 1$ since ε_n and ε'_n are primitive p^n-th roots of unity. The condition $(\varepsilon'_{n+1})^p = \varepsilon'_n$ implies that $pa_{n+1} \equiv a_n \mod p^n\mathbb{Z}$, which is exactly the required condition to glue the $\overline{a_n}$ to a unique $a \in \mathbb{Z}_p$. Since $\overline{a_n} \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ for all $n \in \mathbb{N}$, we have $a \in \mathbb{Z}_p^\times$. $\square$

Proposition 3.42. *If $\varepsilon \in \mathcal{U}$ and $a \in \mathbb{Z}_p^\times$, then $at_\varepsilon = t_{\varepsilon^a} \in B^+$.*

Proof. Write $a = \sum_{i=0}^\infty a_i p^i \in \mathbb{Z}_p$ and $a'_n := \sum_{i=0}^n a_i p^i$. Then the first $n+1$ components of $\varepsilon^{a'_n}$ are identical to those of ε^a, so $\varepsilon^a = \lim_n \varepsilon^{a'_n}$ by Proposition 1.7. Hence

$$\log([\varepsilon^a]) = \log([\lim_{n\to\infty} \varepsilon^{a'_n}]) = \lim_{n\to\infty} \log([\varepsilon^{a'_n}]) = \lim_{n\to\infty} a'_n \log([\varepsilon]) = a\log([\varepsilon])$$

by the continuity of $\log([\cdot])$. $\square$

Proposition 3.43. *Each $t_\varepsilon \in B_{\mathrm{dR}}^+$ is a uniformizer of B_{dR}^+. In particular, $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[\frac{1}{t}]$.*

Proof. [2, 4.4.8] Since θ_{dR}^+ is continuous, it suffices to prove $[\varepsilon] - 1 \in \ker\theta \setminus (\ker\theta)^2$; the only noteworthy part is $[\varepsilon] - 1 \notin (\ker\theta)^2$. Use 1.15 (i) to find a $w \in \mathrm{W}(\mathcal{O}_C^\flat)^\times$ such that $\xi := [s] + pw = [s] + p[w_0] + p^2(\ldots)$ generates $\ker\theta_{\mathcal{O}_C}$ and assume to the contrary that $[\varepsilon] - 1 = \xi^2 x$ for some $x \in \mathrm{W}(\mathcal{O}_C^\flat)$. Recall the adapted Witt structure polynomials

$$\begin{aligned}
\tilde{P}_1(X_0, X_1, Y_0, Y_1) &= X_0^p Y_1^p + X_1^p Y_0^p + pX_1^p Y_1^p,\\
\tilde{S}_1(X_0, X_1, Y_0, Y_1) &= X_1^p + Y_1^p + p^{-1}(X_0^p + Y_0^p - (X_0 + Y_0)^p),\\
\tilde{I}_1(X_0, X_1) &= p^{-1}(-X_0^p - pX_1^p - (-X_0)^p) = \begin{cases} -X_1^p & \text{if } p \neq 2,\\ -(X_0^2 + X_1^2) & \text{if } p = 2.\end{cases}
\end{aligned}$$

We use these to directly calculate

$$\begin{aligned}
\xi^2 &= [s^2] + p[2sw_0] + p^2(\ldots),\\
\xi^2\left(\sum_{i=0}^\infty p^i[x_i]\right) &= [s^2x_0] + p[s^2x_1 + 2sw_0x_0] + p^2(\ldots),\\
[\varepsilon] - 1 &= [\varepsilon - 1] + p(\ldots) && \text{if } p \neq 2,\\
[\varepsilon] - 1 &= [\varepsilon - 1] + p[0^p + \tilde{I}_1(1,0)^p - \varepsilon]^{1/p} + p^2(\ldots)\\
&= [\varepsilon - 1] + p[1 - \varepsilon^{1/p}] + p^2(\ldots) && \text{if } p = 2.
\end{aligned}$$

It follows that $|\varepsilon - 1|_\flat < |p|^{p/(p-1)}$ since

$$\begin{aligned} |\varepsilon - 1|_\flat &= |s^2 x_0|_\flat \le |p|^2 < |p|^{p/(p-1)} && \text{if } p \neq 2, \\ |\varepsilon - 1|_\flat &= |1 - \varepsilon^{1/p}|_\flat^p = |s^2 x_1|_\flat^p \le |p|^4 < |p|^{p/(p-1)} && \text{if } p = 2; \end{aligned}$$

but this contradicts Proposition 3.39, so $[\varepsilon] - 1 \notin (\ker \theta_{\mathcal{O}_C})^2$. $\square$

Definition 3.44.

(i) The *ring of crystalline periods* is the localization $B_{\text{cris}} := B^+_{\text{cris}}[\frac{1}{t}]$.

(ii) We equip B_{dR} with the filtration of a discretely valued field, i.e. $\operatorname{Fil}^n B_{\text{dR}} = t^n B^+_{\text{dR}}$ for $n \in \mathbb{Z}$. This filtration is exhaustive and separated and clearly turns B_{dR} into a filtered ring. If $R \subset B_{\text{dR}}$ is a subring, we equip R with the subspace filtration

$$\operatorname{Fil}^n R = R \cap \operatorname{Fil}^n B_{\text{dR}} = R \cap t^n B^+_{\text{dR}}.$$

This filtration is likewise separated and exhaustive and turns R into a filtered ring.

(iii) We write $B_\bullet, B^+_\bullet$ and $\varphi_\bullet$ for any consistent choice of B_{cris}, B^+_{cris} and φ_{cris}, or $B^+_\rho[\frac{1}{t}]$, B^+_ρ and φ_ρ with $\rho \in (0; |p|]$, where $\varphi_\bullet : B_\bullet \to B_\bullet$ also denotes the (injective) extension of $\varphi_\bullet : B^+_\bullet \to B^+_\bullet$ induced by $\varphi_\bullet(t) = pt$. Similarly, we refer to the canonical extensions of the inclusions $\iota_{\sigma,\rho}$, etc. by the same name $\iota_{\sigma,\rho}$. Note that the maps θ_ρ cannot extend to $B^+_\rho[\frac{1}{t}]$ because $\theta_\rho(t) = 0$.

Remark 3.45. Let $\rho \in (0; |p|]$ and $w \in \mathrm{W}(\mathcal{O}^\flat_C)^\times$ such that $\xi := [s] + pw$ generates $\ker \theta_{\mathcal{O}_C}$. We proved in 3.43 that t and ξ are associated in B^+_{dR}; this turns out to be false in B^+_ρ. Since ξ generates $\ker \theta_\rho$, there is a $u \in B^+_\rho$ such that $t = \xi u$. If t and ξ were associated, we would have $u^{-1} \in B^+_\rho$; this is impossible since $\varphi(\xi) = pt\varphi_\rho(u^{-1}) \in \ker \theta_{\text{dR}}$ contradicts the calculation in Remark 3.32. We do however have $u^{-1} \in B^+_\rho[\frac{1}{t}] \cap B^+_{\text{dR}} = \operatorname{Fil}^0 B^+_\rho[\frac{1}{t}]$ since $t \in B^+_\rho[\frac{1}{t}]^\times$ and $u \in (B^+_{\text{dR}})^\times$. It surprisingly follows that $B^+_\rho \subset \operatorname{Fil}^0 B^+_\rho[\frac{1}{t}]$ is strict.

By a simple inclusion argument, one sees that t and ξ are not associated in B^+_{cris} either, and likewise, if $t = \xi u$ for some $u \in B^+_{\text{cris}} \subset B^+_\rho$, we must have $u^{-1} \in \operatorname{Fil}^0 B_{\text{cris}} \setminus B^+_{\text{cris}}$ so that $B^+_{\text{cris}} \subsetneq \operatorname{Fil}^0 B_{\text{cris}}$.

We close this chapter with a study of how the filtration interacts with the Frobenius map. Interestingly, the filtration does not turn out to be $\varphi_\bullet$-stable; since $\varphi(\xi) \notin \ker \theta$, we have $\varphi_B(\xi/t) \in \operatorname{Fil}^{-1} B_{\text{dR}} \setminus \operatorname{Fil}^0 B_{\text{dR}}$ even though $\xi/t \in \operatorname{Fil}^0 B_{\text{dR}}$. This should however be understood as an advantage rather than as a defect because it is what makes Proposition 3.54 possible.

Proposition 3.46. *Let $\rho = |p|$ and $\varepsilon \in \mathcal{U}$. Then $t_\varepsilon \in \mathring{B}^+_{\rho^p}$ and $t_\varepsilon^{p-1} \in p\mathring{B}^+_{\rho^p}$; in particular, $t_\varepsilon^{p-1} \in pA_{\text{cris}}$ and $A_{\text{cris}}[\frac{1}{t}] = B_{\text{cris}}$.*

Proof. Since $|\varepsilon - 1|_\flat \overset{3.39}{=} \rho^{p/(p-1)} \ge \rho^p$, it follows that $|[\varepsilon] - 1|_{\rho^p} = \rho^{p/(p-1)}$. Then for each summand of t_ε, we have

$$\left| (-1)^{n+1} \frac{([\varepsilon] - 1)^n}{n} \right|_{\rho^p} = \frac{\rho^{np/(p-1)}}{|n|_{\rho^p}} \le \frac{\rho^{np/(p-1)}}{|n!|_{\rho^p}} = \rho^{(n-(p-1)v_p(n!))p/(p-1)} \overset{2.9}{\le} \rho^{p/(p-1)}.$$

Therefore $|t_\varepsilon|_{\rho^p} \le \rho^{p/(p-1)}$, so $t_\varepsilon \in \mathring{B}^+_{\rho^p} \overset{3.33}{\subset} A_{\text{cris}}$ and $|t_\varepsilon^{p-1}|_{\rho^p} \le \rho^p$. By Proposition 3.8, we thus have $t_\varepsilon^{p-1} \in p\mathring{B}^+_{\rho^p} \subset pA_{\text{cris}}$. $\square$

Proposition 3.47. *If $\varepsilon \in \mathcal{U}$, then $\bigcap_{n\geq 0} \ker(\theta_{\mathcal{O}_C} \circ \varphi^n) = ([\varepsilon]-1)\mathrm{W}(\mathcal{O}_C^\flat)$.*

Proof. [3, 3.1.7] Clearly $\theta_{\mathcal{O}_C}(\varphi^n([\varepsilon]-1)) = 1^{p^n} - 1 = 0$ for all $n \in \mathbb{N}_0$. On the other hand, let $w \in \bigcap_{n\geq 0} \ker(\theta_{\mathcal{O}_C} \circ \varphi^n)$ and set $\xi = \frac{[\varepsilon]-1}{[\varepsilon^{1/p}]-1} = \sum_{i=0}^{p-1} [\varepsilon^{i/p}] \in \mathrm{W}(\mathcal{O}_C^\flat)$. Using 1.14, we see that ξ generates $\ker \theta_{\mathcal{O}_C}$ because $\theta(\xi) = \frac{1-1}{\varepsilon_1 - 1} = 0$ and $|\xi_0|_\flat = |\frac{\varepsilon-1}{\varepsilon^{1/p}-1}|_\flat \overset{3.39}{=} |p|^{\frac{p}{p-1}-\frac{1}{p-1}} = |p|$.

Since $\theta_{\mathcal{O}_C}(\varphi^0(w)) = \theta_{\mathcal{O}_C}(w) = 0$, there is a $v^{(0)} \in \mathrm{W}(\mathcal{O}_C^\flat)$ with $w = v^{(0)}\xi$. Assume we have elements $v^{(0)}, \ldots, v^{(n)} \in \mathrm{W}(\mathcal{O}_C^\flat)$ such that $w = \varphi^{-n}(v^{(n)}) \prod_{i=0}^n \varphi^{-i}(\xi)$. Then a direct calculation shows

$$\theta_{\mathcal{O}_C}\left(\varphi^{n+1}\left(\prod_{i=0}^n \varphi^{-i}(\xi)\right)\right) = \prod_{i=0}^n \sum_{i=0}^{p-1} \theta_{\mathcal{O}_C}([\varepsilon]^{ip^{n-i}}) = \prod_{i=0}^n \sum_{i=0}^{p-1} 1^{ip^{n-i}} = p^{n+1} \neq 0,$$

so necessarily $\theta_{\mathcal{O}_C}(\varphi(v^{(n)})) = 0$, which means that there is a $v^{(n+1)} \in \mathrm{W}(\mathcal{O}_C^\flat)$ such that $\varphi(v^{(n)}) = v^{(n+1)}\xi$, i.e. $w = \varphi^{-(n+1)}(v^{(n+1)}) \prod_{i=0}^{n+1} \varphi^{-i}(\xi)$.

Hence $|w_0|_\flat \leq |p|^{1+p^{-1}+\ldots+p^{-n}}$ for all $n \in \mathbb{N}_0$, so $|w_0|_\flat \leq |p|^{\frac{1}{1-p^{-1}}} = |p|^{p/(p-1)} = |\varepsilon - 1|_\flat$. This means that there is a $z \in \mathcal{O}_C^\flat$ with $w_0 z = \varepsilon - 1$, so that $w - ([\varepsilon]-1)[z] \in p\mathrm{W}(\mathcal{O}_C^\flat)$, proving that $\bigcap_{n\geq 0} \ker(\theta_{\mathcal{O}_C} \circ \varphi^n) \subset ([\varepsilon]-1)\mathrm{W}(\mathcal{O}_C^\flat) + p\mathrm{W}(\mathcal{O}_C^\flat)$.

This suffices because as in the proof of 1.14, we can find $a_0, b_0 \in \mathrm{W}(\mathcal{O}_C^\flat)$ such that $w = a_0([\varepsilon]-1) + pb_0$, conclude that $\theta_{\mathcal{O}_C}(\varphi^n(b_0)) = 0$ for all $n \in \mathbb{N}_0$, and hence inductively construct elements $a_0, b_0, \ldots, a_n, b_n \in \mathrm{W}(\mathcal{O}_C^\flat)$ such that

$$w = a_0([\varepsilon]-1) + pb_0 = \ldots = p^n b_n + \sum_{i=0}^n a_i([\varepsilon]-1)p^i$$

for all $n \geq 0$. Consequently $w = \sum_{i=0}^\infty a_i([\varepsilon]-1)p^i \in ([\varepsilon]-1)\mathrm{W}(\mathcal{O}_C^\flat)$, as claimed. □

Lemma 3.48. *Let $\rho = |p|^{p/(p-1)}$ and $\varepsilon \in \mathcal{U}$.*

(i) *Let $\sum_{i=0}^\infty b_i X^i \in \mathbb{Q}[\![X]\!]$ be the inverse of $\sum_{i=0}^\infty \frac{(-1)^i}{i+1} X^i \in \mathbb{Q}[\![X]\!]^\times$. Then we have $v_p(b_n) \geq -\frac{n}{p-1}$ for all $n \in \mathbb{N}_0$.*

(ii) *The elements t_ε and $\frac{[\varepsilon^p]-1}{p}$ are associated in $\mathring{B}_\rho^+$.*

Proof. (i): Note that $b_0 = 1$ and $b_n = -\sum_{i=0}^{n-1} b_i \frac{(-1)^{n-i}}{n-i+1}$ for all $n \in \mathbb{N}$. We proceed inductively, the case $n = 0$ being trivial. It suffices to show that $v_p(n-i+1) \leq \frac{n-i}{p-1}$ for all $0 \leq i < n$ since then

$$v_p(b_n) \geq \min_{0\leq i<n} v_p(b_i) - v_p(n-i+1) \overset{!}{\geq} \min_{0\leq i<n} -\frac{i}{p-1} - \frac{n-i}{p-1} = -\frac{n}{p-1}.$$

When $n - i < p - 1$, the inequality is trivial; for $\frac{n-i}{p-1} \geq 1$, Bernoulli's inequality shows

$$v_p(n-i+1) \leq \log_p(1+(n-i)) \overset{!}{\leq} \log_p((1+(p-1))^{\frac{n-i}{p-1}}) = \frac{n-i}{p-1}.$$

(ii): It suffices to show $|[\varepsilon^p]-1|_\rho \leq \rho^{\frac{1}{p-1}}$ and $\lim_n |[\varepsilon^p]-1|_\rho^n \rho^{-\frac{n}{p-1}} = 0$, since then the series $\sum_{i=0}^\infty b_i([\varepsilon^p]-1)^i$ converges in $\mathring{B}_\rho^+$ by part (i) and the statement follows from

$$t_\varepsilon = \frac{p\log([\varepsilon])}{p} = \frac{\log([\varepsilon^p])}{p} = \frac{[\varepsilon^p]-1}{p} \cdot \sum_{i=0}^\infty (-1)^i \frac{([\varepsilon^p]-1)^i}{i+1}.$$

Indeed, $|[\varepsilon^p]-1|_\rho \le \max\{|\varepsilon^p-1|_\flat,\ \rho\} \overset{3.39}{=} \max\{\rho^p,\ \rho\} = \rho \le \rho^{\frac{1}{p-1}}$. Now if $p \neq 2$, then

$$\lim_{n\to\infty} |[\varepsilon^p]-1|_\rho^n \rho^{-\frac{n}{p-1}} \le \lim_{n\to\infty} \rho^n \rho^{-\frac{n}{p-1}} = 0.$$

If on the other hand $p=2$, we can show, by the same methods as in the proof of 3.43, that there is a $w \in \mathrm{W}(\mathcal{O}_C^\flat)$ with $[\varepsilon^p]-1 = [\varepsilon^p-1]+p[\varepsilon-1]+p^2w$, so that

$$\begin{aligned} |[\varepsilon^p]-1|_\rho &= \max\{|\varepsilon^p-1|_\flat,\ \rho|\varepsilon-1|_\flat,\ \rho^2|w|_\rho\} \\ &= \max\{|p|^{p^2/(p-1)},\ \rho|p|^{p/(p-1)},\ \rho^2|w|_\rho\} \\ &= \max\{\rho^2,\ \rho^2,\ \rho^2|w|_\rho\} = \rho^2, \end{aligned}$$

whence $\lim_n |[\varepsilon^p]-1|_\rho^n \rho^{-\frac{n}{p-1}} = \rho^{2n-n} = 0$. $\square$

Lemma 3.49. *If $\rho = |p|^{p/(p-1)}$ and $\varepsilon \in \mathcal{U}$, then $\mathring{B}_\rho^+ = \mathrm{W}(\mathcal{O}_C^\flat) + \frac{[\varepsilon]-1}{p}\mathring{B}_\rho^+$.*

Proof. Write $\pi_\varepsilon := \frac{[\varepsilon]-1}{p}$. We first show that $B^{b,+} \cap \mathring{B}_\rho^+ = \mathrm{W}(\mathcal{O}_C^\flat)[\pi_\varepsilon]$. Clearly $\pi_\varepsilon \in \mathring{B}_\rho^+$, so that $B^{b,+} \cap \mathring{B}_\rho^+ \supset \mathrm{W}(\mathcal{O}_C^\flat)[\pi_\varepsilon]$. For $x \in B^{b,+} \cap \mathring{B}_\rho^+ \setminus 0$, let $n \in \mathbb{N}_0$ be minimal such that $x_i = 0$ for all $i < -n$. If $n = 0$, we have $x \in \mathrm{W}(\mathcal{O}_C^\flat) \subset \mathrm{W}(\mathcal{O}_C^\flat)[\pi_\varepsilon]$. Otherwise, note that $|x_{-n}|_\flat \le \rho^n = |\varepsilon-1|_\flat^n$, so there exists a (unique) $u_n \in \mathcal{O}_C^\flat$ such that $x_{-n} = u_n(\varepsilon-1)^n$. Hence $x' := x - [u_n]\pi_\varepsilon^n$ is an element of $B^{b,+} \cap \mathring{B}_\rho^+$ with $x'_i = 0$ for all $i < -n+1$; proceeding inductively, we obtain unique $u_1, \dots, u_n \in \mathcal{O}_C^\flat$ such that $x - \sum_{i=1}^n [u_i]\pi_\varepsilon^i \in \mathrm{W}(\mathcal{O}_C^\flat)$.

Now let $x \in \mathring{B}_\rho^+ = \widehat{\mathrm{W}(\mathcal{O}_C^\flat)[\pi_\varepsilon]}$ and write $x = \sum_{i=0}^\infty p^i x^{(i)}$ for some $x^{(i)} \in \mathrm{W}(\mathcal{O}_C^\flat)[\pi_\varepsilon]$. Then there exist $w^{(i)} \in \mathrm{W}(\mathcal{O}_C^\flat)$ and $y^{(i)} \in \mathrm{W}(\mathcal{O}_C^\flat)[\pi_\varepsilon]$ with $x^{(i)} = w^{(i)} + \pi_\varepsilon y^{(i)}$; hence

$$x = \sum_{i=0}^\infty p^i x^{(i)} = \sum_{i=0}^\infty p^i (w^{(i)} + \pi_\varepsilon y^{(i)}) = \sum_{i=0}^\infty p^i w^{(i)} + \pi_\varepsilon \sum_{i=0}^\infty p^i y^{(i)} \in \mathrm{W}(\mathcal{O}_C^\flat) + \pi_\varepsilon \mathring{B}_\rho^+. \quad \square$$

Proposition 3.50. *We have $(\mathrm{Fil}^0 B_\bullet)^{\varphi=1} := \{b \in \mathrm{Fil}^0 B_\bullet \mid \varphi_\bullet(b) = b\} = \mathbb{Q}_p$.*

Proof. [3, 3.4.4] Inspection of Teichmüller lifts shows $(B^{b,+})^{\varphi=1} = \mathbb{Q}_p$. Note that it suffices to consider the case $B_\rho^+[\frac{1}{t}]$ with $\rho = |p|^{p/(p-1)}$; if $x \in (\mathrm{Fil}^0 B_{|p|}^+[\frac{1}{t}])^{\varphi=1}$, then

$$x = \varphi_{|p|}(x) \in (\mathrm{Fil}^0 B_{|p|^p}^+[\tfrac{1}{t}])^{\varphi=1} \subset (\mathrm{Fil}^0 B_\rho^+[\tfrac{1}{t}])^{\varphi=1} = \mathbb{Q}_p.$$

The other cases then follow via inclusion.

Let $x \in (\mathrm{Fil}^0 B_\rho^+[\frac{1}{t}])^{\varphi=1}$ and write $x = t_\varepsilon^{-m} y$ with $\varepsilon \in \mathcal{U}$, $m \in \mathbb{N}_0$ and $y \in B_\rho^+ \setminus t_\varepsilon B_\rho^+$. Then $\varphi_\rho(y) = p^m t_\varepsilon^m x = p^m y$. Let $k \in \mathbb{N}_0$ such that $p^k y \in \mathring{B}_\rho^+$ and write $p^k y = w + \frac{[\varepsilon]-1}{p} b$ for suitable $w \in \mathrm{W}(\mathcal{O}_C^\flat)$ and $b \in \mathring{B}_\rho^+$ using Lemma 3.49. Assume that $m > 0$. Then $\theta_{\mathrm{dR}}(t_\varepsilon^m x) = 0$, so

$$\begin{aligned} \theta_{\mathcal{O}_C}(\varphi^n(w)) &= \theta_{\mathcal{O}_C}\left(\varphi^n\left(p^k y - \frac{[\varepsilon]-1}{p} b\right)\right) \\ &= \theta_{\mathrm{dR}}(p^{mn+k} y) - p^{-1}\theta_{\mathcal{O}_C}(\varphi^n([\varepsilon]-1))\theta_\rho(\varphi_\rho^n(b)) \\ &\overset{3.47}{=} \theta_{\mathrm{dR}}(p^{mn+k} y) = p^{mn+k}\theta_{\mathrm{dR}}(t^m x) \overset{!}{=} 0 \end{aligned}$$

for all $n \in \mathbb{N}_0$; by Proposition 3.47, this means that $w = ([\varepsilon]-1)v$ for some $v \in \mathrm{W}(\mathcal{O}_C^\flat)$. By Lemma 3.48 (ii), t_ε divides $[\varepsilon^p]-1$ in B_ρ^+, so it divides

$$([\varepsilon^p]-1)\varphi(v) = \varphi(w) = \varphi\left(p^k y - \frac{[\varepsilon]-1}{p} b\right) = p^{k+m} y + \frac{[\varepsilon^p]-1}{p}\varphi_\rho(b) \in B_\rho^+$$

as well, which implies that $y \in t_\varepsilon B_\rho^+$, contrary to its construction. Hence $m = 0$.

Let $u \in (\mathring{B}_\rho^+)^\times$ with $t_\varepsilon u = \frac{[\varepsilon^p]-1}{p}$, again using Lemma 3.48 (ii). By a simple limit argument, φ_ρ is norm-decreasing on $\mathring{B}_\rho^+$. Therefore

$$|p^{n-1} t_\varepsilon \varphi_\rho^{n-1}(u\varphi_\rho(b))|_\rho \overset{!}{\leq} \rho^{n-1}|t_\varepsilon|_\rho \overset{3.12}{\leq} \rho^{n-1}|t_\varepsilon|_{|p|^p}^{\log_{|p|^p}(\rho)} \overset{3.46}{\leq} \rho^{n-1}$$

for all $n \in \mathbb{N}$; in particular $\lim_n p^{n-1} t_\varepsilon \varphi_\rho^{n-1}(u\varphi_\rho(b)) = 0$. This shows

$$\begin{aligned}
p^k x &= \lim_{n\to\infty} \varphi_\rho^n(p^k x) \\
&= \lim_{n\to\infty} \varphi^n(w) + \varphi_\rho^{n-1}\left(\frac{[\varepsilon^p]-1}{p}\varphi_\rho(b)\right) \\
&= \lim_{n\to\infty} \varphi^n(w) + \varphi_\rho^{n-1}(t_\varepsilon u \varphi_\rho(b)) \\
&= \lim_{n\to\infty} \varphi^n(w) + p^{n-1} t_\varepsilon \varphi_\rho^{n-1}(u\varphi_\rho(b)) \\
&\overset{!}{=} \lim_{n\to\infty} \varphi^n(w) \overset{3.6}{\in} \mathrm{W}(\mathcal{O}_C^\flat).
\end{aligned}$$

Hence $x \in (B^{b,+})^{\varphi=1} = \mathbb{Q}_p$. □

The restriction to $\mathrm{Fil}^0 B_\bullet$ is crucial because as we will soon show, the Frobenius invariants of $B_\bullet$ turn out to be much larger. The proof of the results below follows the general principles of the arguments in [12, §6.2.2, §6.2.3].

Lemma 3.51. *Assume that C is algebraically closed. Then for all $a, b \in \mathrm{W}(\mathcal{O}_C^\flat)$, there exists a $v \in \mathrm{W}(\mathcal{O}_C^\flat)$ such that $\varphi(v) - av - b = 0$.*

Proof. Let $x^{(0)} = b$. Given any $x^{(i)} \in \mathrm{W}(\mathcal{O}_C^\flat)$, we can always find a $v_i \in \mathcal{O}_C^\flat$ such that $[v_i^p] - a[v_i] = x^{(i)} - px^{(i+1)}$ for some $x^{(i+1)} \in \mathrm{W}(\mathcal{O}_C^\flat)$ because $C^\flat$ is algebraically closed and thus the monic polynomial $X^p - a_0 X - x_0^{(i)} = 0$ always admits a root $v_i \in \mathcal{O}_C^\flat$. Hence

$$\varphi\left(\sum_{i=0}^\infty p^i[v_i]\right) - a\sum_{i=0}^\infty p^i[v_i] = \sum_{i=0}^\infty p^i([v_i^p] - a[v_i]) = \sum_{i=0}^\infty p^i(x^{(i)} - px^{(i+1)}) = x^{(0)} = b. \quad \square$$

Proposition 3.52. *Assume that C is algebraically closed and let $\rho = |p|^{p/(p-1)}$ and $r \in \mathbb{N}$. Then the continuous and $\mathbb{Q}_p$-linear map $p^{-r}\varphi_\rho - \mathrm{id} : \mathrm{Fil}^r B_\rho^+ \to B_\rho^+$ is surjective.*

Proof. By $\mathbb{Q}_p$-linearity, it suffices to find preimages of $y \in \mathring{B}_\rho^+$. By continuity, it furthermore suffices to find an $x \in \mathrm{Fil}^r \mathring{B}_\rho^+$ with $p^{-r}\varphi_\rho(x) \in \mathring{B}_\rho^+$ and $p^{-r}\varphi_\rho(x) - x - y \in p\mathring{B}_\rho^+$ since a preimage is then easily constructed by the usual p-adic approximation argument.

Let $\pi_\varepsilon = \frac{[\varepsilon]-1}{p} \in \mathring{B}_\rho^+$, $q = \sum_{i=0}^{p-1}[\varepsilon^i] \in \mathrm{W}(\mathcal{O}_C^\flat)$, and use 1.15 (ii) to find an $s \in \mathcal{O}_C^\flat$ with $|s|_\flat = |p|$ such that $[s] - p \in \mathrm{W}(\mathcal{O}_C^\flat)$ is a generator of $\ker\theta_{\mathcal{O}_C}$. Note that $\varphi(\pi_\varepsilon) = \pi_\varepsilon q$, $\pi_\varepsilon \in \ker\theta$ and that $\varphi^{-1}(q)$ is a generator of $\ker\theta$ (cf. proof of 3.47), so that there is a $u \in \mathrm{W}(\mathcal{O}_C^\flat)^\times$ with $\varphi^{-1}(q) = ([s]-p)u$. Furthermore, we have $q \in p\mathring{B}_\rho^+$ since

$$|q|_\rho = |([s]^p - p)\varphi(u)|_\rho = |[s]^p - p|_\rho = \max\{|p|^p, \rho\} = \rho.$$

Recall that by Lemma 3.49, we can write any $y \in \mathring{B}_\rho^+$ as $\sum_{i=0}^r y_i\pi_\varepsilon^i + \pi_\varepsilon^{r+1}y'$, with $y_i \in \mathrm{W}(\mathcal{O}_C^\flat)$ and $y' \in \mathring{B}_\rho^+$. Note that for $x := -\pi_\varepsilon^{r+1}y' \in \mathrm{Fil}^r \mathring{B}_\rho^+$, we have

$$p^{-r}\varphi_\rho(x) - x - \pi_\varepsilon^{r+1}y' = \pi_\varepsilon^{r+1}(p^{-r}q^{r+1}\varphi_\rho(-y') + y' - y') \in p^{-r}q^{r+1}\mathring{B}_\rho^+ \subset p\mathring{B}_\rho^+;$$

hence it suffices to find, for each $0 \le i \le r$ and $y_i \in \mathrm{W}(\mathcal{O}_C^\flat)$, an $x_i \in \mathrm{Fil}^r \mathring{B}_\rho^+$ such that $p^{-r}\varphi_\rho(x_i) - x_i - y_i\pi_\varepsilon^i \in p\mathring{B}_\rho^+ + \pi_\varepsilon^{i+1}\mathring{B}_\rho^+$.

For any $w \in \mathrm{W}(\mathcal{O}_C^\flat)^\times$, there is a $v_w \in \mathrm{W}(\mathcal{O}_C^\flat)$ with $\varphi(w)^r\varphi(v_w) - y_i = \varphi^{-1}(q)^{r-i}v_w$ by Lemma 3.51. For $x_{i,w} := v_w\varphi^{-1}(q)^{r-i}\pi_\varepsilon^i \in \mathrm{Fil}^r \mathring{B}_\rho^+$, we then have

$$\begin{aligned} p^{-r}\varphi_\rho(x_{i,w}) - x_{i,w} - y_i\pi_\varepsilon^i &= p^{-r}\varphi(v_w)q^{r-i}\varphi(\pi_\varepsilon^i) - v_w\varphi^{-1}(q)^{r-i}\pi_\varepsilon^i - y_i\pi_\varepsilon^i \\ &= \pi_\varepsilon^i(p^{-r}\varphi(v_w)q^{r-i}q^i - v_w\varphi^{-1}(q)^{r-i} - y_i) \\ &= \pi_\varepsilon^i(p^{-r}\varphi(v_w)q^r - (\varphi(w)^r\varphi(v_w) - y_i) - y_i) \\ &= \pi_\varepsilon^i\varphi(v_w)(p^{-r}q^r - \varphi(w)^r). \end{aligned}$$

If $r = 0$, this is zero regardless of choice of w; otherwise, note that this is divisible by $p^{-1}q - \varphi(w)$ and it suffices to find a $w \in \mathrm{W}(\mathcal{O}_C^\flat)^\times$ such that $p^{-1}q - \varphi(w) \in p\mathring{B}_\rho^+ + \pi_\varepsilon\mathring{B}_\rho^+$.

If $p = 2$, choose $w := 1$ and note that $q = 1 + [\varepsilon]$, so that

$$p^{-1}q - \varphi(w) = \frac{1+[\varepsilon]}{2} - 1 = \frac{[\varepsilon]-1}{2} = \pi_\varepsilon.$$

If on the other hand $p \neq 2$, choose $w := -u$ and recall that $\varphi^{-1}(q) = ([s]-p)u$, so that

$$p^{-1}q - \varphi(-u) = \varphi(p^{-1}\varphi^{-1}(q) + u) = \varphi(p^{-1}([s]-p)u + u) = \frac{[s^p]}{p}\varphi(u).$$

Then $|p^{-1}q - \varphi(-u)|_\rho = |\frac{[s^p]}{p}\varphi(u)|_\rho = |p|^{p-\frac{p}{p-1}} = |p|^{\frac{p}{p-1}(p-2)} \le \rho$ as required. $\square$

Corollary 3.53. *If C is algebraically closed, then for every $\rho \in (0;|p|]$ and $r \in \mathbb{N}$, the map $p^{-r}\varphi_\rho - \mathrm{id} : \mathrm{Fil}^r B_\rho^+ \to B_\rho^+$ is surjective.*

Proof. We know that this holds for $\rho = |p|^{p/(p-1)}$ by 3.52. If $0 < \sigma < \rho$, then for any $y \in B_\sigma^+ \subset B_\rho^+$ there exists an $x \in \mathrm{Fil}^r B_\rho^+$ with $p^{-r}\varphi_\rho(x) - x = y$. Note that $\varphi_\rho(x) \in B_{\rho^p}^+$, so if $\rho^p > \sigma$, we conclude $x = p^{-r}\varphi_\rho(x) - y \in B_{\rho^p}^+ + B_\sigma^+ = B_{\rho^p}^+$ and can continue like this until $\rho^{p^n} \le \sigma$ for some $n \in \mathbb{N}$. But then $x = p^{-r}\varphi_\rho(x) - y \in B_{\rho^{p^n}}^+ + B_\sigma^+ = B_\sigma^+$. $\square$

Proposition 3.54. *If C is algebraically closed, then for each $r \in \mathbb{Z}$, there is a short exact sequence of $\mathbb{Q}_p$-vector spaces*

$$0 \longrightarrow \mathbb{Q}_p t^r \longrightarrow \mathrm{Fil}^r B_\bullet \xrightarrow{p^{-r}\varphi_\bullet - \mathrm{id}} B_\bullet \longrightarrow 0.$$

Proof. Injectivity is trivial. For exactness in the middle, the case $r = 0$ was the subject of Proposition 3.50; when $r \neq 0$, every $x \in \mathrm{Fil}^r B_\bullet$ with $p^{-r}\varphi_\bullet(x) - x = 0$ satisfies

$$(\varphi_\bullet - \mathrm{id})(t^{-r}x) = p^{-r}t^{-r}\varphi_\bullet(x) - t^{-r}x = t^{-r}(p^{-r}\varphi_\bullet(x) - x) = 0,$$

so $t^{-r}x \in \mathbb{Q}_p$ by the exactness part of the case $r = 0$. It only remains to show surjectivity.

We know that $p^{-s}\varphi_\rho - \mathrm{id} : \mathrm{Fil}^s B_\rho^+ \to B_\rho^+$ is surjective for all $s \in \mathbb{N}$ and $\rho \in (0;|p|]$ by 3.53. Therefore if $y \in B_{\mathrm{cris}}^+ \subset B_\rho^+$, there exists an $x \in \mathrm{Fil}^s B_\rho^+$ with $p^{-s}\varphi_\rho(x) - x = y$. Since $x = p^{-s}\varphi_\rho(x) - y \in B_{\rho^p}^+ + B_{\mathrm{cris}}^+ = B_{\mathrm{cris}}^+$, the map $p^{-s}\varphi_{\mathrm{cris}} - \mathrm{id} : \mathrm{Fil}^s B_{\mathrm{cris}}^+ \to B_{\mathrm{cris}}^+$ is surjective as well. Note that one can similarly derive the statement for B_ρ^+ from the statement for B_{cris}^+ if one takes e.g. [12, 6.25.2] as a starting point instead of 3.52.

Finally, let $y \in B_\bullet$. Then there exists an $s \in \mathbb{N}$ such that $t^s y \in B_\bullet^+$ and $r + s \in \mathbb{N}$, as well as an $x \in \mathrm{Fil}^{r+s} B_\bullet^+$ such that $p^{-(s+r)}\varphi_\rho(x) - x = t^s y$. Hence $t^{-s}x \in \mathrm{Fil}^r B_\bullet$ satisfies

$$(p^{-r}\varphi_\bullet - \mathrm{id})(t^{-s}x) = t^{-s}(p^{-(r+s)}\varphi_\bullet(x) - x) = t^{-s}t^s y = y. \qquad \square$$

Theorem 3.55 (Bloch-Katou sequence). *If C is algebraically closed, then there is an exact sequence of $\mathbb{Q}_p$-vector spaces*

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_\bullet^{\varphi=1} \longrightarrow B_{\mathrm{dR}}/B_{\mathrm{dR}}^+ \longrightarrow 0,$$

where the map $B_\bullet^{\varphi=1} \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$ is induced from the inclusion $B_\bullet \hookrightarrow B_{\mathrm{dR}}$.

Proof. Injectivity is clear; for exactness in the middle, use 3.54 and apply the snake lemma to

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathbb{Q}_p \arrow[r] \arrow[d] & \operatorname{Fil}^0 B_\bullet \arrow[r, "\varphi_\bullet - \mathrm{id}"] \arrow[d] & B_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_\bullet^{\varphi=1} \arrow[r] \arrow[d] & B_\bullet \arrow[r, "\varphi_\bullet - \mathrm{id}"] \arrow[d] & B_\bullet \arrow[d] & \\
 & B_\bullet^{\varphi=1}/\mathbb{Q}_p & B_\bullet/(\operatorname{Fil}^0 B_\bullet) & 0 &
\end{tikzcd}$$

to conclude $B_\bullet^{\varphi=1}/\mathbb{Q}_p \cong B_\bullet/(B_\bullet \cap B_{\mathrm{dR}}^+)$, where the isomorphism is clearly induced by $B_\bullet^{\varphi=1} \hookrightarrow B_\bullet$. It now suffices to show that $B_\bullet \to B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$ is surjective. Since

$$B_{\mathrm{dR}}^+ = \theta_{\mathrm{dR}}^{-1}(\theta_{\mathrm{dR}}(B^{b,+})) = \ker \theta_{\mathrm{dR}} + B^{b,+} \subset tB_{\mathrm{dR}}^+ + B_\bullet,$$

we see inductively that $t^{-n}B_{\mathrm{dR}}^+ \subset B_{\mathrm{dR}}^+ + B_\bullet$ for all $n \in \mathbb{N}$, i.e. $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+ + B_\bullet$. $\square$

4 The ring of crystalline periods B_{cris}

We return to the setting of §1.3. Let K be a *p-adic field*, i.e. a non-archimedean complete discretely valued field with perfect residue field k such that $\operatorname{char} K = 0$ and $\operatorname{char} k = p$. We fix an algebraic closure $\overline{K}$ and write $C = \widehat{\overline{K}}$ for its completion, which is again algebraically closed by Krasner's lemma. By construction, the field C is perfect and complete, so since $|C| = |\overline{K}| \subset \mathbb{R}_0^+$ is already dense, C is perfectoid. We thus have associated rings $B^{b,+}$, B_ρ^+, B_ρ, B^+, B, A^0_{cris}, A_{cris}, B^+_{cris}, B_{cris}, B^+_{dR} and B_{dR} as in §3.3 and keep the notation $B_\bullet$ from 3.44.

Denote the residue field of $\mathcal{O}_{\overline{K}}$ by $\overline{k}$. This is an algebraic closure of k and the residue field of the completed maximal unramified extension $\widehat{K^{\mathrm{ur}}} \subset C$, which is another p-adic field that induces the same C and whose absolute Galois group $\operatorname{Gal}(\widehat{K^{\mathrm{ur}}})$ is canonically isomorphic to the inertia group I of $G := \operatorname{Gal}(K)$.

If we write $K_0 := \mathrm{W}(k)[\frac{1}{p}]$ for the *maximal unramified subfield* of K, then K/K_0 is a finite and totally ramified extension. This is well-known if $K/\mathbb{Q}_p$ is finite, but works in the general p-adic field setting too, see e.g. the remark following [2, 4.2.3]. Writing $e := [K : K_0]$, we have $K = K_0(\pi)$ for some $\pi \in \mathcal{O}_K$ that is the root of a monic Eisenstein polynomial $\sum_{i=0}^{e} a_i X^i$ over $\mathcal{O}_{K_0} = \mathrm{W}(k)$, i.e. $a_e = 1$, $a_i \in p\mathrm{W}(k)$ for all $0 \le i < e$ and $a_0 \notin p^2\mathrm{W}(k)$.

Remark 4.1. The group $G = \operatorname{Gal}(K)$ acts on the following rings:

Acts on	Induced from	Induced by
C	$\overline{K}$	completion
$C^\flat$	C	functoriality of $\cdot^\flat$ and localization
$B^{b,+}$	$\mathcal{O}_C^\flat$	Witt vector functor and localization
B_ρ^+	$B^{b,+}$	$\lvert\cdot\rvert_\rho$-isometric action on $B^{b,+}$
B^+	B_ρ^+	G-equivariance of $\iota_{\sigma,\rho}$
B_{dR}	$B^{b,+}$	Proposition 1.3 (iii) and localization
A_{cris}, B^+_{cris}	$\mathrm{W}(\mathcal{O}_C^\flat)$	$\lvert g.[s]\rvert_\flat = \lvert[g.s]\rvert_\flat = \lvert p\rvert$
$B_\bullet$	$B_\bullet^+$	$g.t_\varepsilon = \log([\varepsilon^{\chi(g)}]) = \chi(g)t_\varepsilon$

It is easy to see that the various $\theta_\bullet$, $\varphi_\bullet$, and all inclusions between the rings above are G-equivariant since this can either be checked on $\mathrm{W}(\mathcal{O}_C^\flat)$ or follows from Proposition 1.3 (iii). Since $g.t^n B^+_{\mathrm{dR}} = \chi(g)^n t^n B^+_{\mathrm{dR}} \subset t^n B^+_{\mathrm{dR}}$, the G-action is compatible with the filtration on $B_\bullet$.

Proposition 4.2. *$\mathcal{O}_C$ is a $\mathrm{W}(\overline{k})$-algebra; $\mathcal{O}_C^\flat$ is a $\overline{k}$-algebra; $\mathrm{W}(\mathcal{O}_C^\flat)$ is a $\mathrm{W}(\overline{k})$-algebra. Every Gauß norm $\lvert\cdot\rvert_\rho$ restricts to the p-adic absolute value normalized to $\lvert p\rvert = \rho$ on $\mathrm{W}(\overline{k}) \subset \mathrm{W}(\mathcal{O}_C^\flat)$.*

Proof. Recall that $\overline{k}$ is the residue field of the completed maximal unramified extension $\widehat{K_0^{\mathrm{ur}}} \subset C$ of K_0. Since $\widehat{K_0^{\mathrm{ur}}}$ is a p-adic field, there is an inclusion $\mathrm{W}(\overline{k}) = \mathcal{O}_{\widehat{K_0^{\mathrm{ur}}}} \hookrightarrow \mathcal{O}_C$. The other two algebra structures arise through the tilting and Witt functors. The final remark follows from $\overline{k}^\times \subset (\mathcal{O}_C^\flat)^\times$. □

4.1 $B_\rho^+[\frac{1}{t}]$ and B_{cris} as equivalent period rings

Before we can apply the theory of period rings to $B_\rho^+[\frac{1}{t}]$ and B_{cris}, we need to show that these rings are $(\mathbb{Q}_p, G)$-regular. Unlike in the Hodge-Tate or de Rham cases, this is significantly more technical to establish.

Lemma 4.3. *Let $\rho = |p|$ and extend the Gauß norm $|\cdot|_\rho$ to* $\operatorname{Frac} B_\rho^+$ *by multiplicativity. For $x = \sum_{i=0}^{e-1} \pi^i x_i \in K \otimes_{K_0} \operatorname{Frac} B_\rho^+ = \bigoplus_{i=0}^{e-1} \pi^i \operatorname{Frac} B_\rho^+$, let*

$$|x|_\otimes := \max_{0 \le i < e} \rho^{i/e} |x_i|_\rho.$$

Then $|\cdot|_\otimes$ is an absolute value. More precisely, we show:

(i) *$|\cdot|_\otimes$ is a submultiplicative* $\operatorname{Frac} B_\rho^+$*-norm.*

(ii) *$|\cdot|_\otimes$ is multiplicative on the subring $\bigoplus_{i=0}^{e-1} \pi^i \mathrm{W}(\mathcal{O}_C^\flat) \subset \bigoplus_{i=0}^{e-1} \pi^i \operatorname{Frac} B_\rho^+$.*

(iii) *$|\cdot|_\otimes$ is multiplicative on $K \otimes_{K_0} \operatorname{Frac} B_\rho^+$. In particular, $K \otimes_{K_0} \operatorname{Frac} B_\rho^+$ is a domain.*

Proof. (i): That $|\cdot|_\otimes$ is a $\operatorname{Frac} B_\rho^+$-norm is clear from the definition since $|\cdot|_\rho$ is an absolute value on $\operatorname{Frac} B_\rho^+$ and addition is componentwise. If $x = \sum_{i=0}^{e-1} \pi^i x_i \in \bigoplus_{i=0}^{e-1} \pi^i \operatorname{Frac} B_\rho^+$, then

$$|\pi x|_\otimes = \left| \sum_{i=1}^{e-1} \pi^i x_{i-1} + \sum_{i=0}^{e-1} \pi^i a_i x_{e-1} \right|_\otimes = \max\{|a_0 x_{e-1}|_\rho, \max_{1 \le i < e} \rho^{i/e} |x_{i-1} + a_i x_{e-1}|_\rho\}.$$

Since all $a_i \in p\mathrm{W}(k)$ and $a_0 \notin p^2\mathrm{W}(k)$, Remark 4.2 implies that $|a_0|_\rho = \rho$ and $|a_i|_\rho \le \rho$ for all $0 \le i < e$. Hence

$$\begin{aligned} |a_0 x_{e-1}|_\rho &= \rho |x_{e-1}|_\rho = |\pi|_\otimes \rho^{(e-1)/e} |x_{e-1}|_\rho \\ &\le |\pi|_\otimes |x|_\otimes, \\ \rho^{i/e} |x_{i-1} + a_i x_{e-1}|_\rho &\le \max\{\rho^{i/e} |x_{i-1}|_\rho,\ \rho^{i/e} |a_i x_{e-1}|_\rho\} \\ &\le |\pi|_\otimes \max\{\rho^{(i-1)/e} |x_{i-1}|_\rho,\ \rho^{i/e} \rho^{(e-1)/e} |x_{e-1}|_\rho\} \\ &\le |\pi|_\otimes |x|_\otimes, \end{aligned}$$

proving $|\pi x|_\otimes \le |\pi|_\otimes |x|_\otimes$. The strict triangle inequality then shows

$$|xy|_\otimes \le \max_{0 \le i < e} |x \pi^i y_i|_\otimes \le \max_{0 \le i < e} |x|_\otimes \rho^{i/e} |y_i|_\rho = |x|_\otimes |y|_\otimes$$

for all $x = \sum_{i=0}^{e-1} \pi^i x_i,\ y = \sum_{i=0}^{e-1} \pi^i y_i \in \bigoplus_{i=0}^{e-1} \pi^i \operatorname{Frac} B_\rho^+$.

(ii): Let $x_i = \sum_{j=0}^\infty p^j [x_i^{(j)}],\ y_i = \sum_{j=0}^\infty p^j [y_i^{(j)}] \in \mathrm{W}(\mathcal{O}_C^\flat)$ for all $0 \le i < e$ and set $x = \sum_{i=0}^{e-1} \pi^i x_i$, $y = \sum_{i=0}^{e-1} \pi^i y_i$. Without loss of generality, let $x, y \ne 0$. Choose the unique pair

$$(j_x, i_x) \in \{(j, i) \in \mathbb{N}_0 \times \{0, \dots, e-1\} \mid |x|_\otimes = \rho^{i/e} \rho^j |x_i^{(j)}|_\rho\}$$

that minimizes $ej_x + i_x$ and set $x' = \sum_{i=0}^{e-1} \pi^i \sum_{j \in \mathbb{N}_0, ej+i \ge ej_x+i_x} p^j [x_i^{(j)}]$. Since the condition $ej + i \ge ej_x + i_x$ is equivalent to $j \ge j_x + \lceil \frac{i_x - i}{e} \rceil$, we can explicitly calculate

$$|x - x'|_\otimes \overset{!}{=} \Big| \sum_{i=0}^{e-1} \pi^i \sum_{\substack{j \in \mathbb{N}_0 \\ ej+i < ej_x+i_x}} p^j [x_i^{(j)}] \Big|_\otimes < |\pi^{i_x} p^{j_x} [x_{i_x}^{(j_x)}]|_\otimes = |x|_\otimes.$$

Define i_y, j_y, y' analogously. Like in Proposition 3.4,

$$\begin{aligned} |xy - x'y'|_\otimes &= |(x - x')y - x'(y' - y)|_\otimes \\ &\le \max\{|(x - x')y|_\otimes, |x'(y' - y)|_\otimes\} \\ &\overset{\text{(i)}}{\le} \max\{|x - x'|_\otimes |y|_\otimes, |x'|_\otimes |y' - y|_\otimes\} \\ &< |x|_\otimes |y|_\otimes. \end{aligned}$$

It now suffices to show $|\pi^{i_x+i_y}p^{j_x+j_y}[x_{i_x}^{(j_x)}y_{i_y}^{(j_y)}]|_\otimes \le |x'y'|_\otimes$ since then

$$|xy - x'y'|_\otimes < |x|_\otimes |y|_\otimes = |\pi^{i_x+i_y}p^{j_x+j_y}[x_{i_x}^{(j_x)}y_{i_y}^{(j_y)}]|_\otimes \le |x'y'|_\otimes,$$

which implies $|xy|_\otimes \le |x|_\otimes |y|_\otimes \le |x'y'|_\otimes = |xy|_\otimes$.

For $z = \sum_{i=0}^{e-1} \pi^i z_i \in \bigoplus_{i=0}^{e-1} \pi^i \mathrm{W}(\mathcal{O}_C^\flat)$, define $\ell(z) = \min_{0\le i<e}(i + ev_p(z_i)) \in \mathbb{Z} \cup \{\infty\}$. Then the series representation in $\mathrm{W}(\mathcal{O}_C^\flat)$ implies the following for all z, z' and $0 \le i < e$:

$$\ell(pz) = \ell(z) + e, \qquad \ell(z+z') \ge \min\{\ell(z), \ell(z')\}, \qquad \ell(z_i) + i \ge \ell(z).$$

Furthermore, $\ell(\pi z) \ge \ell(z) + 1$ since

$$\ell(\pi z) = \ell(\sum_{i=1}^{e-1} \pi^i z_{i-1} + \sum_{i=0}^{e-1} \pi^i a_i z_{e-1}) \ge \min\{\ell(\sum_{i=1}^{e-1} \pi^i z_{i-1}), \ell(\sum_{i=0}^{e-1} \pi^i a_i z_{e-1})\},$$
$$\ell(\sum_{i=0}^{e-1} \pi^i a_i z_{e-1}) \ge \ell(\sum_{i=0}^{e-1} \pi^i p z_{e-1}) = \ell(p z_{e-1}) = \ell(z_{e-1}) + e \ge \ell(z) + 1,$$
$$\ell(\sum_{i=1}^{e-1} \pi^i z_{i-1}) \ge \ell(z) + 1.$$

Now note that whenever $ej + i < \ell(z)$ and $a \in \mathcal{O}_C^\flat$, we have

$$|z + \pi^i p^j [a]|_\otimes \ge |\pi^i (z_i + p^j[a])|_\otimes = \rho^{i/e} |z_i + p^j[a]|_\otimes \stackrel{!}{=} \rho^{i/e} |p^j[a]|_\otimes = |\pi^i p^j [a]|_\otimes.$$

Hence setting $x'' = x' - \pi^{i_x} p^{j_x} [x_{i_x}^{(j_x)}]$ and $y'' = y' - \pi^{i_y} p^{j_y} [y_{i_y}^{(j_y)}]$, we have

$$\begin{aligned}\ell(x'y' - \pi^{i_x+i_y}p^{j_x+j_y}[x_{i_x}^{(j_x)}y_{i_y}^{(j_y)}]) &= \ell(\pi^{i_x}p^{j_x}y'' + \pi^{i_y}p^{j_y}x'' + x''y'') \\ &\ge \min\{ei_x + j_x + \ell(y''),\ ei_y + j_y + \ell(x''),\ \ell(x''y'')\} \\ &> e(i_x + i_y) + j_x + j_y,\end{aligned}$$

which implies $|xy|_\otimes = |x'y'|_\otimes \ge |\pi^{i_x+i_y}p^{j_x+j_y}[x_{i_x}^{(j_x)}y_{i_y}^{(j_y)}]|_\otimes = |x|_\otimes |y|_\otimes$.

(iii): Since $|\cdot|_\otimes$ is a norm, eliminating denominators reduces the statement from $\bigoplus_{i=0}^{e-1} \pi^i \operatorname{Frac} B_\rho^+$ to $\bigoplus_{i=0}^{e-1} \pi^i B_\rho^+$. By part (i), multiplication is continuous with respect to $|\cdot|_\otimes$; from the definition of $|\cdot|_\otimes$ it readily follows that componentwise convergence implies convergence with respect to $|\cdot|_\otimes$, which further reduces the result from $\bigoplus_{i=0}^{e-1} \pi^i B_\rho^+$ to $\bigoplus_{i=0}^{e-1} \pi^i B^{b,+}$. Eliminating denominators again finally reduces to part (ii). □

Remark 4.4. Lemma 4.3 (ii) and the multiplicativity part of Proposition 3.4 are instances of the same general theorem regarding the *ramified Witt vectors* $\mathcal{O}_K \otimes_{\mathrm{W}(k)} \mathrm{W}(\mathcal{O}_C^\flat)$, which explains why their proofs are so similar. Ramified Witt vectors can be constructed in essentially the same way the normal Witt vectors are, using the uniformizer π instead of p and p^e-th powers instead of p-th powers. Over perfect coefficient rings they admit a unique series representation $\sum_{i=0}^\infty \pi^i [x_i]$ and one obtains Gauß norms and the results of §3.1 in a completely analogous way. The norm $|\cdot|_\otimes$ above is then the normal Gauß norm $|\cdot|_\rho$, although one should note that our $x_i^{(j)}$ is merely associated, not equal to x_{ej+i}.

This approach is chosen in [6], but it is difficult to fit into our conventional treatment without developing ramified Witt vectors in full detail because the Witt structure polynomials S_n, P_n, I_n, which the proof of Proposition 3.4 relies on, differ in the ramified case.

Proposition 4.5. *The map $K \otimes_{K_0} B_\bullet \to B_{\mathrm{dR}}$ is injective and G-equivariant. Equipping $K \otimes_{K_0} B_\bullet$ with the subspace filtration, we obtain $\operatorname{gr}(K \otimes_{K_0} B_\bullet) \cong \operatorname{gr} B_{\mathrm{dR}} \overset{1.27}{\cong} B_{\mathrm{HT}}$ on the level of graded rings.*

Proof. The map $K \otimes_{K_0} B_\bullet \to B_{\mathrm{dR}}$ factors into $K \otimes_{K_0} B_\bullet \hookrightarrow K \otimes_{K_0} \operatorname{Frac} B_\rho^+ \to B_{\mathrm{dR}}$, so it suffices to prove that $K \otimes_{K_0} \operatorname{Frac} B_\rho^+ \to B_{\mathrm{dR}}$ is injective. But $K \otimes_{K_0} \operatorname{Frac} B_\rho^+$ is a domain by Lemma 4.3, and since it is also a finite-dimensional $\operatorname{Frac} B_\rho^+$-algebra, this means it is a field. Hence $K \otimes_{K_0} \operatorname{Frac} B_\rho^+ \to B_{\mathrm{dR}}$ is a field homomorphism and necessarily injective. The map is equivariant because $\operatorname{Frac} B_\rho^+ \hookrightarrow B_{\mathrm{dR}}$ is equivariant.

The composition $\operatorname{Fil}^n(K \otimes_{K_0} B_\bullet) \hookrightarrow t^n B_{\mathrm{dR}}^+ \twoheadrightarrow t^n B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+ \cong C(\chi^n)$ is surjective since $\theta_{\mathrm{dR}}(B^{b,+}) = C$, and thus induces an isomorphism

$$\operatorname{gr}^n(K \otimes_{K_0} B_\bullet) = \operatorname{Fil}^n(K \otimes_{K_0} B_\bullet)/\operatorname{Fil}^{n+1}(K \otimes_{K_0} B_\bullet) \xrightarrow{\sim} t^n B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+ = \operatorname{gr}^n B_{\mathrm{dR}}. \qquad \square$$

Proposition 4.6. *We have $(\operatorname{Frac} B_\bullet)^G = B_\bullet^G = K_0$, and every $b \in B_\bullet \setminus 0$ such that $\mathbb{Q}_p b$ is G-stable is a unit in $B_\bullet$. In particular, $B_\bullet$ is $(\mathbb{Q}_p, G)$-regular.*

Proof. [2, 9.1.6] The injection $K \otimes_{K_0} B_\bullet \hookrightarrow B_{\mathrm{dR}}$ from Proposition 4.5 shows that

$$\dim_{K_0} B_\bullet^G = \dim_K K \otimes_{K_0} B_\bullet^G \le \dim_K (K \otimes_{K_0} B_\bullet)^G \overset{!}{\le} \dim_K B_{\mathrm{dR}}^G \overset{1.26}{=} 1.$$

But $B_\bullet^G$ contains at the very least the G-invariant canonical copy of $K_0 \subset B^+[\frac{1}{t}] \subset B_\bullet$, so $B_\bullet^G = K_0$. The equality $(\operatorname{Frac} B_\bullet)^G = K_0$ follows after localization of $K \otimes_{K_0} B_\bullet \hookrightarrow B_{\mathrm{dR}}$ by a similar argument; note that B_{dR} is a field and thus contains $\operatorname{Frac} B_\bullet$.

Let $b \in B_\bullet \setminus 0$ such that $G.\mathbb{Q}_p b \subset \mathbb{Q}_p b$. We will show that b is algebraic over $\widehat{K^{\mathrm{ur}}}$; then b is algebraic over $\widehat{K_0^{\mathrm{ur}}}$ as well. Since $L := \widehat{K_0^{\mathrm{ur}}}(b) \subset B_\bullet$ is a finite extension of a p-adic field, it is itself a p-adic field with $\widehat{\overline{L}} = C$. Since $\overline{k}$ is already algebraically closed, the maximal unramified subfield of L is $L_0 := \widehat{K_0^{\mathrm{ur}}}$. Applying 4.5 for the p-adic field L instead of K, we obtain an injection $L \otimes_{L_0} L \hookrightarrow L \otimes_{L_0} B_\bullet \overset{4.5}{\hookrightarrow} B_{\mathrm{dR}}$, which implies that $L \otimes_{L_0} L$ is a domain and hence a field. Therefore $x \otimes 1 - 1 \otimes x \in \ker(L \otimes_{L_0} L \to L)$ is zero for all $x \in L$, which means $L = L_0$, i.e. $b \in L^\times = L_0^\times \subset B_\bullet^\times$.

Since $t_\varepsilon \in B_\bullet^\times$ and $g.t_\varepsilon = \chi(g)t_\varepsilon \in \mathbb{Q}_p t_\varepsilon$ for all $\varepsilon \in \mathcal{U}$, we may multiply with a suitable power of t_ε and assume without loss of generality that $b \in B_{\mathrm{dR}}^+ \setminus \ker \theta_{\mathrm{dR}}$. Since by assumption $G.\mathbb{Q}_p^\times b \subset \mathbb{Q}_p^\times b$, there is a character $\eta : G \to \mathbb{Q}_p^\times$ such that $g.b = \eta(g)b$ for all $g \in G$. The residue class $\overline{b} \in C^\times$ of b spans a G-stable line $\mathbb{Q}_p\overline{b} \subset C$ whose corresponding character is also η because the quotient map $\theta_{\mathrm{dR}} : B_{\mathrm{dR}}^+ \twoheadrightarrow C$ is G-equivariant. Since $\overline{b} \in C$ is invertible, there is an isomorphism of topological groups $\mathbb{Q}_p^\times \overline{b} \cong \mathbb{Q}_p^\times$, which shows that $\eta : G \to \mathbb{Q}_p^\times$ is a *continuous* group homomorphism; hence it is $\mathbb{Z}_p^\times$-valued and all its powers are subject to Theorem 1.23.

Let $I \subset G$ be the inertia subgroup. Then we can use

$$\overline{b} \in C(\eta^{-1})^G = \{x \in C \mid g.x = \eta(g)x \text{ for all } g \in G\} \neq 0$$

to conclude that the group $\eta^{-1}(I)$ is finite, so there exists a finite Galois extension $M/\widehat{K^{\mathrm{ur}}}$ such that $J := \operatorname{Gal}(M) \subset \operatorname{Gal}(\widehat{K^{\mathrm{ur}}}) \cong I$ lies in the kernel of η^{-1}. But M is a p-adic field with $\widehat{\overline{M}} = C$, so Theorem 1.23 as applied to M shows that $\overline{b} \in C(\eta^{-1})^J = C^J = M$; notably, $\overline{b} \in C$ is algebraic over $\widehat{K^{\mathrm{ur}}}$, and by Hensel's Lemma, it admits a unique lift $\beta \in B_{\mathrm{dR}}^+$ which is algebraic over $\widehat{K^{\mathrm{ur}}} \subset B_{\mathrm{dR}}^+$. We claim that $b = \beta$.

First note that G acts on β via η as well: Since there exists an $x \in B_{\mathrm{dR}}^+$ such that $\beta = b + tx$, we have

$$\eta(g)^{-1} \cdot g.\beta = \eta(g)^{-1} \cdot g.(b + tx) = b + \eta(g)^{-1}\chi(g)t \cdot (g.x).$$

In particular, since $\eta(g)^{-1} \cdot g.\beta$ is algebraic over $\widehat{K^{\mathrm{ur}}}$ and the Hensel lift is G-equivariant (cf. proof of 1.25), the element $\eta(g)^{-1} \cdot g.\beta$ is another Hensel lift of $\overline{b}$ over $\widehat{K^{\mathrm{ur}}}$, hence equal to β. In other words, $g.\beta = \eta(g)\beta$.

Assume that $b \neq \beta$. Then there is a minimal $r \in \mathbb{N}_0$ with $b - \beta \in \mathrm{Fil}^r B_{\mathrm{dR}}$. As we have just seen, $\mathbb{Q}_p \cdot (b - \beta)$ is G-stable and again has η as its associated character. Now reduce modulo $\mathrm{Fil}^{r+1} B_{\mathrm{dR}}$ to obtain a non-trivial $\mathbb{Q}_p$-line in $\mathrm{Fil}^r B_{\mathrm{dR}} / \mathrm{Fil}^{r+1} B_{\mathrm{dR}} \overset{1.27}{=} C(\chi^r)$ where G acts via $\chi^r\eta$. Again we have $C((\chi^r\eta)^{-1})^I \neq 0$ because

$$b - \beta \in C((\chi^r\eta)^{-1})^I = \{x \in C \mid g.x = \chi^r(g)\eta(g)x \text{ for all } g \in G\} \neq 0.$$

Since $\widehat{K^{\mathrm{ur}}}$ is a p-adic field with completed algebraic closure C and Galois group I, $\chi^{-r}\eta^{-1}$ has finite image by Theorem 1.23. This is only possible if $r = 0$ because otherwise $\eta^{-1}(I)$ is finite and $\chi^{-r}(I)$ isn't. But $r = 0$ contradicts $\theta_{\mathrm{dR}}(b - \beta) = \overline{b} - \overline{b} = 0$; it follows that $b = \beta$, so that b is algebraic over $\widehat{K^{\mathrm{ur}}}$ as claimed. □

Corollary 4.7. *We obtain functors D_{cris} and D_ρ from $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cont}}(G)$ to the category $\mathrm{Vec}_{K_0}^f$ of finite-dimensional K_0-vector spaces. They are faithful, exact and commute with tensor products and duals.*

It turns out that in terms of admissibility, the choice of $B_\bullet$ does not matter.

Proposition 4.8. *Let $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cont}}(G)$. Then the following properties are equivalent:*

(i) *V is B_{cris}-admissible.*

(ii) *V is $B_\rho^+[\frac{1}{t}]$-admissible for some $\rho \in (0; |p|]$.*

(iii) *V is $B_\rho^+[\frac{1}{t}]$-admissible for all $\rho \in (0; |p|]$.*

We call a representation V satisfying these properties crystalline *and denote the full subcategory of crystalline representations of G by $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G)$. The functors D_ρ and D_{cris} are naturally isomorphic as functors $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G) \to \mathrm{Vec}_{K_0}$.*

Proof. (ii) $\iff$ (iii): Admissibility is a matter of K_0-dimensions. For all $0 < \sigma \leq \rho \leq |p|$, the K_0-linear inclusion $\iota_{\sigma,\rho} \otimes \mathrm{id}_V : (B_\sigma^+[\frac{1}{t}] \otimes_{\mathbb{Q}_p} V)^G \hookrightarrow (B_\rho^+[\frac{1}{t}] \otimes_{\mathbb{Q}_p} V)^G$ shows that $B_\sigma^+[\frac{1}{t}]$-admissible implies $B_\rho^+[\frac{1}{t}]$-admissible. On the other hand, $\varphi_\rho : B_\rho^+[\frac{1}{t}] \hookrightarrow B_{\rho^p}^+[\frac{1}{t}]$ is injective, so we have an injective homomorphism of abelian groups

$$\varphi_\rho \otimes \mathrm{id}_V : (B_\rho^+[\tfrac{1}{t}] \otimes_{\mathbb{Q}_p} V)^G \hookrightarrow (B_{\rho^p}^+[\tfrac{1}{t}] \otimes_{\mathbb{Q}_p} V)^G.$$

This is not K_0-linear, but if V is $B_\rho^+[\frac{1}{t}]$-admissible and $v_1, \dots, v_n \in (B_\rho^+[\frac{1}{t}] \otimes_{\mathbb{Q}_p} V)^G$ is a K_0-basis, then the images $(\varphi_\rho \otimes \mathrm{id}_V)(v_i)$ are still K_0-linearly independent because $\varphi : K_0 \xrightarrow{\sim} K_0$ is bijective and hence for all $\lambda_i \in K_0$,

$$(\varphi_\rho \otimes \mathrm{id}_V)\left(\sum_{i=1}^n \varphi^{-1}(\lambda_i)v_i\right) = \sum_{i=1}^n \lambda_i(\varphi_\rho \otimes \mathrm{id}_V)(v_i) = 0$$

implies $\lambda_i = 0$ for all $1 \leq i \leq n$. Therefore $B_\rho^+[\frac{1}{t}]$-admissible implies $B_{\rho^p}^+[\frac{1}{t}]$-admissible. Combining these two results yields the equivalence.

(i) $\iff$ (iii): Apply analogous arguments to the inclusions $B_{\rho^p}^+[\frac{1}{t}] \hookrightarrow B_{\mathrm{cris}} \hookrightarrow B_\rho^+[\frac{1}{t}]$ from Proposition 3.33.

The final assertion follows from the trivial observation that the maps $\iota_{\sigma,\rho} \otimes \mathrm{id}_V$, $(B_{|p|^p} \hookrightarrow B_{\mathrm{cris}}) \otimes \mathrm{id}_V$ and $(B_{\mathrm{cris}} \hookrightarrow B_{|p|}) \otimes \mathrm{id}_V$ are natural in V. $\square$

We will similarly write $D_\bullet$ when the exact choice of functor doesn't matter. As is usual for period ring functors, $D_\bullet$ takes values in a more structured category.

Definition 4.9. A *filtered ϕ-module over K* is an isocrystal D over K_0 such that the base change $D_K := K \otimes_{K_0} D$ is a filtered K-vector space. In other words, we have a triple $(D, \phi_D, \mathrm{Fil}^\bullet)$ where D is a K_0-vector space, $\phi_D : D \xrightarrow{\sim} D$ is an automorphism of abelian groups such that $\phi_D(k.d) = \varphi(k).\phi_D(d)$ for all $k \in K_0$ and $d \in D$ (ϕ_D is *Frobenius-semilinear*), and $\mathrm{Fil}^\bullet$ is a decreasing, separated and exhaustive filtration on D_K.

A *homomorphism of filtered ϕ-modules over K* is a K_0-linear map $f : D \to D'$ such that $f(\phi_D(d)) = \phi_{D'}(f(d))$ holds for all $d \in D$ and $f_K : D_K \to D'_K$ is a homomorphism of filtered K-vector spaces. f is called *strict* if the extension $f_K : D_K \to D'_K$ is strict, i.e. satisfies $f_K^{-1}(\mathrm{Fil}^n D'_K) = \mathrm{Fil}^n D_K$ for all $n \in \mathbb{Z}$.

Filtered ϕ-modules over K form a category which we denote by MF_K^ϕ, with an obvious forgetful functor $\mathrm{MF}_K^\phi \to \mathrm{MF}_K$ to the category of filtered K-vector spaces.

Lemma 4.10. *If $\phi : D \to D'$ is an injective Frobenius-semilinear map between K_0-vector spaces with $\dim_{K_0} D = \dim_{K_0} D' < \infty$, then ϕ is bijective.*

Proof. Since $\varphi : K_0 \xrightarrow{\sim} K_0$ is an automorphism, the image of ϕ is a K_0-linear subspace of D'. Hence it suffices to show that images of linearly independent elements remain linearly independent, which follows by the exact same argument as in the proof of 4.8. In fact, the map $\varphi_\rho \otimes \mathrm{id}$ we considered there was Frobenius-semilinear. $\square$

Remark 4.11. Like MF_K, the category MF_K^ϕ is additive and admits kernels and cokernels, given by the vector space $\ker f$ (respectively $\mathrm{coker}(f)$) with the induced Frobenius (which is bijective by 4.10) and the subspace (resp. quotient) filtration on the base change. It similarly fails to be abelian solely because the inverse of a bijective homomorphism need not respect the filtration after base change. This problem again does not exist for strict homomorphisms and we can nevertheless define a notion of exactness as in MF_K by saying that a sequence of filtered ϕ-modules

$$0 \to D' \to D \to D'' \to 0$$

is *exact* if the underlying sequence of vector spaces is exact and the sequence

$$0 \to D'_K \to D_K \to D''_K \to 0$$

is exact in MF_K, i.e. exact after applying any Fil^n or equivalently inducing the subspace filtration on D'_K and the quotient filtration on D''_K.

We also have a notion of tensor products and duals in MF_K^ϕ. The tensor product of $D, D' \in \mathrm{MF}_K^\phi$ is given by the vector space $D \otimes_{K_0} D'$, the Frobenius-semilinear map $\phi_{D \otimes D'} := \phi_D \otimes \phi_{D'}$, and the filtration

$$\mathrm{Fil}^n(D \otimes_{K_0} D')_K = \mathrm{Fil}^n(D_K \otimes_K D'_K) := \sum_{i+j=n} \mathrm{Fil}^i D_K \otimes \mathrm{Fil}^j D'_K.$$

The dual is given by the vector space $\mathrm{Hom}_{K_0}(D, K_0)$, the Frobenius-semilinear map $\phi_{D^*} := (f \mapsto \varphi^{-1} \circ f \circ \phi_D)$, and the usual dual filtration

$$\mathrm{Fil}^n(\mathrm{Hom}_{K_0}(D, K_0))_K = \mathrm{Fil}^n \mathrm{Hom}_K(D_K, K) := \{f : D_K \to K \mid \mathrm{Fil}^{1-n} D_K \subset \ker f\}.$$

We record the following basic result on filtered vector spaces for later use.

Lemma 4.12. *Let $V, W \in \mathrm{Fil}_K$ and $V' \subset V$ a subspace with the subspace filtration.*

(i) *If $f : V \xrightarrow{\sim} W$ is an isomorphism in Fil_K, then the restriction $f_{|V'} : V' \xrightarrow{\sim} f(V')$ is an isomorphism in Fil_K as well.*

(ii) *If $\dim_K W < \infty$, then $V' \otimes_K W$ carries the subspace filtration relative to $V \otimes_K W$.*

Proof. (i): Write $W' := f(V')$ and let $i_V : V' \hookrightarrow V$ and $i_W : W' \hookrightarrow W$ be the canonical inclusions. Then we have $i_W \circ f_{|V'} = f \circ i_V$, so

$$f_{|V'}(\mathrm{Fil}^n V') = i_W^{-1}(i_W(f_{|V'}(\mathrm{Fil}^n V'))) = i_W^{-1}(f(i_V(\mathrm{Fil}^n V'))) \subset i_W^{-1}(\mathrm{Fil}^n W) = \mathrm{Fil}^n W',$$

$$f_{|V'}^{-1}(\mathrm{Fil}^n W') = f_{|V'}^{-1}(i_W^{-1}(\mathrm{Fil}^n W)) = i_V^{-1}(f^{-1}(\mathrm{Fil}^n W)) = i_V^{-1}(\mathrm{Fil}^n V) = \mathrm{Fil}^n V'.$$

(ii): Let $\iota : V' \hookrightarrow V$ be the inclusion. Since W is finite-dimensional, there exists a basis $w_1, \dots, w_n \in W$ of W such that $\{w_1, \dots, w_n\} \cap \mathrm{Fil}^k W$ is a basis of $\mathrm{Fil}^k W$ for all $k \in \mathbb{Z}$. For each $1 \le j \le n$, let $i_j := \max\{k \in \mathbb{Z} \mid w_j \in \mathrm{Fil}^k W\}$. Since $\iota \otimes \mathrm{id}_W$ is injective,

$$\begin{aligned}
(\iota \otimes \mathrm{id}_W)^{-1}(\mathrm{Fil}^k(V \otimes_K W)) &= (\iota \otimes \mathrm{id}_W)^{-1}\left(\sum_{j=1}^n (\mathrm{Fil}^{k-i_j} V \otimes_K \mathrm{Fil}^{i_j} W)\right) \\
&= \sum_{j=1}^n (\iota \otimes \mathrm{id}_W)^{-1}\left(\bigoplus_{l=1}^j (\mathrm{Fil}^{k-i_j} V \otimes_K K w_j)\right) \\
&\overset{!}{=} \sum_{j=1}^n \bigoplus_{l=1}^j (\iota \otimes \mathrm{id}_W)^{-1}(\mathrm{Fil}^{k-i_j} V \otimes_K K w_j) \\
&= \sum_{j=1}^n \bigoplus_{l=1}^j \mathrm{Fil}^{k-i_j} V' \otimes K w_j \\
&= \mathrm{Fil}^k(V' \otimes W).
\end{aligned}$$

□

Proposition 4.13. *Let $V, W \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cont}}(G)$ and let $f : V \to W$ be a homomorphism of p-adic representations.*

(i) *$D_\bullet(V) = (B_\bullet \otimes_{\mathbb{Q}_p} V)^G$ is a filtered ϕ-module when equipped with $\phi = \varphi_\bullet \otimes \mathrm{id}_V$ and the subspace filtration relative to*

$$K \otimes_{K_0} (B_\bullet \otimes_{\mathbb{Q}_p} V)^G \subset (K \otimes_{K_0} B_\bullet \otimes_{\mathbb{Q}_p} V)^G \overset{4.5}{\hookrightarrow} (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^G = D_{\mathrm{dR}}(V).$$

$D_\bullet(f) = \mathrm{id}_{B_\bullet} \otimes_{\mathbb{Q}_p} f$ is furthermore a homomorphism of filtered ϕ-modules over K.

(ii) *After restriction to $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}$, the natural isomorphisms between D_{cris} and all D_ρ of functors $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}} \to \mathrm{Vec}_{K_0}^f$ from Proposition 4.8 are also natural isomorphisms of functors $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}} \to \mathrm{MF}_K^\phi$.*

Proof. (i): $\varphi_\bullet$ is G-equivariant, so ϕ indeed restricts to an injective Frobenius-semilinear map $D_\bullet(V) \to D_\bullet(V)$, which is bijective by 4.10. $D_\bullet(f)$ is compatible with ϕ because $\phi \circ D_\bullet(f) = (\varphi_\bullet \otimes f) = D_\bullet(f) \circ \phi$; compatibility with filtrations follows from the fact that $D_\bullet(f)_K$ and $D_{\mathrm{dR}}(f)$ are both restrictions of $\mathrm{id}_K \otimes \mathrm{id}_{B_\bullet} \otimes f$.

(ii): We need to show that the inverses of the components of the natural isomorphism (which are base changes of the canonical inclusions) commute with ϕ and respect filtrations after tensoring with K. Compatibility with ϕ readily follows from 3.19 and 3.33 (ii). Compatibility with the filtrations follows from the fact that the base changes of the canonical inclusions induce the subspace filtration, are therefore strict, and hence are isomorphisms if and only if they are bijective. □

Corollary 4.14. *For all $V \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G)$, there is an isomorphism of filtered K-vector spaces $K \otimes_{K_0} D_\bullet(V) \cong D_{\mathrm{dR}}(V)$ which is functorial in V. In particular, crystalline representations are de Rham and $K \otimes_{K_0} D_\bullet(\cdot) \cong D_{\mathrm{dR}}$ as functors $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G) \to \mathrm{Fil}_K$.*

Proof. The map $K \otimes_{K_0} D_\bullet(V) \hookrightarrow D_{\mathrm{dR}}(V)$ is clearly functorial in V and a strict homomorphism of filtered modules over K since it induces the filtration on $K \otimes_{K_0} D_\bullet(V)$. Therefore it suffices to verify

$$\begin{aligned}
\dim_{\mathbb{Q}_p} V &= \dim_{K_0} D_\bullet(V) \\
&= \dim_K (K \otimes_{K_0} D_\bullet(V)) \\
&\leq \dim_K (K \otimes_{K_0} B_\bullet \otimes_{\mathbb{Q}_p} V)^G \\
&\leq \dim_K (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^G \\
&= \dim_K D_{\mathrm{dR}}(V) \\
&\leq \dim_{\mathbb{Q}_p} V.
\end{aligned}$$
□

Corollary 4.15. *The functor $D_\bullet : \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G) \to \mathrm{MF}^{\phi}_K$ is exact and commutes with duals and tensor products.*

Proof. Combine 4.7, 4.14 and 1.28. □

4.2 Properties of crystalline representations

We close with an overview of assorted results on crystalline representations, largely based on [2, §8, §9.3]. Having established that our notions agree with the customary ones, we do not prove all of these properties in detail and refer to the literature instead.

Proposition 4.16. *A representation $V \in \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Q}_p}(\mathrm{Gal}(K))$ is crystalline if and only if it is crystalline as a representation of $\mathrm{Gal}(\widehat{K^{\mathrm{ur}}})$.*

Proof. Denote the $D_\bullet$ functors for K and $\widehat{K^{\mathrm{ur}}}$ by $D^K_\bullet$ and $D^{\widehat{K^{\mathrm{ur}}}}_\bullet$ respectively. Since $\widehat{K^{\mathrm{ur}}_0} \otimes_{K_0} D^K_\bullet(V) \to D^{\widehat{K^{\mathrm{ur}}}}_\bullet(V)$ is easily seen to be compatible with the Frobenius, it suffices to see that its base change is an isomorphism of filtered $\widehat{K^{\mathrm{ur}}}$-vector spaces. This is proven in a similar way to the de Rham case (see e.g. [2, 6.3.8] and [2, 2.4.6]), by reducing to the fact that finite-dimensional K_0-subspaces of $B^+_\bullet$ obtain their natural K_0-vector space topology as the subspace topology. Since we may choose $B^+_\bullet = B^+_\rho$ for some $\rho \in (0; |p|]$, this follows from the fact that $|\cdot|_\rho$ induces the p-adic topology on K_0. □

The analogue of 4.16 for B_{dR} holds for arbitrary (notably ramified) finite extensions L/K as well, by reducing to the Galois case and applying the classic Galois descent isomorphism $L \otimes_K D^K_{\mathrm{dR}}(V) = L \otimes_K D^L_{\mathrm{dR}}(V)^{\mathrm{Gal}(L/K)} \cong D^L_{\mathrm{dR}}(V)$. This argument does not extend to the crystalline case and indeed being crystalline need not be checkable after under such extensions, as we will see in 4.19.

Proposition 4.17. *A representation $V \in \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Q}_p}(G)$ is unramified if and only if it is crystalline and potentially unramified.*

Proof. An unramified representation is trivially potentially unramified and by 4.16 it is crystalline. On the other hand, let V be crystalline and let L/K be a finite Galois extension such that V is unramified as a representation of $\mathrm{Gal}(L)$. By 4.16, we may assume that k is algebraically closed, so that $B_\bullet^{\mathrm{Gal}(L)} = L_0 = K_0$; then

$$\begin{aligned} D^K_\bullet(V) &= D^L_\bullet(V)^{\mathrm{Gal}(L/K)} \\ &\overset{1.20}{\subset} (B_\bullet^{\mathrm{Gal}(L)} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}(L/K)} \\ &\overset{!}{=} (K_0 \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}(L/K)} \\ &= K_0 \otimes_{\mathbb{Q}_p} V^{\mathrm{Gal}(L/K)}, \end{aligned}$$

so a dimension comparison shows $V = V^{\mathrm{Gal}(L/K)}$. But then the action of the inertia group of $\mathrm{Gal}(K)$ must factor through the inertia group of $\mathrm{Gal}(L)$, i.e. V is unramified. □

Corollary 4.18. *A one-dimensional continuous representation $\rho : G \to \mathbb{Q}_p^\times$ is crystalline with Hodge-Tate weight $n \in \mathbb{Z}$ if and only if $\rho\chi^n$ is unramified.*

Proof. Every χ^{-n} is crystalline since $t^n \in D_\bullet(\chi^{-n}) \cong \{x \in B_\bullet \mid g.x = \chi(g)^n \cdot x\}$. After multiplying with χ^n, it suffices to show that ρ is crystalline with Hodge-Tate weight 0 if and only if ρ is unramified; but this follows from 4.17 and Theorem 1.23. □

In particular, no finitely ramified character corresponds to a crystalline representation. However, since the de Rham property is insensitive to finite extension, finitely ramified characters correspond to de Rham representations, showing that the converse of 4.14 does not hold. We can also now illustrate how being crystalline cannot be checked on finite ramified extensions.

Example 4.19. Let L/K be a finite and ramified Galois extension. Then the group ring $\mathbb{Q}_p[\mathrm{Gal}(L/K)]$ is a representation $V \in \mathrm{Rep}^{\mathrm{cont}}_{\mathbb{Q}_p}(G)$. The induced representation for $\mathrm{Gal}(L)$ is trivial (notably unramified and crystalline), so V is potentially unramified; but by Proposition 4.17, V can only be crystalline if it is unramified, which is clearly false.

A major defect of the functors D_{HT} and D_{dR} was that they are faithful, but not full, making it difficult to find a simple description of the categories $\mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(G)$ and $\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G)$. This situation is remarkably different for the functor D_{cris}.

Proposition 4.20. *The functor $D_\bullet : \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G) \to \mathrm{MF}^\phi_K$ is fully faithful. The pseudoinverse on its essential image is $V_\bullet : \mathrm{MF}^\phi_K \to \mathrm{Mod}_{\mathbb{Q}_p[G]}$ with*

$$\begin{aligned} V_\bullet(D) &:= (B_\bullet \otimes_{K_0} D)^{\varphi=1} \cap \mathrm{Fil}^0(B_\bullet \otimes_{K_0} D)_K, \\ V_\bullet(f : D \to D') &:= (\mathrm{id}_{B_\bullet} \otimes_{K_0} f)_{|V_\bullet(D)}, \end{aligned}$$

where $(B_\bullet \otimes_{K_0} D)^{\varphi=1} := \{x \in B_\bullet \otimes_{K_0} D \mid (\varphi_\bullet \otimes \phi_D)(x) = x\}$.

Proof. The ring $B_\bullet$ is a filtered ϕ-module over K via $\varphi_\bullet$ and the subspace filtration for $B_\bullet \otimes_{K_0} K \subset B_{\mathrm{dR}}$, so the above definition makes sense. Recall that for all $V \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G)$ there is a functorial isomorphism $\alpha_V : B_\bullet \otimes_{K_0} D_\bullet(V) \xrightarrow{\sim} B_\bullet \otimes_{\mathbb{Q}_p} V$ of $B_\bullet$-semilinear representations by the general theory of period rings and note that $B_\bullet \otimes_{\mathbb{Q}_p} V \in \mathrm{MF}^\phi_K$ via $\varphi_\bullet \otimes \mathrm{id}_V$ and $\mathrm{Fil}^n(K \otimes_{K_0} B_\bullet) \otimes_{\mathbb{Q}_p} V$. We claim that α_V is an isomorphism in MF^ϕ_K, so that there is a natural isomorphism

$$V_\bullet(D_\bullet(V)) \overset{!}{\cong} (\mathrm{Fil}^0 B_\bullet)^{\varphi=1} \otimes_{\mathbb{Q}_p} V \overset{3.50}{=} \mathbb{Q}_p \otimes_{\mathbb{Q}_p} V \cong V.$$

For $b \in B_\bullet$ and $v = \sum_{i=1}^n b_i \otimes v_i \in (B_\bullet \otimes V)^G \subset B_\bullet \otimes V$, we indeed have

$$(\varphi_\bullet \otimes \mathrm{id}_V)(\alpha_V(b \otimes v)) = (\varphi_\bullet \otimes \mathrm{id}_V)\left(\sum_{i=1}^n bb_i \otimes v_i\right) = \sum_{i=1}^n \varphi_\bullet(b)\varphi_\bullet(b_i) \otimes v_i,$$
$$\alpha_V((\varphi_\bullet \otimes \phi_{D_\bullet(V)})(\varphi_\bullet(b) \otimes v) = \alpha_V\left(\varphi_\bullet(b) \otimes \sum_{i=1}^n \varphi_\bullet(b_i) \otimes v_i\right) = \varphi_\bullet(b)\sum_{i=1}^n \varphi_\bullet(b_i) \otimes v_i.$$

We write $B_{\bullet,K} := B_\bullet \otimes_{K_0} K$. By 4.12 (ii), $B_{\bullet,K} \otimes_{\mathbb{Q}_p} V$ carries the subspace filtration with respect to $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ and $(B_\bullet \otimes_{K_0} D_\bullet(V))_K \cong B_{\bullet,K} \otimes_K D_{\mathrm{dR}}(V)$ carries the subspace filtration with respect to $B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V)$. That $\alpha_{V,K} : B_{\bullet,K} \otimes_K D_{\mathrm{dR}}(V) \xrightarrow{\sim} B_{\bullet,K} \otimes_{\mathbb{Q}_p} V$ is an isomorphism in Fil_K therefore follows from 4.12 (i) and the fact that $\alpha_{V,K}$ is the restriction of the filtered isomorphism $\alpha_{V,\mathrm{dR}} : B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$ from 1.29.

Let $V, V' \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G)$, $D := D_\bullet(V), D' := D_\bullet(V') \in \mathrm{MF}^\phi_K$, and $f : D \to D'$ a homomorphism in MF^ϕ_K. Denote by $\alpha_V, \alpha_{V'}$ the crystalline comparison isomorphisms and let

$$f' = \alpha_{V'} \circ f \circ \alpha_V^{-1} : B_\bullet \otimes_{K_0} V \to B_\bullet \otimes_{K_0} V'.$$

By unwinding definitions, one sees that f' is a base change of a homomorphism $V \to V'$; applying $V_\bullet$ then results in the original map f. $\square$

One might therefore hope to explicitly describe the essential image of $D_\bullet$ to obtain an equivalence of categories. This is in fact possible; the idea is that the generally unrelated Frobenius and filtration should satisfy some kind of compatibility condition.

Definition 4.21. Let $D \in \mathrm{MF}^\phi_K$, $\dim_{K_0} D = d < \infty$ and $\det D := \bigwedge^d D \in \mathrm{MF}^\phi_K$.

(i) The *Hodge number* of $D \neq 0$ is $t_H(D) := \max\{i \in \mathbb{Z} \mid \mathrm{Fil}^i(\det D)_K \neq 0\}$. For $D = 0$, we set $t_H(D) = 0$.

(ii) The *Newton number* of $D \neq 0$ is the uniquely determined valuation $v_p(\lambda)$, where $\lambda \in K_0$ is such that $\phi_{\det D}(x) := (\wedge^d \phi_D)(x) \overset{!}{=} \lambda x$ for some $x \in \det D \setminus 0$. For $D = 0$, we set $t_N(D) = 0$.

(iii) We call $D \in \mathrm{MF}^\phi_K$ *weakly admissible* if $t_H(D) = t_N(D)$ and $t_H(D') \leq t_N(D')$ for all subobjects $D' \in \mathrm{MF}^\phi_K$ of D.

Note that t_H and t_N are really attached to the underlying filtered K-vector space and K_0-isocrystal respectively. Both numbers are well-defined; on the one hand we have $\mathrm{Fil}^{dn}(\det D)_K = 0$ for any $n \in \mathbb{Z}$ with $\mathrm{Fil}^n D_K = 0$, and on the other hand we have

$\varphi(\mu)/\mu \in \mathrm{W}(k)^\times \subset K_0$ for all $\mu \in K_0^\times$, so that $\phi_{\det D}(\mu x) = \varphi(\mu)\phi_{\det D}(x) = \lambda\frac{\varphi(\mu)}{\mu}\mu x$ still gives the same Newton number $v_p(\lambda\frac{\varphi(\mu)}{\mu}) = v_p(\lambda)$.

Another commonly used definition of Hodge and Newton numbers makes use of the *Dieudonné-Manin classification* (cf. [5]), which states that any K_0-isocrystal D decomposes into a direct sum $D = \bigoplus_{\alpha\in\mathbb{Q}} D_\alpha$, where each D_α is *pure of slope* α, i.e. admits a $\mathbb{Z}_p$-lattice $M \subset D_\alpha$ such that $p^{-s}\phi_{D_\alpha}^r(M) = M$ and $\alpha = \frac{s}{r}$, where $\gcd(s,r) = 1$ and $r > 0$.

Equivalently, $\widehat{D_\alpha} := \widehat{K_0^{\mathrm{ur}}} \otimes_{K_0} D_\alpha$ with the filtration $\mathrm{Fil}^n \widehat{D}_{\widehat{K^{\mathrm{ur}}}} := \mathrm{Fil}^n D_K \otimes_K \widehat{K^{\mathrm{ur}}}$ and the Frobenius map $\phi_D \otimes \mathrm{id}_{\widehat{K_0^{\mathrm{ur}}}}$ is isomorphic to a direct sum of finitely many copies of the $\widehat{K_0^{\mathrm{ur}}}$-isocrystal $\Delta_\alpha := \bigoplus_{i=0}^{r-1} \widehat{K_0^{\mathrm{ur}}} \cdot F^i$, whose Frobenius is

$$\phi_{\Delta_\alpha}\left(\sum_{i=0}^{r-1} a_i F^i\right) = \varphi(a_{r-1})p^s F^0 + \sum_{i=1}^{r-1} \varphi(a_{i-1})F^i.$$

It is easily seen that $\dim_{\widehat{K_0^{\mathrm{ur}}}} \Delta_\alpha = r$ and $t_N(\Delta_\alpha) = s$.

Proposition 4.22. *For all $D \in \mathrm{MF}_K^\phi$ with $\dim_{K_0} D < \infty$, we have $t_H(D) = t_H(\widehat{D})$ and $t_N(D) = t_N(\widehat{D})$.*

Proof. Since $\widehat{\det D} \cong \det \widehat{D}$, this reduces to the one-dimensional case, which is trivial. □

Proposition 4.23. *Let $D \in \mathrm{MF}_K^\phi$ be finite-dimensional.*

(i) $t_H(D) = \sum_{i\in\mathbb{Z}} i \cdot \dim_K(\mathrm{Fil}^i D_K / \mathrm{Fil}^{i+1} D_K)$.

(ii) $t_N(D) = \sum_{\alpha\in\mathbb{Q}} \alpha \cdot \dim_{K_0} D_\alpha = \sum_{\alpha\in\mathbb{Q}} \alpha \cdot \dim_{\widehat{K_0^{\mathrm{ur}}}} \widehat{D_\alpha}$.

Proof. The case $D = 0$ is trivial, so assume $D \neq 0$.

(i): Clearly $t_H(D) = t_H(\det D) = \max\{\sum_{j=1}^d i_j \mid \mathrm{Fil}^{i_1} D_K \wedge \ldots \wedge \mathrm{Fil}^{i_d} D_K \neq 0\}$. Note that $\mathrm{Fil}^{i_1} D_K \wedge \ldots \wedge \mathrm{Fil}^{i_d} D_K \neq 0$ if and only if all $\mathrm{Fil}^{i_j} D_K \neq 0$ and there exists a linearly independent choice of vectors $v_j \in \mathrm{Fil}^{i_j} D_K$. The choice of i_j and v_j that realizes the maximum must necessarily satisfy $v_j \in \mathrm{Fil}^{i_j} D_K \setminus \mathrm{Fil}^{i_j+1} D_K$, which implies the formula.

(ii): By 4.22, we can consider $t_N(\widehat{D})$ instead. Note that the canonical isomorphism $\det(\bigoplus_{i=1}^m \Delta_{\alpha_i}) \cong \bigotimes_{i=1}^m \det \Delta_{\alpha_i}$ is compatible with the Frobenius. If $\lambda_1, \ldots, \lambda_m \in \widehat{K_0^{\mathrm{ur}}}$ are such that $\phi_{\det \Delta_{\alpha_i}} = (v \mapsto \lambda_i v)$, then $\otimes_{i=1}^m \phi_{\det \Delta_{\alpha_i}} = (v \mapsto (\prod_{i=1}^m \lambda_i)v)$. Hence it suffices to consider the case $D = \Delta_\alpha$ for some $\alpha = \frac{s}{r} \in \mathbb{Q}$ with $\gcd(s,r) = 1$ and $r > 0$, which indeed satisfies $t_N(\Delta_\alpha) = s = \frac{s}{r} \cdot \dim_{\widehat{K_0^{\mathrm{ur}}}} \Delta_\alpha$ by our previous remarks. □

The main result is now the following theorem due to Colmez and Fontaine.

Theorem 4.24. *For each $D \in \mathrm{MF}_K^\phi$, the following properties are equivalent:*

(i) *D is* admissible*, i.e. there is a $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G)$ such that $D \cong D_{\mathrm{cris}}(V)$.*

(ii) *D is* weakly admissible*, i.e. $\dim_{K_0} D < \infty$, $t_H(D) = t_N(D)$ and $t_H(D') \leq t_N(D')$ for all subobjects $D' < D$.*

In particular, if we denote the full subcategory of weakly admissible filtered ϕ-modules by $\mathrm{MF}_K^{\phi,\mathrm{wadm}}$, the functor $D_\bullet : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G) \to \mathrm{MF}_K^{\phi,\mathrm{wadm}}$ is an equivalence of categories and its pseudoinverse is the functor $V_\bullet$ from 4.20.

Proof ((i) $\Longrightarrow$ (ii)). [2, 9.3.4] By 4.16, V is crystalline as a representation of $\mathrm{Gal}(\widehat{K^{\mathrm{ur}}})$; by 4.22, Hodge and Newton numbers can be computed on $\widehat{D}$. Since all subobjects of D give rise to subobjects of $\widehat{D}$ after base change, it suffices to consider the case where k is algebraically closed.

By unwinding definitions, one derives from 4.15 that $D_\bullet$ and exterior powers commute. Since k is algebraically closed, all unramified characters of G are trivial, so by 4.18, $\det D_\bullet(V) \cong D_\bullet(\det V) \cong D_\bullet(\mathbb{Q}_p(\chi^n)) \cong K_0 t^{-n}$ for some $n \in \mathbb{Z}$; therefore we have $t_H(D_\bullet(V)) = t_N(D_\bullet(V)) = -n$.

Let $D' < D$ be a subobject in MF_K^ϕ and $d := \dim_{K_0} D'$. Note that taking exterior powers preserves subobject relations between filtered ϕ-modules, the crystalline property on representations, and commutes with $D_\bullet$. Since we are only interested in $t_H(D')$ and $t_N(D')$ at this point, we may therefore consider $\det D' < \bigwedge^d D = D_\bullet(\bigwedge^d V)$ instead, i.e. assume that $\dim_{K_0} D' = 1$.

Let $v_1, \dots v_n \in V$ be a $\mathbb{Q}_p$-basis of V and let $e' \in D'$ be a K_0-basis of D', so that $\phi_{D'}(e') = \lambda e'$ for some $\lambda \in K_0^\times$. Then $e' = \sum_{i=1}^n b_i \otimes v_i \in D_\bullet(V)$ for some $b_i \in B_\bullet$; since $\sum_{i=1}^n \lambda b_i \otimes v_i = \phi_{D'}(e')$, we see that $\varphi(b_i) = \lambda b_i$ for all $1 \le i \le n$.

Let $s := t_H(D')$. Since $e' \in \mathrm{Fil}^s D' \setminus \mathrm{Fil}^{s+1} D'$, we have $b_i \in \mathrm{Fil}^s B_\bullet \setminus \mathrm{Fil}^{s+1} B_\bullet$ for some $1 \le i \le n$. Assume that $s > v_p(\lambda)$, contrary to the theorem. Then $b_i \in \mathrm{Fil}^{v_p(\lambda)} B_\bullet$, so $t^{-v_p(\lambda)} b \in \mathrm{Fil}^1 B_\bullet$ and $\varphi_\bullet(t^{-v_p(\lambda)} b_i) = u t^{-v_p(\lambda)} b_i$ for some $u \in \mathrm{W}(k)^\times$. Since k is algebraically closed, we have $u = \varphi(u')/u'$ for some $u' \in \mathrm{W}(k)^\times$ (see e.g. [2, 9.3.3]). But then $t^{-v_p(\lambda)} b_i / u' \in \mathrm{Fil}^1 B_\bullet \cap B_\bullet^{\varphi=1} \overset{3.50}{=} \mathrm{Fil}^1 B_\bullet \cap \mathbb{Q}_p = 0$, which contradicts $b_i \notin \mathrm{Fil}^{s+1} B_\bullet$. $\square$

Note that in view of the theorem, more recent texts often use the latter condition as the definition of admissibility. The direction (ii) $\Longrightarrow$ (i) is significantly more difficult and admits a variety of proofs, all of them complicated. See [3, 4.3.11] for pointers to the various approaches. The original proof in [4] proceeds by showing that there is always a filtration for the underlying isocrystal of D that results in an admissible object, and then stepwise adjusts this filtration in a way that preserves admissibility, until one finally obtains the filtration on D. An important lemma in this proof is [2, 9.3.9], which notably uses the sequence from 3.55.

As a corollary of Theorem 4.24, we obtain that the category $\mathrm{MF}_K^{\phi,\mathrm{wadm}}$ is stable under tensor products. Due to the complicated structure of subobjects of a tensor product, this would be very difficult to show elementarily.

We end this section with a collection of common results on weakly admissible filtered ϕ-modules that we use in §4.3. Beware that while these properties may seem like trivial consequences of 4.24, many of them are actually required for reduction steps.

Proposition 4.25. *Let $D, D', D'' \in \mathrm{MF}_K^\phi$ be finite-dimensional.*

(i) *If $0 \to D' \to D \to D'' \to 0$ is exact in MF_K^ϕ, then $t_H(D) = t_H(D') + t_H(D'')$.*

(ii) *If $D, D' \in \mathrm{MF}_K^\phi$, then $t_H(D \otimes D') = \dim_{K_0} D' \cdot t_H(D) + \dim_{K_0} D \cdot t_H(D')$.*

(iii) *If $D \in \mathrm{MF}_K^\phi$, then $t_H(D^\vee) = -t_H(D)$.*

Completely analogous statements hold for t_N.

Proposition 4.26. *Homomorphisms in $\mathrm{MF}_K^{\phi,\mathrm{wadm}}$ are strict. All kernels and cokernels exist in $\mathrm{MF}_K^{\phi,\mathrm{wadm}}$ and agree with their counterparts in MF_K^ϕ. In particular, $\mathrm{MF}_K^{\phi,\mathrm{wadm}}$ is an abelian category.*

Proposition 4.27. *$D \in \mathrm{MF}_K^\phi$ is weakly admissible if and only if $\widehat{D} \in \mathrm{MF}_{\widehat{K^{\mathrm{ur}}}}^\phi$ is weakly admissible.*

Proposition 4.28. *Let $D \in \mathrm{MF}_K^{\phi,\mathrm{wadm}}$ and let $(D_i)_{i\in I}$ be a family of weakly admissible subobjects of D. Then $\bigcap_{i\in I} D_i \in \mathrm{MF}_K^\phi$ is also weakly admissible.*

Proposition 4.29. *Let $0 \to D' \to D \to D'' \to 0$ be a short exact sequence in MF_K^ϕ. If two of D, D', D'' are weakly admissible, then so is the third.*

Proposition 4.30. *Let $D, D' \in \mathrm{MF}_K^\phi$ with $\dim_{K_0} D, \dim_{K_0} D' < \infty$ and let $f : D \to D'$ be a bijective homomorphism in MF_K^ϕ.*

(i) *We have $t_N(D) = t_N(D')$ and $t_H(D) \le t_H(D')$.*

(ii) *f is an isomorphism in MF_K^ϕ if and only if $t_H(D) = t_H(D')$.*

4.3 Applications to p-divisible groups

Finally, let us discuss a result of Fontaine from [11, §6] that was a historical motivation for the concept of crystalline representations: A connection between two classifications of p-divisible groups over $\mathcal{O}_K$. For a classic introduction to p-divisible groups, see [15]; a detailed treatment can be found in [8]; for a general overview with some further historical background, see e.g. [2, §7.2, §7.3].

We denote the category of p-divisible groups over a ring R by BT_R and the category of p-divisible groups over a ring R up to isogeny by $\mathrm{BT}_R^{\mathrm{isog}}$. We consider a p-divisible group Γ over R an inductive family of finite flat commutative group schemes $(\Gamma_n)_{n\in\mathbb{N}_0}$ over R and denote its height by $\mathrm{ht}\,\Gamma$.

Remark 4.31. The p-divisible groups Γ over $\mathcal{O}_K$ can be classified in two major ways.

- On the one hand, there is the *Tate module* $T(\Gamma) := \varprojlim_n \Gamma_n(\overline{K}) \in \mathrm{Mod}_{\mathbb{Z}_p[G]}$, where the inverse limit is taken over the maps j_n such that $\Gamma_{n+1} \xrightarrow{j_n} \Gamma_n \to \Gamma_{n+1}$ is the multiplication by p. The Tate module is free of rank $\mathrm{ht}\,\Gamma$ and carries the profinite topology; its formation is functorial in $\Gamma \in \mathrm{BT}_{\mathcal{O}_K}$, yielding a fully faithful functor from $\mathrm{BT}_{\mathcal{O}_K}$ to the category of finitely generated $\mathbb{Z}_p$-modules with continuous G-action, see [15, p. 181, Corollary 1]. Note that $T(\Gamma)$ only depends on the generic fiber $\Gamma \times_{\mathcal{O}_K} K$. Passing to the isogeny viewpoint, we also obtain a fully faithful functor
$$T' := T(\cdot) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p : \mathrm{BT}_{\mathcal{O}_K}^{\mathrm{isog}} \to \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cont}}(G).$$

- On the other hand, we can pass to the special fiber $\Gamma \times_{\mathcal{O}_K} k$ and study the contravariant *Dieudonné functor* $\mathbb{M} : \mathrm{BT}_k \to \mathrm{Mod}_{\mathcal{D}_k}^{\mathrm{Wff}}$, an exact anti-equivalence between the categories of p-divisible groups over k and the left modules over the (non-commutative) Dieudonné ring $\mathcal{D}_k$ which are free of finite rank over $\mathrm{W}(k)$. It is exact and commutes with base changes to perfect extensions l/k.

  Passing to isogenies again, $\mathbb{M}'(\Gamma) := \mathbb{M}(\Gamma \times_{\mathcal{O}_K} k) \otimes_{\mathrm{W}(k)} K_0$ is a K_0-isocrystal by Lemma 4.10. Furthermore, the base change $\mathbb{M}'(\Gamma)_K$ admits a distinguished subspace L_Γ, which is the canonical image of the cotangent space $t_\Gamma^*(K)$, giving $\mathbb{M}'(\Gamma)$

the structure of a filtered ϕ-module over K via

$$\mathrm{Fil}^n\,\mathbb{M}'(\Gamma)_K = \begin{cases} \mathbb{M}'(\Gamma)_K & \text{if } n < 1, \\ L_\Gamma & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Note that this MF_K^ϕ-structure can be equivalently described in terms of the pair $(L_\Gamma, \mathbb{M}'(\Gamma))$ and morphisms $\mathbb{M}'(\Gamma) \to \mathbb{M}'(\Gamma')$ that map L_Γ into $L_{\Gamma'}$; this is done in older sources such as the one we cite. This gives rise to a fully faithful functor

$$\mathbb{M}' := \mathbb{M}(\cdot \times_{\mathcal{O}_K} k) \otimes_{\mathrm{W}(k)} K_0 : \mathrm{BT}_{\mathcal{O}_K}^{\mathrm{isog}} \to \mathrm{MF}_K^\phi\,.$$

Given these two approaches to the classification of p-divisible groups over $\mathcal{O}_K$, a natural question to ask is whether there exists a functor $D : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cont}}(G) \to \mathrm{MF}_K^\phi$ such that $D \circ T' \cong \mathbb{M}'$. In Theorem 4.37, we will see that this functor is D_{cris}.

Lemma 4.32. *If $\Gamma \in \mathrm{BT}_{\mathcal{O}_K}$, then $\mathbb{M}'(\Gamma) \in \mathrm{MF}_K^\phi$ is weakly admissible.*

Proof. The case where k is algebraically closed is proven in [14, 1.4]. For the general case, note that $\mathbb{M}'$ commutes with base change to $\widehat{K^{\mathrm{ur}}}$ and use 4.27. □

Lemma 4.33. *Let $0 \to D' \to D \to D'' \to 0$ be a short exact sequence in $\mathrm{MF}_K^{\phi,\mathrm{wadm}}$. The following propositions are equivalent:*

(i) *D lies in the essential image of $\mathbb{M}'$.*

(ii) *D' and D'' lie in the essential image of $\mathbb{M}'$.*

In particular, if this is the case, there exist $\Gamma, \Gamma', \Gamma'' \in \mathrm{BT}_{\mathcal{O}_K}^{\mathrm{isog}}$ such that the sequence $0 \to \Gamma'' \to \Gamma \to \Gamma' \to 0$ is exact and $\mathbb{M}'(\Gamma) = D$, $\mathbb{M}'(\Gamma') = D'$, $\mathbb{M}'(\Gamma'') = D''$.

Proof. Again, the case where k is algebraically closed is proven in [14, 1.8]. The general case follows from the fact that $\mathbb{M}'$ commutes with base change to $\widehat{K^{\mathrm{ur}}}$. □

Lemma 4.34. *If $D \subset B^+$ is a finite-dimensional φ_B-stable K_0-subspace, equipped with the structure of a filtered ϕ-module over K via the subspace filtration with respect to $D_K \subset B^+ \otimes_{K_0} K \subset B_{\mathrm{dR}}$, then $t_H(D) \leq t_N(D)$.*

Proof. [11, 6.5] Assume to the contrary that $t_H(D) > t_N(D) = r$, let $e \in \mathrm{Fil}^{t_H(D)} \det D$ be a basis element of $\det D$ such that $\varphi(e) = p^r e$. For each $f \in \mathrm{Hom}_{\mathrm{MF}_K^\phi}(\det D, B^+)$, we then have $\varphi_B(f(e)) = f(\phi_{\det D}(e)) = f(p^r e) = p^r f(e)$. Since $f(e) \in \mathrm{Fil}^{t_H(D)} B^+$, we have hence $t^{-r} f(e) \in (\mathrm{Fil}^{t_H(D)-r} B^+)^{\varphi=1} \subset (\mathrm{Fil}^1 B_\bullet)^{\varphi=1} \overset{3.54}{=} 0$, so $f = 0$.

Let $n = \dim_{K_0} D$. The above result in particular applies to the map $f_{g_1} \wedge \ldots \wedge f_{g_n}$, where $g_1, \ldots, g_n \in G$ and $f_{g_i}(d) = g_i.d$. Hence for any K_0-basis $d_1, \ldots, d_n \in D$, we have $\det(g_i.d_j)_{i,j} = 0$. By a standard result (proven e.g. in [11, 6.7]), it follows that $d_1, \ldots, d_n$ are B_{dR}^G-linearly (i.e. K-linearly) independent over B_{dR}; this contradicts 4.5. □

Definition 4.35. The space of *Witt bivectors over $\mathcal{O}_C^\flat$* is the $\mathbb{Q}_p$-subspace

$$BW(\mathcal{O}_C^\flat) := \{\sum_{i \in \mathbb{Z}} p^i [x_i] = \lim_{n \to \infty} \sum_{i=-n}^{\infty} p^i [x_i] \in B^+ \mid \limsup_{i \to \infty} |x_{-i}|_\flat^{1/p^i} < 1\} \subset B^+.$$

These series really converge in B^+ since with $q := \limsup_{i\to\infty} |x_{-i}|_\flat^{1/p^i} < 1$, we have

$$\lim_{i\to\infty} |p^{-i}[x_{-i}]|_\rho = \lim_{i\to\infty} \rho^{-i}|x_{-i}|_\flat \leq \lim_{i\to\infty} \rho^{-i} q^{p^i} = 0$$

for all $\rho \in (0;1)$. Beware that it is non-trivial that $BW(\mathcal{O}_C^\flat)$ is a $\mathbb{Q}_p$-subspace since it is unknown whether general elements of B^+ admit representations of the form $\sum_{i\in\mathbb{Z}} p^i[x_i]$, whether such representations are unique, and whether they are stable under sums and products. However, the above convergence conditions together with 3.3 result in the following formula:

$$\sum_{i\in\mathbb{Z}} p^i[x_i] + \sum_{i\in\mathbb{Z}} p^i[y_i] = \sum_{i\in\mathbb{Z}} p^i[\lim_{n\to\infty} \tilde{S}_n(x_{i-n},\dots,x_i,y_{i-n},\dots,y_i)^{1/p^n}]$$

Clearly $BW(\mathcal{O}_C^\flat)$ is stable under φ_B. Equipping $BW(\mathcal{O}_C^\flat) \otimes_{K_0} K \subset B_{\mathrm{dR}}$ with the subspace filtration, we see that $BW(\mathcal{O}_C^\flat)$ is an infinite-dimensional filtered ϕ-module over K. Unwinding definitions reveals that the ring $BW(\mathcal{O}_C^\flat)$ above is simply the ring $\mathrm{BW}(\mathrm{Res}(\mathcal{O}_C))$ from [8, p. 228] and similarly $\theta_{\mathrm{dR}|BW(\mathcal{O}_C^\flat)} : BW(\mathcal{O}_C^\flat) \to C$ is simply the map $\mathrm{bw}_{\mathcal{O}_C} : \mathrm{BW}(\mathcal{O}_C^\flat) \to C$. See also [6, §1.10.2] for a treatment more in line with our approach. We can therefore make use of the following lemma:

Lemma 4.36. *There is a natural isomorphism of functors* $\mathrm{BT}_{\mathcal{O}_K} \to \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cont}}(G)$

$$T'(\cdot) \cong \mathrm{Hom}_{\mathrm{MF}_K^\phi}(\mathbb{M}'(\cdot), BW(\mathcal{O}_C^\flat)).$$

Proof. This is a rewording of [8, Théorème 1, p. 232] with $\mathcal{S} = \mathcal{O}_C$. □

We are now able to prove the main result, which is Theorem 6.2 from [11].

Theorem 4.37. *There is a natural isomorphism of contravariant functors* $\mathrm{BT}_{\mathcal{O}_K} \to \mathrm{MF}_K^\phi$

$$\mathbb{M}' \overset{!}{\cong} D_\bullet(T'(\cdot)^\vee) = \mathrm{Hom}_{\mathbb{Q}_p[G]}(T'(\cdot), B_\bullet).$$

Proof. [11, 6.2] Let $\Gamma \in \mathrm{BT}_{\mathcal{O}_K}$ and functorially identify $T'(\Gamma) = \mathrm{Hom}_{\mathrm{MF}_K^\phi}(\mathbb{M}'(\Gamma_k), BW)$ using 4.36. We are looking for a functorial isomorphism

$$\eta_\Gamma : \mathbb{M}'(\Gamma) \to \mathrm{Hom}_{\mathbb{Q}_p[G]}(\mathrm{Hom}_{\mathrm{MF}_K^\phi}(\mathbb{M}'(\Gamma), BW), B_\bullet) = D_\bullet(T'(\Gamma)^\vee);$$

since $BW \subset B^+ \subset B_\bullet$, the obvious choice is $\eta_\Gamma(m) := (f \mapsto f(m))$, which is clearly $\mathbb{Q}_p$-linear and equivariant. For functoriality, let $\alpha : \Gamma \to \Gamma'$ be a homomorphism of p-divisible groups and note that for all $m' \in \mathbb{M}'(\Gamma')$, we indeed have

$$\begin{aligned}
&(\mathrm{Hom}_{\mathbb{Q}_p[G]}(\mathrm{Hom}_{\mathrm{MF}_K^\phi}(\mathbb{M}'(\alpha), BW), B_\bullet) \circ \eta_{\Gamma'})(m') \\
&= (\mathrm{Hom}_{\mathbb{Q}_p[G]}(\mathrm{Hom}_{\mathrm{MF}_K^\phi}(\mathbb{M}'(\alpha), BW), B_\bullet))((f' \mapsto f'(m'))) \\
&= (f' \mapsto f'(m')) \circ \mathrm{Hom}_{\mathrm{MF}_K^\phi}(\mathbb{M}'(\alpha), BW) \\
&= (f' \mapsto f'(m')) \circ (f \mapsto f \circ \mathbb{M}'(\alpha)) \\
&= (f \mapsto f(\mathbb{M}'(\alpha)(m'))) \\
&= (\eta_\Gamma \circ \mathbb{M}'(\alpha))(m').
\end{aligned}$$

We claim that $D_\bullet(T'(\Gamma)^\vee)$ is also weakly admissible and that η_Γ is injective. In that case, η_Γ is strict by 4.26 and

$$\dim_{K_0} D_\bullet(T'(\Gamma)^\vee) \leq \dim_{\mathbb{Q}_p} T'(\Gamma) = \operatorname{ht}\Gamma = \dim_{K_0} \mathbb{M}'(\Gamma) \overset{!}{\leq} \dim_{K_0} D_\bullet(T'(\Gamma)^\vee)$$

shows that $T'(\Gamma)$ is crystalline and η_Γ is an isomorphism

Consider $\ker \eta_\Gamma = \bigcap_{v\in T'(\Gamma)} \ker(v : \mathbb{M}'(\Gamma) \to BW) \in \mathrm{MF}_K^\phi$, via the identification from Lemma 4.36 and the fact that $\phi_B \circ v = v \circ \phi_{\mathbb{M}'(\Gamma)}$. Every $v \in T'(\Gamma)$ (trivially) gives rise to a short exact sequence $0 \to \ker v \to \mathbb{M}'(\Gamma) \to \operatorname{coim} v \to 0$ in MF_K^ϕ. Note that

$$t_H(\operatorname{coim} v) \overset{4.30}{\leq} t_H(\operatorname{im} v) \overset{4.34}{\leq} t_N(\operatorname{im} v) \overset{4.30}{=} t_N(\operatorname{coim} v), \tag{$*$}$$

so since $\mathbb{M}'(\Gamma)$ is weakly admissible by 4.32 and $\ker v < \mathbb{M}'(\Gamma)$, we have

$$\begin{aligned} t_N(\ker v) &\overset{4.25}{=} t_N(\mathbb{M}'(\Gamma)) - t_N(\operatorname{coim} v) \\ &\overset{!}{=} t_H(\mathbb{M}'(\Gamma)) - t_N(\operatorname{coim} v) \\ &\overset{(*)}{\leq} t_H(\mathbb{M}'(\Gamma)) - t_H(\operatorname{coim} v) \\ &\overset{4.25}{=} t_H(\ker v) \\ &\overset{!}{\leq} t_N(\ker v), \end{aligned}$$

so $\ker v$ is weakly admissible as well. Hence 4.28 shows that $\ker \eta_\Gamma \in \mathrm{MF}_K^{\phi,\mathrm{wadm}}$ and 4.29 implies that $\operatorname{coim} \eta_\Gamma \in \mathrm{MF}_K^{\phi,\mathrm{wadm}}$.

Now 4.33 applies to the sequence $0 \to \ker \eta_\Gamma \to \mathbb{M}'(\Gamma) \to \operatorname{coim} \eta_\Gamma \to 0$, which is therefore the image under $\mathbb{M}'$ of an exact sequence $0 \to \Gamma'' \to \Gamma \to \Gamma' \to 0$ in $\mathrm{BT}_{\mathcal{O}_K}^{\mathrm{isog}}$. But since $T'(\Gamma'') \cong T'(\Gamma)$, it follows that $\Gamma' = 0$, hence $\ker \eta_\Gamma = 0$. □

In particular, we see that for any $\Gamma \in \mathrm{BT}_{\mathcal{O}_K}$, the representation $T(\Gamma) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline. However, not all crystalline representations arise as the Tate module of some p-divisible group. The Hodge-Tate weights of representations that arise in this way must always lie in $\{0, 1\}$ by [15, p. 180]. By a conjecture of Fontaine, later proven by Kisin in [13, Theorem 0.3], the converse holds as well.

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Eidesstattliche Versicherung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Masterarbeit unter Betreuung durch Prof. Dr. Jan Kohlhaase selbstständig verfasst und Zitate kenntlich gemacht habe. Andere als die angegebenen Quellen und Hilfsmittel wurden von mir nicht benutzt. Diese Masterarbeit hat in dieser oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegen.

Essen, den 12.12.2022

Sebastian Melzer