SEMINAR ON STACKS

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(This is a modification of a seminar Timo Richarz and I ran a few years ago, a large part of the program thus owes a lot to Timo.)

In both complex and algebraic geometry it is of fundamental interest to construct and study parameter spaces of geometric objects like vector bundles, (elliptic) curves etc. Those *moduli spaces* are usually not complex manifolds or algebraic varieties themselves but what one calls a *stack* (german: *Stack*; french: *champ*). They enlarge our notion of spaces:

$$\text{(analytic/algebraic spaces)} \subset \text{(analytic/algebraic stacks)}.$$

Aim of the seminar is to learn what these are by studying prototypical examples of stacks like

(1) stacky curves;

(2) moduli spaces of elliptic curves;

(3) moduli spaces of vector bundles.

In order to attend the seminar, you should be familiar with either complex or algebraic geometry as covered by a two semester course. Depending on the background of the participants the seminar will be either more complex analytic or more algebraic.

Overview. In Talk 1 we try to construct moduli spaces of vector bundles and elliptic curves which should serve as guideline throughout the seminar. In Talks 2-8, we develop the language of stacks. By now the standard reference for stacks is the Stacks Project [StaPro] and for almost all of this seminar you can find a lot in there. The recent book project of Jarod Alper [Al24] is a very valuable introduction.

Below we give references to my notes [He04, He10] (beware of misprints!) and Vistoli [Vi04] - but it might be helpful to consult the other sources as well. Also you should definitely contact me to discuss about your talk well in advance.

The theory will be applied in Talks 9-10 to constructing a moduli space for vector bundles on curves and computing its cohomology.

Talk 1: Two motivating examples: Vector bundles and elliptic curves

Start by recalling some basic facts on (holomorphic) vector bundles [Hu05, §2.2]. For us it is important to have the following three different points of view on vector bundles: (1) as manifolds fibered in vector spaces; (2) as locally free $\mathcal{O}_X$-modules; (3) as Gl_n-torsors. Explain why all three concepts are equivalent, and that the isomorphism class of objects (3) are given by the (non-abelian) cohomology set $H^1(X, \mathrm{Gl}_n)$.

Next we try to construct moduli spaces: If you have some familiarity with the classifying space of vector bundles in topology, it would be nice to explain some of its properties. Follow the reasoning in [He04, §1]: a first try could be to search for a complex manifold BGl_n such that as sets

$$\mathrm{Maps}_{\mathbb{C}}(T, \mathrm{BGl}_n) = \langle \text{rank } n \text{ vector bundles on } T\rangle / \simeq .$$

Explain why BGl_n can not be a complex manifolds (not even a topological space!). Another try: let X be a smooth projective connected $\mathbb{C}$-curve (i.e. a compact Riemann surface), and let us try to construct a space Bun_X^n such that

$$\mathrm{Maps}_{\mathbb{C}}(T, \mathrm{Bun}_X^n) = \langle \text{rank } n \text{ vector bundles on } X \times T\rangle / \simeq .$$

Explain why this is (almost) possible for $n = 1$: we denote $\mathrm{Pic}_X = \mathrm{Bun}_X^1$. Then $\mathrm{Pic}_X = \coprod_{d\in\mathbb{Z}} \mathrm{Pic}_X^d$ where Pic_X^d is the space of line bundles of degree d. Explain why Pic_X^0 is a complex torus of dimension $g = \mathrm{genus}(X)$: this follows from the exponential sequence [Hu05, Def 2.2.12]. For $n \geq 2$ a manifold like Bun_X^n does not exist: explain this in the case $X = \mathbb{P}^1_{\mathbb{C}}$ and $n = 2$.

Recall the definition of an elliptic curve as a 1-dimensional complex torus $\mathbb{C}/\Lambda$, cf. [Hu05, §2.1]. Explain that all are homeomorphic to $S^1 \times S^1$, but have different complex structures. Our aim is to find a space which parametrizes all elliptic curves. Follow [Se73, Ch VII]: define the modular group $G = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$, and its action on the upper half plane $\mathbb{H}$. Explain §1, Thm 1 and draw the picture. Explain the relation to $\mathbb{Z}$-lattices $\Lambda \subset \mathbb{C}$ of rank 2. In particular, §2, Rmk after Prop 3. Conclude that the set $\mathbb{H}/G$ is the set of isomorphism classes of elliptic curves. Now explain the holomorphic G-invariant function $j\colon \mathbb{H} \to \mathbb{C}$ (§3.3), and conclude that it defines a bijection $\mathbb{H}/G \simeq \mathbb{C}$. Follow the reasoning in [Ed03] which shows that $\mathbb{C}$ is not the moduli space of elliptic curves. The problems arise because the action of G on $\mathbb{H}$ is not free. For $N \geq 2$, let us define the congruence subgroup $G_N \subset G$ as the image of $\ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N))$ in G. Then the action of G_N on $\mathbb{H}$ is nice for $N \geq 3$ (proper discontinuous), and the quotient $\mathbb{H}/G_N$ is a complex manifold. What do points of $\mathbb{H}/G_N$ correspond to in terms of elliptic curves? Conclude that we would be able to define the moduli space of elliptic curves, if we could pass to quotients under finite groups acting badly.

Talk 2: Moduli spaces I - Definition and Examples

With the previous examples in mind we define now what a moduli space should be in general, cf. [Vi04, §2.1]. For this we need the Yoneda lemma saying that an object of a category is determined by its functor of points [Vi04, §2.1.2]. Define representable functors, universal objects and give many examples[1] [Vi04, §2.1.3]. In particular: affine space, projective space and Grassmannians. Explain pullbacks carefully. If you have a background in topology it would be nice to mention Eilenberg-Maclane spaces representing cohomology. End with the remark that the Yoneda lemma allows us to define for example open or closed subfunctors of a functor on Top/Man/Sch.

Talk 3: Moduli spaces II - Gluing and descent

We now give necessary conditions for the existence of moduli spaces: gluing and descent. If F is a representable set-valued functor on Top/Man/Sch, then elements of $F(X)$ are called objects, and these have to satisfy a sheaf property with respect to open coverings. Recall the non-example of isomorphism classes of vector bundles from the first talk where the sheaf property fails [He04, §1] (because we passed to isoclasses!). Discuss the gluing condition for open coverings [He04, Def 1.1.1, Rmk 1.2.1] and for quotient spaces (if $Y \to X$ is a submersive map of topological spaces or manifolds, then objects (e.g. vector bundles) on X should be the same as objects on Y which are constant along the fibers). Explain how for quotient spaces fiber products take the role of intersections, cf. [Vi04, §4.1.1]. Next define the notion of a Grothendieck topology (resp. site) [Vi04, §2.3.1], and the sheaf condition on functors [Vi04, §2.3.3]. Give examples from

(1) topology: open coverings, covering spaces (=étale topology);

(2) differentiable manifolds: submersions (in order to have fiber products) [He04, Rmk 1.2.6];
(3) algebra: Zariski topology, étale topology, fpqc topology (if $A \to B$ is a faithfully flat ring map, then $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is submersive [**?**, Tag 02JY]). State that these are subcanonical [Vi04, Thm 2.55].

[1]The examples are written in the language of schemes, and should be translated to complex manifolds.

Talk 4: Stacks - Fibered categories or pseudo functors

Recall from the last talk that passing to the set of isomorphism classes caused the representability problems in the case of vector bundles. Solution: We do not pass to isomorphism classes, and hence we have to allow our functors to have values in categories. Define stacks [He04, Def 1.1, Rmk 1.2.1] and give examples: representable functors, the moduli space of vector bundles (give details here). Explain how to avoid problems with functorial pullbacks [He10, Rmk 1.2]. For details consult [Vi04, §3.1, §3.2, §], and explain why the two approaches via fibered categories or pseudo functors are equivalent.

Although it is tempting and very helpful to think in terms of pseudo-functors, the language of fibered categories is often much more convenient to work with.

Talk 5: The 2-Yoneda lemma and representable morphisms

Prove the 2-Yoneda lemma for stacks ([Vi04, §3.6] or [He04, Lem 1.3]). Now follow [He04, §2]: Compute an example of a fibered-product of stacks. Use this to define representable morphisms. Give basic properties and examples/non-examples. Representability and pullback. The diagonal morphism. Define what a topological or differentiable (or algebraic) stack is [He04, Def. 2.3] (or [Vi04, 2.6.2]). Explain also the notion of a Deligne-Mumford stack (=DM-stack), cf. [BN06, §3].

Talk 6: Differentiable/Algebraic stacks: properties and examples

Recall the definition of the previous talk. Again we follow [He04, §2]: Now we can make sense of basic geometric properties of stacks like connectedness, smoothness, dimension for differentiable stacks (or algebraic stacks). Show that these are well defined, give examples. Properties of morphism of algebraic stacks, closed/open substacks. Existence of universal objects. As an example give the analytic description of elliptic curves and their moduli stack $\mathcal{M}_{1,1} = [\mathbb{H}/G]$, cf. Talk 2.

Talk 7: Groupoids and quotients

Recall that $\mathcal{M}_{1,1}$ can be written as the quotient of a complex manifold under a finite group action. We follow [He04, §2, §3] and look more closely at group actions: Explain the groupoid defined by an atlas, and conversely the stack defined by a groupoid. Conclude that stacks also solve the problem of passing to quotients for bad group actions. In particular, note that the stack quotient $[X/G]$ is smooth whenever X is, although the topological quotient X/G may have singularities. Explain the example of the classifying space (explain why GL_n-torsors are the same as vector bundles), the example of $\mathcal{M}_{1,1}$ and point out that it is a DM-stack. State [BN06, Prop 3.5] saying that any topological (or complex analytic) DM-stack is locally of the form $[X/G]$.

Talk 8: fpqc-descent

Depending on the audience, we probably should insert time to prove descent for the fpqc topology, which is fundamental - and not so difficult to prove once you believe in it. References would be [Vi04, §4.2 and §2.3.6]).The main results are that quasi-coherent sheaves form a stack in the fpqc topology ([Vi04, Theorem 4.23])

Talk 9–10?: Vector bundles on curves

We end the seminar by solving the problem posed in Talk 1 of constructing the moduli stack of rank n vector bundles Bun non a curve, cf. [He10, Ex 1.14] (more references follow). Discuss the moduli stack Bun_2 on $\mathbb{P}^1$, and its open/closed substacks. Let $\mathcal{P}\mathrm{ic} = Bun_1$ be the Picard groupoid. Explain that $\mathcal{P}\mathrm{ic} \to \mathrm{Pic}$ is the coarse moduli space, and makes $\mathcal{P}\mathrm{ic}$ into a $\mathbb{C}^*$-gerbe which splits according to the choice of a point on X.

We should prove that Bun_n is a smooth stack and compute the connected components. We can deduce that the subspace of semistable bundles is connected as well, which is harder to do directly.

It would be nice to indicate a sample of a cohomology computation for stacks, e.g. [He04, Ex 4.9] or [He10]

A more detailed description of these talks will follow.

References

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