

Recall:
§ Simplicial sets & geom realization

DEF: simplex set Δ :

- $\text{obj} = \{i \in \mathbb{N} \neq \emptyset \text{ tot ord sets}\}$
- $\text{mor} = \text{order pres maps}$

notation: $[n] = \{0 < 1 < \dots < n\}$
 $\forall \text{ obj in } \Delta \text{ is (uniquely) isom to some } [n]$

DEF: a simplicial set is a functor $X: \Delta^{\text{op}} \rightarrow \text{Set}$

notation: $X([n]) =: X_n \leftarrow$ "set of n -simplices of X "
 $0\text{-simplices} = \text{vertices}$
 $1\text{-simplices} = \text{edges}$

• Face & degeneracy maps:

• $d_{n,i}: [n-1] \rightarrow [n]$ "miss i "

$\hookrightarrow d_{n,i}: X_n \rightarrow X_{n-1}$ "face map"

e.g: $d_{n,0}: 0 < \dots < n-1 \mapsto 1 < \dots < n$

• $\sigma_{n,i}: [n+1] \rightarrow [n]$ "hits i twice"

$\sigma_{n,0}: 0 < \dots < n+1 \mapsto 0 = 0 < 1 < \dots < n$

$\hookrightarrow \sigma_{n,i}: X_n \rightarrow X_{n+1}$ "degeneracy maps"

image = "degenerate $(n+1)$ -simplices"

Ex: The "standard n -simplex" is $\Delta^n = \text{Hom}_{\Delta}(-, [n])$

RMK: By Yoneda:

$$X_n = X([n]) \cong \text{Hom}_{\text{Fun}}(\text{Hom}(-, [n]), X) = \text{Hom}_{\text{Set}}(\Delta^n, X)$$

Ex: Let $\mathcal{C}$ be a small category

$N(\mathcal{C})$ = "nerve of $(\mathcal{C})$ " is the Set defined by

$$N(\mathcal{C})_k = \left\{ \text{chain of composable sequences of morphisms} \right. \\ \left. X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \rightarrow \dots \xrightarrow{f_k} X_k \right. \\ \left. = \text{Fun}([k], \mathcal{C}) \right\}$$

DEF: "Geometric realization functor"

$| \cdot | : \mathcal{S}et \longrightarrow Top$ mapping a $\mathcal{S}et X$ to a CW-complex.

Formally: $X \cong \varinjlim_{\Delta^n \rightarrow X} \Delta^n \Rightarrow |X| := \varinjlim_{\Delta^n \rightarrow X} |\Delta^n|$,

where $|\Delta^n|$ is the standard topological n -simplex.

Intuitively, $|X|$ is a CW-complex whose set of n -faces is in bijection with the non-degenerate n -simplices of X & the glueing maps are obtained by the face maps.

RMK: $| \cdot |$ can be also defined by adjunction:

$Sing : Top \longrightarrow \mathcal{S}et$ "singular simplex functor"

$X \longmapsto S(X), S_n(X) := \{ |\Delta^n| \rightarrow X \text{ continuous} \}$

$\Rightarrow | \cdot | \dashv Sing$

• Similarly, the nerve also admits a left adjoint

$h : \mathcal{S}et \longrightarrow Cat$ "homotopy category",

$h \dashv N$

§ Classifying space of a small category

$$\begin{array}{ccc} \text{Cat} = \text{small categories} & \xrightarrow{N} & \mathcal{S}\text{et} \xrightarrow{|-|} \text{Top} \\ \mathcal{C} & \xrightarrow{\quad} & B\mathcal{C} = |N\mathcal{C}| \\ & & = \text{"classifying space of } \mathcal{C} \text{"} \end{array}$$

Explicitly:

- 0-cells: object of $\mathcal{C}$
- 1-cells: non-identity morphisms.
edges = source, target
- 2-cells: 2-chains of composable morph ($\neq \text{id}$)

$$C_0 \xrightarrow{f} C_1 \xrightarrow{g} C_2 \rightsquigarrow \begin{array}{ccc} & C_1 & \\ f \nearrow & & \searrow g \\ C_0 & \xrightarrow{g \circ f} & C_2 \end{array} \quad \text{edges} = f, g, g \circ f$$

- n-cells: n-chains of composable morph ($\neq \text{id}$)

$$C_0 \rightarrow C_1 \rightarrow \dots \rightarrow C_{n-1} \rightarrow C_n \quad \text{edges} = \text{ignore one of the } C_i \text{'s}$$

(if some morph becomes trivial, get a degenerate edge)

RMK: $B\mathcal{C} \cong B\mathcal{C}^{\text{op}}$ (canonical iso)

RMK: $B: \text{Cat} \longrightarrow \langle \text{CW-complexes, cellular maps} \rangle$ is characterized by the following properties:

(1) (naturality)

$$F: \mathcal{C} \rightarrow \mathcal{D} \text{ funct} \rightsquigarrow BF: B\mathcal{C} \rightarrow B\mathcal{D} \text{ cellular map,}$$

$$BF \circ BG = B(F \circ G), \quad B(1_{\mathcal{C}}) = 1_{B\mathcal{C}}$$

(2) • $B[n] = \Delta^n$

$$\bullet \varphi: [i] \rightarrow [n] \text{ funct} \rightsquigarrow B\varphi: \Delta^i \rightarrow \Delta^n$$

vertex $f \mapsto \varphi(f)$

$$(3) \text{ Let } \mathbb{I} := \left\langle \begin{array}{l} \text{OBJ: } [n] \rightarrow \mathcal{C} \\ \text{mor } [i] \rightarrow [n] \rightarrow \mathcal{C} \end{array} \right\rangle \Rightarrow B\mathcal{C} = \text{colim}_{[i] \rightarrow \mathcal{C} \in \mathbb{I}} B[i]$$

Properties:

- $C \subseteq D$ sub-cob $\Rightarrow BC \subseteq BD$ subcomplex
- $C = \coprod_{\alpha} C_{\alpha} \Rightarrow BC = \coprod_{\alpha} BC_{\alpha}$
- $B(C \times D) \cong \underbrace{BC \times BD}_{\uparrow \text{compactly generated topology}}$

Examples:

- $B\emptyset = \emptyset$
- $B[0] = \{0\} = \text{one-point space}$
- $B[1] = [0, 1] = \begin{matrix} 0 & \xrightarrow{\quad} & 1 \end{matrix}$
- $B[2] = \Delta^2 = \begin{matrix} & 1 & \\ & \nearrow \searrow & \\ 0 & & 2 \end{matrix}$

Ex: $C_2 = \langle \begin{matrix} \circlearrowleft \\ * \end{matrix} 1 \mid \sigma^2 = 1 \rangle$. BC_2 :

- 0-cell: $* = e_0$ edges: $\emptyset$
- 1-cell: $* \xrightarrow{\sigma} * = e_1$ $* = * e_0$
- 2-cell: $* \xrightarrow{\sigma} * \xrightarrow{\sigma} * = e_2$ $* \xrightarrow{e_1} *$, $* \xrightarrow{e_0} *$, $* \xrightarrow{e_1} *$
- 3-cell: $* \rightarrow * \rightarrow * \rightarrow *$ e_3 e_2, e_1, e_1, e_2

$\leadsto$ one n -cell $e_n \forall n$, attached to e_{n-1} by a map of degree 2

$\leadsto BC_2 \cong \mathbb{RP}^{\infty}$, $(BC_2)_{n-\infty} \cong \mathbb{RP}^n$

Ex: G grp can be seen as a 1-obj category

$\hookrightarrow BG$: $\begin{matrix} \bullet 1 \text{ vertex} \\ \bullet 1 \text{ cell } \forall \text{ nontrivial element of } G \end{matrix}$

§ Homotopy theoretic properties of BC

- $F_i: \mathcal{C} \rightarrow \mathcal{D}$ $i=0,1$, $\eta: F_0 \Rightarrow F_1$ nat transf
 $\Leftrightarrow \eta: \mathcal{C} \times [1] \longrightarrow \mathcal{D}$, $\eta|_{\mathcal{C} \times 0} = F_0$, $\eta|_{\mathcal{C} \times 1} = F_1$
 $\hookrightarrow B\eta: B\mathcal{C} \times \Delta^1 \rightarrow B\mathcal{D}$, $B\eta|_{B\mathcal{C} \times 0} = BF_0$, $B\eta|_{B\mathcal{C} \times 1} = BF_1$
 $\Rightarrow \eta$ induces a homotopy $BF_0 \sim BF_1$
 $\Downarrow$

- A pair of adjoint functors $L: \mathcal{C} \rightarrow \mathcal{D} \dashv R: \mathcal{D} \rightarrow \mathcal{C}$ induces a homotopy equivalence $B\mathcal{C} \simeq B\mathcal{D}$
 $(LR \Rightarrow 1_{\mathcal{D}}, 1_{\mathcal{C}} \Rightarrow RL)$.

In particular, an equivalence of categories induces a homotopy equivalence

- If $\mathcal{C}$ is not small, but equivalent to $\mathcal{C}_0$ small, then we set $B\mathcal{C} := B\mathcal{C}_0$. By the previous rule this is well defined up to homotopy equivalence

- If $\mathcal{C}$ has an initial obj I , the constant functor $\mathcal{C} \rightarrow [0]$

has a left adjoint $[0] \rightarrow \mathcal{C}$
 $0 \mapsto I$

$\Rightarrow B\mathcal{C} \sim \Delta^0$ is contractible.

Similarly if $\mathcal{C}$ has a final object.

$$\boxed{\pi_0}$$

$$X \text{ top sp} \leadsto \pi_0(X) = \{\text{conn comp of } X\}$$

If $X = (C, \sim)$ is a CW complex

$$\Rightarrow \pi_0(X) = \{\text{vertices}\} / \{\text{edges}\}$$

$$\uparrow v_0 \sim v_1 \text{ iff } \exists e_1, \dots, e_n \text{ edges}$$

$$v_0 \xrightarrow{e_1} w_1 \xrightarrow{e_2} \dots \xrightarrow{e_n} w_n = v_1$$

In the case of BC , this becomes:

$$\pi_0(\mathcal{C}) := \pi_0(BC) = \text{obj}(\mathcal{C}) / \sim,$$

where $\sim$ is the eqv rel $C \sim C'$ if $\exists C \rightarrow C'$ or $C' \rightarrow C$.

$$\boxed{\pi_1}$$

REMINDER:

- Let Γ be a connected graph, let $T \subseteq \Gamma$ be a maximal tree (contractible graph). Then
 $\pi_1(\Gamma) = \langle e \in e(\Gamma) \setminus e(T) \rangle$

$\updownarrow$
 contains every vertex

- The gluing of a 2-cell allows to contract the path on its boundary

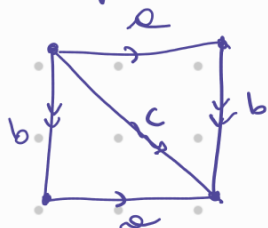
PROP: Let T be a maximal tree in $\mathcal{C}$.

Then $\pi_1(BC)$ is generated by $[f]$, $f \in \text{Mor } \mathcal{C}$, modulo:

$$(1) [t] = 1 \quad \forall t \in T, \quad [1_c] = 1 \quad \forall c \in \mathcal{C}$$

$$(2) [f] \cdot [g] = [f \circ g]$$

Example:



$$\leadsto T = \bullet$$

$$\leadsto \pi_1 = \langle a, b, c \rangle / \substack{c = ab \\ c = ba} = \langle ab \mid ab = ba \rangle$$

Ex: $\pi_1(BG) = G$ (Moreover, $\pi_i(BG) = 0 \ \forall i \geq 2 \Rightarrow BG = K(G, 1)$)

$$\xrightarrow{\quad} \overbrace{H_*(BG, \mathbb{M})}^{\text{cellular hom}} = \overbrace{H_*(G, \mathbb{M})}^{\text{group hom}}$$

• $\pi_1(BM) = M^{-1}M$ (M monoid)

Homology

REMINDER: cellular homology of a CW-complex X

$$C_n(X) = H_n(X^n, X^{n-1}) = \bigoplus_{e_\alpha^{n\text{-cell}}} \mathbb{Z} e_\alpha^n$$

$$d_n: C_n(X) = \bigoplus \mathbb{Z} e_\alpha^n \longrightarrow C_{n-1}(X) = \bigoplus \mathbb{Z} e_\mu^{n-1}$$

$$e_\alpha^n \mapsto \sum_\mu d_{\alpha\mu} \cdot e_\mu^{n-1}$$

$d_{\alpha\mu}$ = incidence number of e_α^n on e_μ^{n-1} , is the degree of the map

$$S^{n-1} \xrightarrow[\text{attaching of } e_\alpha^n]{\Sigma_\alpha} X^{n-1} \twoheadrightarrow X^{n-1} / X^{n-1} \setminus e_\mu^{n-1} \cong D_\mu^{n-1} / S_\mu^{n-2} \cong S^{n-1}$$

$\hookrightarrow L^{\alpha\mu}$

$d_{\alpha\mu} := \deg(L^{\alpha\mu}) \leftarrow$ only well defined up to a sign!
(need to choose an orientation when attaching the cells)

In the case of BC :

$$C_n(BC) = \bigoplus_{c_0 \rightarrow \dots \rightarrow c_n} \mathbb{Z}(f_0 \dots f_{n-1})$$

$$d_n: (f_1 \dots f_n) \mapsto \sum_{i=1}^n (-1)^i (f_1 \dots f_{i-1} \cancel{f_i} f_{i+1} \dots f_n)$$

$$\left. \begin{array}{l} \\ \\ \end{array} \right\} \rightarrow H_*(BC, \mathbb{Z})$$

more generally, we can take homology with coefficients in $M: \mathcal{C} \rightarrow \mathbf{Ab}$:

$$C_n(\mathcal{C}, M) = \bigoplus_{c_0 \rightarrow \dots \rightarrow c_n} M(c_0) \underbrace{(f_0 \dots f_{n-1})}_{\text{just to keep track of indexing}}$$

$$d_n: M(c_0)(f_0 \dots f_{n-1}) \longrightarrow M(c_1)(f_1 \dots f_{n-1}) \oplus \bigoplus_i M(c_0)(f_0 \dots \cancel{f_i} \dots f_{n-1})$$

$$\begin{array}{ccc} x \in M(c_0) & \mapsto & M(f_0)(x) \oplus \bigoplus_i (-1)^i \cdot x \\ \text{indexed} & & \\ \text{by } (f_0 \dots f_{n-1}) & & \end{array}$$

e.g.:

$$\dots \rightarrow C_1(BC) = \bigoplus_{c_0 \rightarrow c_1} M(c_0)(f) \xrightarrow{(*)} \bigoplus_{c_0} M(c_0)$$

$$\begin{array}{ccc} x \in M(c_0) & \mapsto & (M(f)(x), -x) \\ \text{indexed} & & \\ \text{by } f: c_0 \rightarrow c_1 & & \end{array}$$

$$\Rightarrow H_0(\mathcal{C}, M) = \text{coker}(*) = \text{colim}_{c \in \mathcal{C}} M(c)$$

RMK: If M is "morphism-inverting", then $H_i(\mathcal{C}, M)$ is canonically isomorphic to the homology groups of BC w/ local coefficients M , $H_i(BC; M)$.

§ Homotopy fiber & comm categories

Want: long exact seq of homotopy groups

Alg topology reminder: Homotopy fiber sequence
 $(X, x_0), (Y, y_0)$ pointed spaces

$f: X \rightarrow Y$ pointed map

$F(f) := \{(\alpha, \gamma) \mid \begin{matrix} \alpha \in X \\ \gamma: [0, 1] \rightarrow Y, \gamma(0) = y_0, \gamma(1) = f(\alpha) \end{matrix}\}$ "homotopy fiber of f "
 $\uparrow$ pointed: $(\alpha_0, \text{const path } y_0)$

PROP: $F(f) \xrightarrow{p_1} X \xrightarrow{f} Y$ is an h-exact sequence, i.e.
 $\forall B \in \text{Top}^0, [B, F(f)]^0 \xrightarrow{p_{1*}} [B, X]^0 \xrightarrow{f_*} [B, Y]^0$ is
 an exact sequence of pointed spaces
 $(f_*^{-1}(\text{base point}) = \text{im}(p_{1*}))$

Iterating:

$\dots F(f_{n-1}) \xrightarrow{f_n} F(f_{n-2}) \rightarrow \dots \rightarrow F(f_2) \xrightarrow{f_3} F(f_1) \xrightarrow{f_2} F(f_0) = F(f) \xrightarrow{f_1} X \xrightarrow{f} Y$
 is h-exact

DEF: $\Omega X = \{\text{loops in } X \text{ pointed at } x_0\} = \text{Hom}(S^1, X)^0$

$\hookrightarrow [SX, Y]^0 \cong [X, \Omega Y]^0$ homeom

$\downarrow$
 $\pi_{m+1}(X, x_0) \cong \pi_m(\Omega X, x_0)$

PROP:
$$\begin{array}{ccc} F(f_2) & \xrightarrow{f_3} & F(f_1) \\ \downarrow & & \downarrow \text{equiv hom} \\ \Omega X & \xrightarrow{\Omega f} & \Omega Y \end{array}$$

pf:

$$\begin{aligned} F(f_1) &= \{(\alpha, \sigma) \mid \begin{matrix} \alpha \in F(f) & \sigma(0) = x_0 \\ \sigma: [0, 1] \rightarrow X & \sigma(1) = f(\alpha) \end{matrix}\} \\ &= \{(\alpha, \sigma, \gamma) \mid \begin{matrix} \gamma: [0, 1] \rightarrow Y & \gamma(1) = f(\alpha) \\ \sigma: [0, 1] \rightarrow X & \sigma(1) = f_1(\alpha, \gamma) = \alpha \end{matrix}\} \\ &\cong \{(\gamma, \sigma) \mid \begin{matrix} \gamma: [0, 1] \rightarrow Y & \gamma(1) = f(\sigma(1)) \\ \sigma: [0, 1] \rightarrow X \end{matrix}\} \xrightarrow{f_2} F(f) = \{(\alpha, \gamma) \mid \begin{matrix} \gamma: I \rightarrow Y, \\ \gamma(1) = f(\alpha) \end{matrix}\} \\ &\quad (\gamma, \sigma) \mapsto (\sigma(1), \gamma) \end{aligned}$$

$$\Omega Y \xrightarrow{j(f)} F(f_1) \rightarrow \text{homotopy eqv}$$

$$\sigma \mapsto (\sigma, \text{cost } \kappa_0)$$

Construct analogously $j(f_1): \Omega X \rightarrow F(f_2)$

□

Consequence:

$$\dots \rightarrow F(f_2) \xrightarrow{j_2} F(f_1) \xrightarrow{j_1} F(f) \xrightarrow{j_0} X \xrightarrow{j} Y$$

is hom eqv to

$$\begin{array}{ccccccc} F(\Omega f) & \xrightarrow{(\Omega f)'} & \Omega X & \xrightarrow{\Omega j} & \Omega Y & \longrightarrow & F(f) \xrightarrow{j_0} X \longrightarrow Y \\ \downarrow & \nearrow \Omega j' & & & & & \\ \Omega F(f) & & & & & & \end{array}$$

THM: $\exists \downarrow$ homeo.

Iterating, we get

$$\dots \rightarrow \Omega^n F(f) \rightarrow \Omega^n X \rightarrow \Omega^n Y \rightarrow \Omega^{n-1} F(f) \rightarrow \dots \rightarrow F(f) \rightarrow X \rightarrow Y$$

exact sequence

Apply $[S^0, -]^0$ to get the long exact homotopy sequence of $X \xrightarrow{j} Y$:

$$\dots \rightarrow \pi_2(F(f), *) \rightarrow \pi_2(X, \kappa_0) \rightarrow \pi_2(Y, \eta_0)$$

$$\rightarrow \pi_1(F(f), *) \rightarrow \pi_1(X, \kappa_0) \rightarrow \pi_1(Y, \eta_0)$$

$$\rightarrow \pi_0(F(f), *) \rightarrow \pi_0(X, \kappa_0) \rightarrow \pi_0(Y, \eta_0)$$

Upshot: Given $f: X \rightarrow Y$ pointed map and defining $F(f)$ the homotopy fiber of f as above, we get a long exact sequence

$$\dots \rightarrow \pi_2(F(f), *) \rightarrow \pi_2(X, \kappa_0) \rightarrow \pi_2(Y, \eta_0)$$

$$\rightarrow \pi_1(F(f), *) \rightarrow \pi_1(X, \kappa_0) \rightarrow \pi_1(Y, \eta_0)$$

$$\rightarrow \pi_0(F(f), *) \rightarrow \pi_0(X, \kappa_0) \rightarrow \pi_0(Y, \eta_0)$$

What's the analogue for categories?

$F: \mathcal{C} \rightarrow \mathcal{D}$ functor
 $d \in \mathcal{D}$

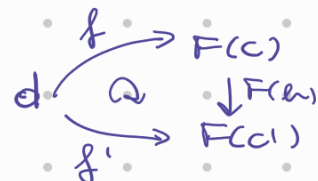
$(f: X \rightarrow Y)$

$(y_0 \in Y \text{ base point})$

COMMA CATEGORY

• OB: (c, f) , $c \in \mathcal{C}$, $f: d \rightarrow F(c)$

• MOR: $d \xrightarrow{f} F(c) \leftrightarrow h: c \rightarrow c'$
 $d \xrightarrow{f'} F(c')$



$d \setminus F \xrightarrow[\text{forgetful funct}]{f} \mathcal{C}$

$(f_1: F(f) \rightarrow X)$

$(c, f) \mapsto c$

$\hookrightarrow d \setminus F \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{D}$
 $(c, f) \mapsto F(c)$

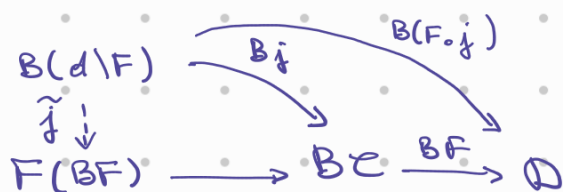
$(F(f) \xrightarrow{f_1} X \xrightarrow{f} Y)$

$d \setminus F \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{D}$
 $\uparrow \eta$
 $*_d = \text{const funct } d$

$\eta_{(c, f)}: d \xrightarrow{f} F(c)$

$\Rightarrow B(F \circ j) \sim \text{constant map i.e. } B(F \circ j) \text{ contractible}$

$\Rightarrow \exists \tilde{j}$ making the following diagram commute



Topologically:

$$F(f) \rightarrow X \xrightarrow{f} Y$$

$$\uparrow \tilde{f}$$

$$\tilde{z} \xrightarrow{\sim} *y_0$$

$$H: Z \times I \rightarrow Y \quad H_0 \equiv y_0 \quad H_1 = f \circ f$$

$$\text{Let } \gamma_z: I \rightarrow Y$$

$$t \mapsto H(z, t)$$

$\Rightarrow \gamma_z$ is a path in Y st $\gamma(0) = y_0$

$$\tilde{f}: Z \rightarrow F(f) = \{(\alpha, \alpha) \mid \alpha: I \rightarrow Y \quad \begin{matrix} \alpha(0) = y_0 \\ \alpha(1) = f(x) \end{matrix}\}$$

$$z \mapsto (f(z), \gamma_z) \leftarrow \begin{matrix} \gamma_z(0) = y_0 \\ \gamma_z(1) = f \circ f(z) \end{matrix} \quad \checkmark$$

NOTE: There is a dual notion

$F/d: \langle (c, f) \mid \begin{matrix} c \in \mathcal{C} \\ f: F(c) \rightarrow d \end{matrix} \rangle$
and the analogous constructions hold.

For later reference, we also need a "global" version of some categories:

$$\text{OBJ} = \bigcup_{d \in \mathcal{D}} \text{obj}(d \backslash F) = \{(c, f) : c \in \mathcal{C}, f: d \rightarrow F(c)\}$$

$\mathcal{D} \backslash F:$

$$\text{MOR: } \begin{array}{ccc} d & \rightarrow & F(c) \\ \downarrow & & \\ d' & \rightarrow & F(c') \end{array} \longleftrightarrow$$

$$c \xrightarrow{a} c', \quad d \xleftarrow{b} d':$$

$$\begin{array}{ccccc} d & \longrightarrow & F(c) & & \\ b \uparrow & \curvearrowright & \downarrow F(a) & & \\ d' & \longrightarrow & F(c') & & \end{array}$$

Forgetful functor

$$f': \mathcal{D} \backslash F \rightarrow \mathcal{C} \times \mathcal{D}^{\text{op}}$$

$$(c, f) \mapsto (c, \underset{d}{f} F(c))$$

? • When is BF a ^{homotopy} equivalence? $\leadsto$ THM A

• When is $\tilde{f}$ a hom equivalence? $\leadsto$ THM B

$\hookrightarrow$ get long exact sequence
in homotopy

§ Quillen's theorems

THM A:

$F: \mathcal{C} \rightarrow \mathcal{D}$ functor st $d \backslash F$ contractible $\forall d \in \mathcal{D}$
 $\Rightarrow BF: B\mathcal{C} \rightarrow B\mathcal{D}$ homotopy eqv.

THM B:

$F: \mathcal{C} \rightarrow \mathcal{D}$ functor st $\forall d \rightarrow d' \in \mathcal{D}$ the induced functor $d' \backslash F \rightarrow d \backslash F$ is a homotopy equivalence
 $\Rightarrow \forall d \in \mathcal{D}$ the geometric realization of
$$d \backslash F \xrightarrow{j} \mathcal{C} \xrightarrow{F} \mathcal{D} \quad \text{is a homotopy fibration seq.}$$

$\Rightarrow$ Long exact sequence

$$\dots \rightarrow \pi_{i+1} B\mathcal{D} \xrightarrow{\partial} \pi_i B(d \backslash F) \xrightarrow{j_*} \pi_i B\mathcal{C} \xrightarrow{F_*} \pi_i B\mathcal{D} \rightarrow \dots$$

To prove the thm's, we need some properties of bisimplicial sets

DEF: $X: (\Delta \times \Delta)^{op} \rightarrow \text{Sets}$ = "bisimplicial set"

• horizontal/vertical face maps & degeneracy maps:

$$\partial_i^h: X_{p,q} \rightarrow X_{p-1,q} \quad \partial_i^v: X_{p,q} \rightarrow X_{p,q-1}$$

$$\sigma_i^h: X_{p,q} \rightarrow X_{p+1,q} \quad \sigma_i^v: X_{p,q} \rightarrow X_{p,q+1}$$

s.t. horizontal maps commute w/ vertical maps

$\hookrightarrow \forall p, q, X_{p,*}$ & $X_{*,q}$ are simplicial sets

Also in this case we get a geom realization

functor: $X \mapsto |X|$ (glue a product $\Delta^p \times \Delta^q$ for any (non deg) element of $X_{p,q}$ and use both vertical and horizontal face maps to construct the gluing)

$$\text{diag}: \text{biset} \longrightarrow \text{Set} \\ X \longmapsto \text{diag}(X), \quad (\text{diag } X)_p = X_{p,p}$$

Property: $|X| \cong_{\text{homot}} |\text{diag } X|$

THM: $f: X \rightarrow Y$ map of bisimpl sets.

1) $|X_{p,\bullet}| \rightarrow |Y_{p,\bullet}|$ homotopy eqv $\forall p \Rightarrow |X| \rightarrow |Y|$ homotopy eqv.

2) If $Y = N\mathcal{J}$ (i.e. $Y_{p,q} = (N\mathcal{J})_p \forall p, q$) and

$f^{-1}(i, \bullet) \rightarrow f^{-1}(j, \bullet)$ is a homotopy equivalence $\forall i \rightarrow j$ in $\mathcal{J}$

$\Rightarrow B(f^{-1}(i, \bullet)) \rightarrow |X| \rightarrow B\mathcal{J} = |Y|$ is a homotopy fibration sequence $\forall i \in \mathcal{J}$.

LEMMA: $F: \mathcal{C} \rightarrow \mathcal{D}$ functor $\Rightarrow \mathcal{D}/F \rightarrow \mathcal{C}$ is a homotopy equivalence

pf: Let X be the bisimpl set:

$$X_{p,q} = \left\{ (d_p \rightarrow d_{q-1} \rightarrow \dots \rightarrow d_0 \rightarrow F(c_0), c_0 \rightarrow \dots \rightarrow c_q) \right\}$$

Let $N\mathcal{C}$ be seen as a bisimplicial set, constant in horizontal index, i.e. $(N\mathcal{C})_{p,q} := (N\mathcal{C})_q$

We have a "projection" $\pi: X \longrightarrow N\mathcal{C}$
 $\left(\begin{smallmatrix} d_p \rightarrow \dots \rightarrow d_0 \rightarrow F(c_0) \\ c_0 \rightarrow \dots \rightarrow c_q \end{smallmatrix} \right) \longmapsto (c_0 \rightarrow \dots \rightarrow c_q)$

NOTE: $(\text{diag } X)_p = \left\{ \begin{smallmatrix} d_p \rightarrow \dots \rightarrow d_0 \rightarrow F(c_0) \\ c_0 \rightarrow \dots \rightarrow c_p \end{smallmatrix} \right\} \cong N(\mathcal{D}/F)_p$

i.e. $|\text{diag } X| \simeq |X_{\dots}| \simeq B(\mathcal{D}/F)$

consider the forgetful functors $\mathcal{D} \xleftarrow{P_2} \mathcal{D}/F \xrightarrow{P_1} \mathcal{C}$

Under the above equivalence:

$$|X| \xrightarrow{\pi} B\mathcal{C} \quad \longleftrightarrow \quad B(\mathcal{D}/F) \xrightarrow{BP_1} B\mathcal{C}$$

Therefore: Bp_1 homotopy eqv $\Leftrightarrow \pi$ homotopy eqv.

$$\pi^{-1}(*, c_0 \rightarrow \dots \rightarrow c_q) \cong \{(d_p \rightarrow \dots \rightarrow d_0 \rightarrow F(c_0))\} \cong (N(D \setminus F(c_0)))_p$$

i.e. $\pi^{-1}(*, c_0 \rightarrow \dots \rightarrow c_q) \cong \underbrace{N(D \setminus F(c_0))}_{\text{contractible, because } D \setminus F(c_0) \text{ has a final object}} \quad \forall c_0 \rightarrow \dots \rightarrow c_q \in NC$

$\Rightarrow$ all the fibers are homotopy equivalent

$\Rightarrow$ By THM 2: $B(D \setminus F(c_0)) \rightarrow |X| \xrightarrow{|\pi|} BC$ is a homotopy fibration. Since the fiber is contractible, this shows that $|\pi|$ is a homotopy equivalence. $\square$

pf of THM A:

$$C \leftarrow D \setminus F \longrightarrow D^{\text{op}}$$

$\underbrace{\hspace{10em}}$
hom eqv by
the lemma

$$D \setminus 1_D$$

Consider $D \setminus F \xrightarrow{\quad} D \setminus D = \{d \rightarrow d'\}$

$$(d \rightarrow F(c), c) \mapsto (d \rightarrow F(c))$$

Get a comm diagram:

$$\begin{array}{ccccc}
 \mathcal{C} & \xleftarrow{\simeq} & \mathcal{D} \setminus F & \longrightarrow & \mathcal{D}^{\circ P} \\
 \downarrow F & & \downarrow & & \parallel \\
 \mathcal{D} & \xleftarrow{\simeq} & \mathcal{D} \setminus \mathcal{D} & \longrightarrow & \mathcal{D}^{\circ P}
 \end{array}$$

Remains to show: also the remaining hor maps are homotopy eqv

Analogous to the lemma:

$$B(\mathcal{D} \setminus F) \simeq B\mathcal{X} \xrightarrow{\pi} B\mathcal{D}^{\circ P}$$

$$\Rightarrow \pi^{-1}(d_q \rightarrow \dots \rightarrow d_0) = N(d_0 \setminus F) \text{ contractible}$$

$\Rightarrow$ conclude in the same way $\square$

pf of THMB:

Analogous to the lemma, we take $N(\mathcal{D}^{\circ P})$ as a bisimplicial complex by $N(\mathcal{D}^{\circ P})_{p,q} := N(\mathcal{D}^{\circ P})_p$.

$$\text{We have a projection } p: \begin{array}{c} X \\ \left(\begin{array}{c} d_p \rightarrow \dots \rightarrow d_0 \rightarrow F(c_0)_1 \\ c_0 \rightarrow \dots \rightarrow c_q \end{array} \right) \end{array} \longrightarrow \begin{array}{c} N(\mathcal{D}^{\circ P}) \\ (d_p \rightarrow \dots \rightarrow d_0) \end{array}$$

$$p^{-1}(d_p \rightarrow \dots \rightarrow d_0, *) \cong \{(c_0 \rightarrow \dots \rightarrow c_q, f: d_0 \rightarrow F(c_0))\} \cong N(d_0 \setminus F)_q,$$

$$\text{i.e. } p^{-1}(d_p \rightarrow \dots \rightarrow d_0, *) \cong N(d_0 \setminus F) \quad \forall d_p \rightarrow \dots \rightarrow d_0 \in N(\mathcal{D}^{\circ P})$$

By hypothesis $B(d \setminus F) \simeq B(d' \setminus F) \quad \forall d \rightarrow d'$. Therefore we are under the hyp of the THM and we can conclude $B(d \setminus F) \rightarrow |X| \rightarrow B\mathcal{D}^{\circ P}$ is a homotopy fibration sequence.

Like in the lemma, $|X| \simeq B(\mathcal{D} \setminus F)$ and

$$B(d \setminus F) \rightarrow |X| \hookrightarrow B(d \setminus F) \rightarrow B(\mathcal{D} \setminus F) \text{ induced by the inclusion } i: d \setminus F \rightarrow \mathcal{D} \setminus F$$

$$|X| \rightarrow B\mathcal{D}^{\circ P} \hookrightarrow B(\mathcal{D} \setminus F) \rightarrow B\mathcal{D}^{\circ P} \text{ induced by } f: \mathcal{D} \setminus F \rightarrow \mathcal{D}^{\circ P}$$

$$\Rightarrow B(d \setminus F) \rightarrow B(D \setminus F) \rightarrow B D^{\circ p} \text{ hom fibration}$$

$$\quad \quad \quad \downarrow \text{By LEMMA}$$

$$\quad \quad \quad B\mathbb{C}$$

In order to conclude, consider:

$$\begin{array}{ccccc} B(d \setminus F) & \longrightarrow & B(D \setminus F) & \xrightarrow[\sim]{p_1} & B\mathbb{C} \\ \downarrow & (1) & \downarrow & (2) & \downarrow \pi \\ B(d \setminus D) & \longrightarrow & B(D \setminus D) & \xrightarrow[\sim]{p_1} & B D \\ & & \downarrow \text{D} \setminus \text{Id} & & \\ s \downarrow & (3) & s \downarrow & & \\ * & \longrightarrow & B(D^{\circ p}) & & \end{array}$$

$\boxed{\begin{smallmatrix} (1) \\ (3) \end{smallmatrix}}$ is homotopy cartesian by what we showed

$\Rightarrow (1)$ is homotopy cartesian

$\Rightarrow \boxed{(1)(2)}$ is homotopy cartesian,

i.e.

$B(d \setminus F) \rightarrow B\mathbb{C} \rightarrow B D$ is a homotopy fibration sequence. $\square$