# The Chern Character Isomorphism

Recall: $X$: smooth quasi-proj algebraic variety $/k$

$$K_0(X) := \frac{\text{Abelian gp generated by } [\mathcal{E}] \leftarrow \text{finite locally free sheaves}/X}{\langle [\mathcal{E}] = [\mathcal{F}] + [\mathcal{G}] \mid 0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0 \text{ exact} \rangle}$$

$$G_0(X) := \frac{\text{Abelian gp generated by } [\mathcal{F}] \leftarrow \text{coherent sheaves}/X}{\langle [\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}''] \mid 0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \text{ exact} \rangle}$$

Cartan homomorphism: $K_0(X) \xrightarrow{\sim} G_0(X)$

Any coherent sheaf on $X$ can be resolved by finite loc. free sheaves. (Serre)

Chow group of degree $d$: $CH^d(X) := Z^d(X)/\sim_{rat}$

Chow ring: $CH^*(X) = \bigoplus_{d=1}^{n} CH^d(X)$ multiplicative structure is given by intersection (well-defined by moving lemma)

proper pushforward. } well-behaved
flat pullback

Chern class: $c_i : K_0(X) \longrightarrow CH^i(X)$ Not a group hom.

Total Chern class: $c = 1 + \sum_{i=1}^{n} c_i : (K_0(X), +) \longrightarrow (CH^*(X)^\times, \cap)$

Goal: Construct a ring hom.

Chern character: $ch : K_0(X) \longrightarrow CH^*(X) \otimes \mathbb{Q}$

and show it becomes an isom. after tensoring with $\mathbb{Q}$.

## §1 Chern classes and Chern characters

Thm (EH, Thm 5.3)

There is a unique way to define $c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + \cdots + c_n(\mathcal{E}) \in CH^*(X)$ for a v.b. $\mathcal{E}$ s.t.

a) (Line bundles) $c(\mathcal{L}) = 1 + c_1(\mathcal{L})$ for a line bundle $\mathcal{L}$

b) (Bundles with enough sections) If $\tau_0, \cdots, \tau_{r-i} \in \Gamma(X, \mathcal{E})$ s.t. $D = V(\tau_0 \wedge \cdots \wedge \tau_{r-i})$ has codim $i$, then $c_i(\mathcal{E}) = [D] \in CH^i(X)$

c) (Whitney's formula) If $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ exact, then

$$c(\mathcal{F}) = c(\mathcal{E})\, c(\mathcal{G})$$

d) (Functoriality) If $\varphi: Y \xrightarrow{\text{smooth}} X$, then $\varphi^*(c(\mathcal{E})) = c(\varphi^*(\mathcal{E}))$

Concrete description of $c_i$:

Thm (Projective bundle formula)

$\mathcal{E}$: rank $r$ bundle $/X$ $\quad \pi: \mathbb{P}(\mathcal{E}) \to X \quad \zeta := c_1((\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1))^\vee)$

Then: (1) $\pi^*: CH^*(X) \to CH^*(\mathbb{P}(\mathcal{E}))$ injective.

(2) $\zeta$ satisfies $f(\zeta) = \zeta^r + \pi^* c_1 \zeta^{r-1} + \cdots + \pi^* c_r = 0$. and $CH^*(\mathbb{P}(\mathcal{E})) = CH^*(X)[\zeta]/f(\zeta)$

Define $c_i(\mathcal{E}) := c_i$

Splitting principle: we can show any identity among Chern classes of general vector bundles by considering splitting vector bundles

Lemma (Splitting construction)

$\mathcal{E}$: rank $r$ $/X$ $\quad$ Then there is a smooth $Y$ and $\varphi: Y \to X$ s.t.

1) $\varphi^*: A(X) \to A(Y)$ is injective

2) $\varphi^*\mathcal{E}$ admits a filtration $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{r-1} \subset \mathcal{E}_r = \varphi^*\mathcal{E}$ by vector subbundles $\mathcal{E}_i \subset \varphi^*\mathcal{E}$ with $\mathcal{E}_i/\mathcal{E}_{i-1}$ locally free of rank 1.

Proof. Consider $\pi: \mathbb{P}(\mathcal{E}) \to X$ we have

$$0 \to \mathcal{E}_1 \to \pi^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \to 0$$

$\mathcal{E}_1$ is locally free of rank $r-1$.

Consider $\pi_1 : \mathbb{P}(\mathcal{E}_1) \to \mathbb{P}(\mathcal{E})$ with

$$0 \to \mathcal{E}_2 \to \pi_1^*\mathcal{E}_1 \to \mathcal{O}(1) \to 0$$

By iterating this construction, it stops at $\mathbb{P}(\mathcal{E}_{r-2}) =: Y$ Flag bundle $Fl(\mathcal{E})$ $\square$

By Whitney. $c(\varphi^*\mathcal{E}) = \prod_{i=1}^{r} c(\mathcal{E}_i/\mathcal{E}_{i-1}) = \prod_{i=1}^{r}(1+\alpha_i)$ Let $\alpha(\mathcal{E}) = \{\alpha_1, \cdots, \alpha_r\}$

Chern roots

Corollary (Properties of $c_i$)

1) (Duals) $c_i(\mathcal{E}^*) = (-1)^i c_i(\mathcal{E})$

2) (tensor product) $\mathcal{E}$ : rank $r$ $\mathcal{L}$ : line bundle $\mathcal{F}$ : rank $s$

$$c_k(\mathcal{E}\otimes\mathcal{L}) = \sum_{i=0}^{k}\binom{r-k+i}{i} c_1(\mathcal{L})^i c_{k-i}(\mathcal{E})$$

$$c_1(\mathcal{E}\otimes\mathcal{F}) = s\cdot c_1(\mathcal{E}) + r\, c_1(\mathcal{F})$$

3) (exterior power) $\mathcal{E}$ : chern roots $\alpha_1, \cdots, \alpha_r$

$$c(\Lambda^p\mathcal{E}) = \prod_{i_1<\cdots<i_p}(1+(\alpha_{i_1}+\cdots+\alpha_{i_p}))$$

<u>Def'n</u> If $\mathcal{E}$ is a vector bundle with chern roots $\alpha_1, \cdots, \alpha_r$, then the Chern character is : $ch(\mathcal{E}) = \sum_{i=1}^{r} e^{\alpha_i} \in CH^*(X)\otimes\mathbb{Q}$.

The $k$-th graded piece $ch_k(\mathcal{E}) = \sum_{i=1}^{r}\frac{\alpha_i^k}{k!}$ can be expressed as a polynomial in the elementary symmetric polynomial of $\alpha_i$ ( which is $c_i(\mathcal{E})$)

For example,

$$ch_0(\mathcal{E}) = \mathrm{rank}(\mathcal{E})$$

$$ch_1(\mathcal{E}) = c_1(\mathcal{E})$$

$$ch_2(\mathcal{E}) = \frac{c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})}{2}$$

$$ch_3(\mathcal{E}) = \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3)$$

If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$, then $\alpha(\mathcal{E}) = \alpha(\mathcal{E}') \cup \alpha(\mathcal{E}'')$.

Thus, $ch(\mathcal{E}) = ch(\mathcal{E}') + ch(\mathcal{E}'')$

If $\alpha(\mathcal{F}) = \{\alpha_1, \cdots, \alpha_f\}$ $\qquad \alpha(\mathcal{G}) = \{\beta_1, \cdots, \beta_g\}$

then $\alpha(\mathcal{F} \otimes \mathcal{G}) = \{\alpha_i \otimes \beta_j \mid 1 \le i \le f, 1 \le j \le g\}$

Thus, $ch(\mathcal{F} \otimes \mathcal{G}) = ch(\mathcal{F})\, ch(\mathcal{G})$

$\rightsquigarrow$ $\quad ch : K_0(X) \longrightarrow CH^*(X) \otimes \mathbb{Q}$. ring homomorphism.

§2 Topological filtration on $G_0(X)$

<u>Def'n</u> The topological filtration (or coniveau) on $G_0(X)$ is

$$\cdots \subset F^{i+1}_{top} G_0(X) \subset F^i_{top} G_0(X) \subset \cdots \subset G_0(X)$$

where $F^i_{top} G_0(X)$ is generated by the classes $[\mathcal{F}]$ with
$\operatorname{codim}_X(\operatorname{Supp}(\mathcal{F})) \geq i$.

<u>Want to show:</u> $F^i_{top} G_0(X) = \langle [\mathcal{O}_Z] \mid Z \subset X$ irred. closed with $\operatorname{codim}_X Z \geq i \rangle$ $\quad (*)$

Denote it by $(*)$.

For this, we need a dévissage theorem. The following is taken from the paper Sur les faisceaux algébriques et les faisceaux analytiques cohérents, by Grothendieck

<u>Def'n</u> $\mathcal{C}$ : Ab Category $\quad \mathcal{C}' \subset \mathcal{C}$ subcategory.

We say that $\mathcal{C}'$ is an exact sub-category of $\mathcal{C}$ if for any exact $0 \to A' \to A \to A'' \to 0$ in $\mathcal{C}$, any two terms in $\mathcal{C}'$ will imply the third term in $\mathcal{C}'$ and any direct summand of $A \in \mathcal{C}'$ lies in $\mathcal{C}'$.

<u>Thm</u> (Dévissage) $X$ : alg. var, $Y_i \subset X$ irred. components, $\mathcal{O}_{Y_i}$-mod $\mathcal{F}_{Y_i}$ with $\operatorname{Supp}(\mathcal{F}_{Y_i}) = Y_i$.
$K(X)$ := Ab Category of coherent sheaves on $X$. Then any exact subcategory $K \subset K(X)$ containing $\{\mathcal{F}_i\}$ is the same as $K(X)$.

Let $K$ be the minimal exact subcategory of $F^i_{top} G_0(X)$ containing all $\mathcal{O}_Z$ for $Z \subset X$ irred. closed w/ codim $\geq i$. Then $K \simeq F^i_{top} G_0(X)$ and $(*)$ holds. $\square$

Another description of rat'l equivalence:

$Z \subset X \times \mathbb{A}^1_k$ closed of $\dim = n+1$.

$$\begin{array}{ccccc} Z^0 & \longrightarrow & Z & \longleftarrow & Z^1 \\ \downarrow & & \downarrow & & \downarrow \\ X \times \{0\} & \xrightarrow{i_0} & X \times \mathbb{A}^1_k & \xleftarrow{i_1} & X \times \{1\} \end{array}$$

$R_n(X) \hookrightarrow Z_n(X)$ consists of: $\langle \sum_i n_i ([Z_i^0] - [Z_i^1]) \rangle$

where $\sum n_i [Z_i] \in Z_{n+1}(X \times \mathbb{A}^1_k)$ and $Z_i$ intersects

$\dim(Y \cap Z) \le \dim Y + \dim Z - \dim X \longrightarrow$ properly with $X \times \{0\}$ and $X \times \{1\}$.

Consider the map $\gamma : Z^p(X) \longrightarrow F^p_{top}(G_0(X))$

$$[Y] \longmapsto [\mathcal{O}_Y]$$

Let $Z \in Z^{p-1}(X \times \mathbb{A}^1_k)$, $\partial^i [Z] := [Z] \cup [X \times \{i\}] \in Z^p(X)$ $\quad i = 0, 1$

$$Z^{p-1}(X \times \mathbb{A}^1_k) \xrightarrow{\partial^0 - \partial^1} Z^p(X) \twoheadrightarrow CH^p(X) \to 0$$

**Lemma**: $Y, Z$ closed subvar. (codim $p, q$) Suppose $Y, Z$ intersect properly. Then

$$\gamma[Y] \cup \gamma[Z] = \gamma([Y] \cup [Z]) \quad \text{in} \quad G_0(X)^{\ge p+q} / G_0(X)^{\ge p+q+1}.$$

**Prop.** $\gamma$ induces a group hom. $\gamma : CH^p(X) \longrightarrow F^p / F^{p+1}$

**Proof.** To show: $\gamma(\partial^0 [Z]) = \gamma(\partial^1 [Z])$

$$\iff [\mathcal{O}_Z] \cup [\mathcal{O}_{X \times \{0\}}] = [\mathcal{O}_Z] \cup [\mathcal{O}_{X \times \{1\}}]$$

$$\iff i_0^* = i_1^*$$

$$\begin{array}{ccc} G_0(X \times \mathbb{A}^1) & \overset{i_0^*}{\underset{i_1^*}{\rightrightarrows}} & G_0(X) \\ p^* \uparrow \wr & \overset{id}{\underset{id.}{\nearrow\!\!\!\nearrow}} & \\ G_0(X) & & \end{array}$$

$\square$

Want: $\gamma$ is an isomorphism.

Consider. $CH^i(X) \xrightarrow{\gamma} F^i / F^{i+1} \xrightarrow{c_i} CH^i(X)$

§ Ch isomorphism.

<u>Def'n</u> $\mathcal{E}$ : v.b. with $\partial(\mathcal{E}) = \{\alpha_1, \cdots, \alpha_r\}$

The Todd class of $\mathcal{E}$ is $Td(\mathcal{E}) = \prod_{i=1}^{n} \frac{\alpha_i}{1-e^{-\alpha_i}} = 1 + $ (Higher degrees)

If $f: X \to Y$ flat, $f^* ch(\mathcal{E}) = ch(f^*(\mathcal{E}))$ by functoriality.

But push-forward does not commute with ch.

Thm (GRR) $f: X \to Y$ proper, $Y$ sm/k. Then $\forall\, \alpha \in K_0(X)$.

$$ch(f_*\alpha) \cdot Td(T_Y) = f_*(ch(\alpha) \cdot Td(T_X)) \quad \text{in} \quad CH^*(Y)_{\mathbb{Q}}.$$

Give $CH_*(X)$ natural filtration: $F_k CH_*(X) = \sum_{i \le k} CH_i(X)$

Assume $f: V \hookrightarrow X$ regular closed embedding. $\dim V = k$. $\operatorname{codim}_X V = i$.

Td takes sums to products $\Rightarrow$ $Td(N)^{-1} = f^* Td(T_Y)^{-1} \cdot Td(T_X)$

if $\alpha = [\mathcal{O}_V]$ $\Longrightarrow$ $ch([\mathcal{O}_V]) = f_*([V] \cdot Td(N)^{-1}) = f_*([V] + \alpha)$ $\quad \alpha \in F_{k-1} CH_*(V)$

$$\Longrightarrow \quad c_j([\mathcal{O}_Z]) = 0 \quad \text{if } j < i$$

$$c_i([\mathcal{O}_Z]) = (-1)^i (i-1)! \, [Z]$$

<u>Rmk</u> Considering regular embedding is enough. In fact, for any embedding $f: V \hookrightarrow X$, let $S \subsetneq V$ s.t. $V-S \hookrightarrow X-S$ regular.

Since $CH_*(S) \to CH_*(X) \to CH_*(X-S) \to 0$ exact. and

the image of $CH_*(S)$ is in $f_*(F_{k-1} CH_*(V))$.

Therefore, for $CH^i(X) \xrightarrow{\gamma} gr^i G_0(X) \xrightarrow{c_i} CH^i(X)$

$$\gamma \circ c_i = (-1)^i (i-1)! \, id$$

$$c_i \circ \gamma = (-1)^i (i-1)! \, id.$$

$\Longrightarrow$ (*) $c_i \otimes \mathbb{Q}$ : $gr^i G_0(X) \otimes \mathbb{Q} \xrightarrow{\sim} CH^i(X) \otimes \mathbb{Q}$ isom of $\mathbb{Q}$-v.s.

**Thm** $ch \otimes \mathbb{Q} : K_0(X)_{\mathbb{Q}} \longrightarrow CH^*(X)_{\mathbb{Q}}$ is an isom of rings.

**Proof.** Consider $ch_i = \pi_i \circ ch : K_0(X) \xrightarrow{ch} CH^*(X)_{\mathbb{Q}} \xrightarrow{\pi_i} CH^i(X)_{\mathbb{Q}}$

$ch_i =$ i-th power sum part of ch

$$= \frac{(-1)^i}{(i-1)!} c_i + \underbrace{(\text{lower degree } c_j)}_{\text{vanish on } F^i_{top} K_0(X)}$$

$\rightsquigarrow$ $ch_i =$ multiple of $c_i$ $\rightsquigarrow$ isom $gr^i K_0(X)_{\mathbb{Q}} \to CH^i(X)_{\mathbb{Q}}$. $\square$