An overview on p-adic L-functions

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Classical L-functions

Classically, an L-function is a meromorphic function on $\mathbb{C}$ (often entire) associated to a mathematical object X (usually coming from geometry, representation theory, number theory . . .).

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The construction of a complex L-function usually goes *mutatis mutandis* as follows:

(i) Write a series (a so-called Dirichlet series) of the form

$$L(s) = L(X, s) = \sum_{n=1}^{+\infty} \frac{a_n}{n^s}$$

where the coefficients $\{a_n\}_{n\geq 1} \subset \mathbb{C}$ satisfy growth conditions that ensure that L defines a holomorphic function on the right half-plane $\{\mathrm{Re}(s) > r\} \subset \mathbb{C}$ for some $r \in \mathbb{R}$, $r \geq 1$. The coefficients $\{a_n\}$ encode information about the object X.

Classical L-functions

(ii) Find a gamma factor (i.e. a suitable meromorphic function on $\mathbb{C}$, often related to the usual gamma function) such that the function ˆs• Lˆs• ˆs• extends to a meromorphic function on $\mathbb{C}$ satisfying a functional equation of the form

$$ \hat{}s\bullet \quad c_L \quad \hat{}k \quad s\bullet $$

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(iii) Use the above functional equation to extend L to a meromorphic function on C. which we will denote again by L LˆX; s•.

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A prototypical example of this phenomenon is the so-called *analytic class number formula* (due to Dirichlet, Kummer, Dedekind, . . .).

If K is a number field (i.e. a finite field extension of $\mathbb{Q}$) one can attach to K the so-called Dedekind zeta function ζ_K, prove the analytic continuation and functional equation and finally show that

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}\,\mathrm{Reg}_K\,h_K}{w_K\sqrt{|d_K|}}$$

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Many important open conjectures in number theory can be phrased in terms of L-functions.

Dirichlet characters

Let $N \in \mathbb{Z}$, $N \geq 2$. A Dirichlet character defined modulo N is a function $\chi : \mathbb{Z} \to \mathbb{C}$ such that

- $\chi(1) = 1$, $\chi(N) = 0$
- $\chi(n) = \chi(m)$ if $n \equiv m \mod (N)$
- $\chi(nm) = \chi(n)\chi(m)$ for all $n, m \in \mathbb{Z}$

We say that χ is trivial if $\chi(\mathbb{Z}) \subseteq \{0, 1\}$.

The constant function $\mathbf{1} : \mathbb{Z} \to \mathbb{C}$ ($\mathbf{1}(n) = 1$ for all $n \in \mathbb{Z}$) is the unique (and trivial!) Dirichlet character modulo 1.

Dirichlet L-functions

The Dirichlet L-series associated to χ is

$$L(\chi, s) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}$$

This series converges for $\mathrm{Re}(s) > 1$. Actually $L(\chi, s)$ defines a holomorphic function for $\mathrm{Re}(s) > 0$ if χ is not trivial. In this case $L(\chi, 1) \neq 0$.

If χ is trivial then $(s - 1) \cdot L(\chi, s)$ can be continued to a holomorphic function for $\mathrm{Re}(s) > 0$ (not vanishing at $s = 1$).

These two different behaviours are the key ingredients that allowed Dirichlet to prove his theorem about primes in arithmetic progressions in 1837.

Riemann ζ function

When $\chi = \mathbf{1}$ then $L(\mathbf{1}, s) = \zeta(s)$ is the Riemann zeta function.

Euler proved (in 1737) that it admits a product expansion as

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In 1859 Riemann proved that:

(i) there is an entire function ξ such that when $\mathrm{Re}(s) > 1$ it holds

$$\xi(s) = \frac{1}{2} \cdot s(s-1) \cdot \pi^{-s/2} \cdot \Gamma\left(\frac{s}{2}\right) \cdot \zeta(s)$$

(ii) the function ξ satisfies $\xi(s) = \xi(1-s)$ for all $s \in \mathbb{C}$.

Hence ζ can be continued to a meromorphic function on $\mathbb{C}$ with a unique simple pole at $s = 1$

Riemann hypothesis

It is not too hard to prove that

(i) $\Gamma(n+1) = n!$ and $\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$ for $n \in \mathbb{N}$

(ii) Γ has simple poles at $s = -n$ for $n \in \mathbb{N}$ and is holomorphic elsewhere.

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Since $\zeta(s) \neq 0$ when $\mathrm{Re}(s) > 1$, we obtain that for $\mathrm{Re}(s) < 0$ it can happen $\zeta(s) = 0$ if and only if $s = -2n$ for $n \in \mathbb{Z}_{\geq 1}$. These are the so-called *trivial zeroes* of ζ. The interesting zeroes of ζ lie in the strip $S = \{0 \leq \mathrm{Re}(s) \leq 1\}$ and $\zeta(s_0) = 0$ for some $s_0 \in S$ if and only if $\zeta(1 - s_0) = 0$.

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Conjecture (Riemann, 1859)

The non-trivial zeroes of ζ all lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

Bernoulli numbers and special values

Thanks to Euler we know that for all $n \in \mathbb{Z}_{\geq 1}$

$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n} B_{2n}}{2\,(2n)!}$$

where $B_k \in \mathbb{Q}$ denotes the k-th Bernoulli number.
These rational numbers are defined via the equality of formal power series in $\mathbb{Q}[[X]]$.

$$\frac{X}{\exp(X) - 1} = \sum_{k \geq 0}^{\infty} B_k \frac{X^k}{k!}$$

This also means that, for $n \in \mathbb{Z}_{\geq 1}$

$$\zeta(1 - 2n) = -\frac{B_{2n}}{2n} \in \mathbb{Q}$$

is a rational number!

Generalized Bernoulli numbers

If χ is a Dirichlet character modulo N, we define generalized Bernoulli numbers $B_{n,\chi} \in \mathbb{Q}(\chi)$ via a modified generating function

$$\sum_{a=1}^{N} \frac{\chi(a)\, X \exp(aX)}{\exp(NX) - 1} = \sum_{n \geq 0} B_{n,\chi} \frac{X^n}{n!}$$

And one can prove that for $k \geq 1$

$$L(\chi, 1-k) = -\frac{B_{k,\chi}}{k} \in \mathbb{Q}(\chi)$$

is an algebraic number!

The p-adic topology on $\mathbb{Q}$

$\mathbb{R}$ = completion of $\mathbb{Q}$ with respect to the Euclidean absolute value (Archimedean)

Are there other absolute values on $\mathbb{Q}$? If p is a prime number and $x = r/s \in \mathbb{Q}$, we can set

$$|x|_p = c^{v_p(r) - v_p(s)}$$

where $c \in (0,1)$. This new absolute value satisfies a strong triangular inequality (we say it is non-Archimedean)

$$|x - y|_p \leq \max(|x|_p, |y|_p)$$

$\mathbb{Z}_p$, $\mathbb{Q}_p$ and beyond . . .

One can complete $\mathbb{Q}$ with respect to $|\cdot|_p$, obtaining a field denoted by $\mathbb{Q}_p$, called field of p-adic numbers. An element $\alpha \in \mathbb{Q}_p$ can be written uniquely as

$$\alpha = \sum_{n \geq M} a_n p^n$$

with $a_n \in \{0, 1, \ldots, p-1\}$. Inside $\mathbb{Q}_p$ we have the subring

$$\mathbb{Z}_p := \{\alpha \in \mathbb{Q}_p : |\alpha|_p \leq 1\} \supset \mathbb{Z}$$

known as the ring of p-adic integers.

We can thus see Dirichlet characters taking values in an algebraic closu $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ (after fixing an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$) and study them p-adically.

Towards p-adic Dirichlet L-functions

In particular it makes sense to ask whether there exist a (continuous/analytic) function

$$L_{p,\chi} : \mathbb{Z}_p \to \bar{\mathbb{Q}}_p$$

such that for $k \geq 1, k \in \mathbb{Z} \subset \mathbb{Z}_p$ it holds

$$L_{p,\chi}(1-k) = L(\chi, 1-k) \cdot \{\text{explicit factor at } p\}$$

The existence of such a function is suggested by the many congruences satisfied by Bernoulli numbers.

Kubota-Leopoldt p-adic L-function

Theorem (Kubota-Leopoldt, 1964)

Let χ be a (p-adic) Dirichlet character. Then there is a continuous function $L_{p,\chi} : \mathbb{Z}_p \setminus \{1\} \to \bar{\mathbb{Q}}_p$ such that for all $k \in \mathbb{Z}_{\geq 1}$ it holds

$$L_{p,\chi}(1-k) = -(1 - \chi\omega^{-k}(p) \cdot p^{k-1}) \cdot \frac{B_{k,\chi}}{k} =$$

$$= (1 - \chi\omega^{-k}(p) \cdot p^{k-1}) \cdot L(\chi\omega^{-k}, 1-k)$$

where $\omega : \mathbb{Z}_p \to \mathbb{Z}_p$ denotes the Teichmüller character

$$\omega(s) = \lim_{n \to +\infty} s^{p^n} \in \mu_{p-1} \cup \{0\} \subset \mathbb{Z}_p$$

Moreover if χ is non-trivial, $L_{p,\chi}$ extends to a continuous function on $\mathbb{Z}_p$.

One construction of $L_{p,\chi}$

- Write $\chi = \psi\eta$ with ψ primitive of conductor p^m and η primitive of conductor N with $p \nmid N$.
- Define a p-adic *pseudomeasure* $\mu_{p,\eta}$ on $\mathbb{Z}_p^\times$ and let

$$L_{p,\chi}(s) = \int_{\mathbb{Z}_p^\times} \psi\omega^{-1}(x) \langle x \rangle^{-s} \, d\mu_{p,\eta}$$

- Show that

$$L_{p,\chi}(1-k) = (1 - \chi\omega^{-k}(p) \, p^{k-1}) \, L(\chi\omega^{-k}, 1-k)$$

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Remark

A *measure* on $\mathbb{Z}_p^\times$ with values in $\mathbb{Z}_p$ can be thought as an element of

$$\mathrm{Hom}^{cts}_{\mathbb{Z}_p}(C(\mathbb{Z}_p^\times, \mathbb{Z}_p), \mathbb{Z}_p) \cong \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$$

L-functions attached to modular forms

Let $f \in S_k(N, \chi)$ be a normalized eigenform of level N, weight k and character χ. Then f has a q-expansion as

$$f = \sum_{n=1}^{\infty} a_n q^n \qquad q = \exp(2\pi i z), \ \mathrm{Im}(z) > 0$$

and the L-function associated to f is not surprisingly defined (at least for $\mathrm{Re}(s) > k/2 + 1$)

$$L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{p} \frac{1}{1 - a_p p^{-s} + \chi(p) p^{k-1-2s}}$$

$$= \prod_{p \mid N} \frac{1}{1 - a_p p^{-s}} \prod_{p \nmid N} \frac{1}{(1 - \alpha_p^1 p^{-s})(1 - \alpha_p^2 p^{-s})}$$

It extends to a holomorphic function on $\mathbb{C}$ and satisfies a functional equation $s \mapsto k - s$.

Triple product L-functions - classical case

Let f, g, h be normalized eigenforms of level N_f, N_g, N_h, character χ_f, χ_g, χ_h, weight k, l, m respectively. Let $N = \operatorname{lcm}(N_f, N_g, N_h)$. Write

$$f = \sum_{n \geq 1} a_n q^n \qquad g = \sum_{n \geq 1} b_n q^n \qquad h = \sum_{n \geq 1} c_n q^n$$

and set

$$L(f \otimes g \otimes h, s)_p = \prod_{\sigma \in \{1,2\}^{\{1,2,3\}}} \frac{1}{\left(1 - \alpha_p^{(\sigma(1))} \beta_p^{(\sigma(2))} \gamma_p^{(\sigma(3))} p^{-s}\right)} \quad \text{for } p \nmid N$$

$$L(f \otimes g \otimes h, s) = \prod_{p \nmid N} L(f \otimes g \otimes h, s)_p$$

Garrett and Harris-Kudla proved that $L(f \otimes g \otimes h, s)$ admits analytic continuation to $\mathbb{C}$ and functional equation $s \leftrightarrow k + l + m - 2 - s$.

Triple product p-adic L-functions

My PhD project is related to the construction of a p-adic L-function of three variables (k, l, m) that should interpolate (the algebraic part) of the special values

$$L\left(f_k \otimes g_l \otimes h_m, \frac{k+l+m-2}{2}\right)$$

where f, g, h are suitable p-adic families of eigenforms specializing to classical eigenforms in classical weights.

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This construction has been already achieved in many cases and with different approaches (some people involved: Andreatta, Bertolini, Darmo
Greenberg, Hsieh, Iovita, Rotger, Seveso, Venerucci, . . .) and we would
like to generalise it to more general settings.