Welschinger Invariants

I Moduli spaces

1. Space of (real) almost complex structures
2. Space of pseudo-holomorphic maps
3. Holomorphic bundles, Riemann-Roch
4. Moduli spaces - Theorem of regular values

II The Welschinger invariant

1. Definition and main theorem
2. Local description near a cusp
3. Stratification and local patching
4. Pf of main theorem

III Recursive formulas

1. The invariant Θ and its well-definition
2. The recursive relation and its proof
3. Generalization and computations

I.1 Almost complex structures

Def • X a (even dim'l) real manifold. An almost complex structure on X is an automorphism J of T_X with $J^2 = -\mathrm{id}$

• For (X, ω) symplectic manifold, say J is tamed by ω if $\omega(v, Jv) > 0, \forall v$; say J is compatible with ω if also $\omega(J\cdot, J\cdot) = \omega$

• Multiplication $\times i : \mathbb{C}^n \to \mathbb{C}^n$ gives a J on $T\mathbb{C}^n$ which is invariant under holomorphic isos. Thus a complex manifold has a canonical almost complex structure

• The Nijenhuis-Nirenberg operator NN is a diff. operator with $NN(J) = 0$ (integrable) $\Leftrightarrow$ J comes from a complex manifold structure

• for $\dim X = 2$ all J are integrable

$$\{\text{almost complex structures}\} = \{\text{complex structures}\}$$

Def. (X, ω) a symplectic 4-manifold

$\mathcal{J}_\omega = \left\{ \begin{array}{l} \text{almost } \mathbb{C}\text{-structures on } X \\ \text{tamed by } \omega \end{array} \right\}$

- A real structure on (X, ω) is an automorphism $c_X : X \to X$ with $c_X^2 = id$, $c_X^*(\omega) = -\omega$. $\mathbb{R}X := X^{c_X}$

$\mathbb{R}\mathcal{J}_\omega = (\mathcal{J}_\omega)^{c_X}$, $c_X(J) := -dc_X J \, dc_X$

(c_X is antiholomorphic for $J \in \mathbb{R}\mathcal{J}_\omega$)

Theorem $\mathcal{J}_\omega$ is a separable Banach manifold and $\mathbb{R}\mathcal{J}_\omega$ is a separable submanifold

Remarks: • "Banach manifold": manifold modeled on Banach spaces

• one need to consider J as a section of $\mathrm{End}\, T_X$ after a completion wrt a $L^{k,p}$ - sobolev norm: L^p for 1^{st} k derivatives

2. Pseudo-holomorphic maps

Fix: • real symplectic 4-mnfld (X, ω, c_X) and $x = (x_1, \dots, x_m)$ a real configuration of distinct points of X

• $S := (S^2, \omega_S)$ oriented 2-sphere

$\mathcal{J}_S = \{$(almost) $\mathbb{C}$-str. on S, compatible with $\omega_S\}$

• $z = (z_1, \dots, z_m)$ distinct pts of S

• $d \in H_2(X, \mathbb{Z})$ with $c_1(X) \cdot d > 0$, $c_X^*(d) = -d$

Def $\mathcal{S}^d(x) = \{ u : S \to X \mid u_*([S]) = d,\ u(z) = x \}$
(Sobolev complete)

$\cup$

$\mathcal{P}^d(x) = \{(u, j, J) \in \mathcal{S}^d(x) \times \mathcal{J}_S \times \mathcal{J}_\omega \mid du \circ j = J \circ du\}$

(j, J) pseudo-holomorphic maps

$$\mathcal{P}^{d}(X) \supset \mathcal{P}^{d*}(X)$$

$$\shortparallel$$

$\{(u, j, J) \mid u$ is non-multiple$\}$

u, does not factor as

$$S \xrightarrow{f} S \xrightarrow[u']{} X$$

multiple cover

<u>Prop</u> $\mathcal{S}^{d}(X)$ is a Banach manifold

and $\mathcal{P}^{d*}(X)$ is a separable

Banach submanifold

Action of diffeomorphisms

$$\mathrm{Diff}(S,z) = \left\{ \begin{array}{c} \text{diffeos } \varphi \text{ of } S \text{ with} \\ \varphi(z_i) = z_i \end{array} \right\}$$

$$= \mathrm{Diff}(S,z)^+ \amalg \mathrm{Diff}(S,z)^-$$

Diff^+ acts on $\mathcal{P}^{d,*}(x)$ by

$$\varphi(u,j,J) = (u\circ\varphi^{-1}, (\varphi^{-1})^* j, J)$$

Diff^- by

$$\varphi(u,j,J) = (c_X\circ u\circ\varphi^{-1}, (\varphi^{-1})^* j, -dc_X\, J\, dc_X)$$

Note if $\varphi \in \mathcal{D}iff$ has a fixed point (u, j, J) in $\mathcal{P}^{d*}(x)$, then $\varphi \in \mathcal{D}iff^-$, $\varphi^2 = id$ and gives a real structure c_S on S

Def $\mathcal{P}^{d*}_{(x)} \supset \mathbb{R}\mathcal{P}^{d*}_{(x)} := \bigcup_{c_S} \left(\mathcal{P}^{d*}_{(x)}\right)^{c_S}$

Prop $\mathbb{R}\mathcal{P}^{d*}(x)$ is a separable Banach submanifold of $\mathcal{P}^{d*}(x)$

3 Holomorphic bundles

The Gromov operator is another diff. op. that for $(u, j, J) \in P^d(A)$ gives a $\bar{\partial}$-operator for $u^* T_X$, T_S and associated bundles. We can thus speak of the sheaves of holomorphic sections of these bundles on $\mathbb{CP}^1 = (S, j)$ and apply standard results, such as Riemann Roch. A real structure decomposes the cohomology into ± 1 eigenspaces

4 Moduli spaces

Def $\mathcal{M}^d(X) := \mathcal{P}^{d*}(A) / \mathrm{Diff}(S, z)^+$

with induced projection

$$\pi : \mathcal{M}^d(X) \to \mathcal{J}_\omega$$
$$(u, j, J) \mapsto J$$

Prop $\mathcal{M}^d(X)$ is a separable Banach manifold with an action of $\mathbb{Z}/2 = \mathrm{Diff}/\mathrm{Diff}^+$

Def $\mathbb{R}\mathcal{M}^d(X) = \left(\mathcal{M}^d(X)\right)^{\mathbb{Z}/2}$

with

$$\begin{array}{c} \mathbb{R}\mathcal{M}^d(X) \\ \downarrow \pi_{\mathbb{R}} \\ \mathbb{R}\mathcal{J}_\omega \end{array}$$

Theorem 1 The set of regular values of π intersects $\mathbb{R}J_\omega$ is a dense 2^{nd} category subset

Note. $\pi^{-1}(J) = \mathcal{M}^d(x)_J$

$=$ moduli space of m-pointed pseudo-holomorphic maps $(S, j, \bar{z}) \to (X, J, x)$

• The map π is Fredholm of index $2(c_1(X)d-1-m)$, so for a regular value J, $\mathcal{M}^d(X)_J$ is a real manifold of this dimension

• Define the holomorphic sheaf N_u on $\mathbb{CP}^1$ by

$$0 \to T_{S,j} \to u^* T_{X,J} \to N_u \to 0$$

$$N_u(-z) := N_u \otimes \mathcal{O}_{\mathbb{P}^1}(-z)$$

Then $c_1(X)d-1-m = \chi(N_u(-z))$

• For $J \in \mathbb{R}\mathcal{J}_\omega$ regular value

$\pi_{\mathbb{R}}$ has index $\chi_{\mathbb{R}}(N_u(-z)^+) = \chi(N_u(-z))$

so $\mathbb{R}\mathcal{M}^d(X)_J$ is a real manifold of dimension χ

II Welschinger invariant and main theorem

1. Take $u : (S, j) \to (X, J)$ in $\mathbb{R}\mathcal{M}^{\ell}(\alpha)$ (a real rational curve). Let $C = u(S) \subset X$. Say C has only odp singularities if u is an immersion, $\forall s \in S$, $u^{-1}(u(s)) = s$ or $= s \amalg s'$, and in the 2^{nd} case $du(T_s S) \neq du(T_{s'} S)$. Call $y = u(s)$ an o.d.p. on C

3 cases
- non isolated real double pt.
  $y \in \mathbb{R}X$, $du(T_s S)$ and $du(T_{s'} S)$ are real, i.e. $dc_{X,y}$ invariant
- isolated. $\sim$
  $y \in \mathbb{R}X$ $du(T_s S)$ and $du(T_s S)$ are J-conjugate: $dc_{X,y}(du T_s) = du T_{s'}$
- $y \notin \mathbb{R}X$

<u>Note</u> C has $\delta = \frac{1}{2}(d^2 - c_1(X)d + 2)$ doublepts

<u>Def</u> the <u>mass</u> $m(C) = \#\{\text{isolated odp's on } C\}$

- Write $\mathbb{R}X = \mathbb{R}X_1 \sqcup \dots \sqcup \mathbb{R}X_N$ ← connected components

$r_i = \#\{x_i \in \mathbb{R}X_i\}$, $r = (r_1, \dots, r_N)$

- Take $|x| = c_1(X)d - 1$

$\Rightarrow$ for J generic $\mathcal{M}^d(x)_J$ has dim $= 0$ and for all $u \in \mathcal{M}^d(x)_J$, C_u has only odp's

<u>Def</u> $n_d(m) = \#$ real C in $\mathcal{M}^d(x)_J$ with $m(C) = m$

$$\chi^d_r(x, J) = \sum_{m=0}^{\delta} (-1)^m n_d(m)$$

Theorem 2 Fix d, r. For all real conf. x of type r and all J such that $\chi^d_r(x, J)$ is defined, $\chi^d_r(x, J)$ depends only on d, r.

Idea of proof

a) one shows that the subsets of $\mathbb{R}\mathcal{M}^d(x)$ of curves with higher order cusps or more than 1 cusp or 1 ordinary tacnode or 1 ord. triple has codim ≥ 2.

The loci: 1 ordinary cusp, 1 ordinary triple pt. one ordinary tacnode have codim 1. There is also the codim 1 boundary locus of maps from $S \vee S$ to X.

b) take two pts in $\mathbb{R}\mathcal{M}(x)$ and connect them by a real path that avoids

The codim ≥ 2 loci in (a)
and show $\chi^d_r(X, J)$ is unchanged
in crossing the codim 1 loci.

Main tool: Combine a local analysis of the local deformation space of a curve singularity with a gluing construction that extends the local picture to a global one

Gluing: Take a $u:(S, j) \to (X, J)$

a point $s \in S$ a $s \in B_2 \subseteq S$

$t = u(s) \in B_4 \subset X$ and restrict to

$u_{|B} : (B_2, j) \to (B_4, J)$. Taking B_2, B_4
a small change in j, J makes $j|B_2$, $J|B_4$
standard

Take a family of small deformations of u, $u_\lambda|_{B_2} : B_2 \to B_4$

Shrinking B_4, can extend $u_\lambda|_{B_2}$ to a family $u_\lambda : S \to X$ and $J|_{B_4}$, $j|_{B_2}$ to J_λ on X, j_λ on S such that u_λ is a family of pseudo-holomorphic maps

Local analysis

- A map $u : (S, j) \to (X, J)$ has a cusp at $s \in S$ if $du_{|s} = 0$

order = order of vanishing of du defined as:

$$0 \to T_S^h \to u^* T_X^h \to N_u \to 0$$

u has a cusp of order n at s if $N_{u,s}$ has a torsion submodule of length n.

Write $N_u = N_u' \oplus \bigoplus_{\text{cusps}} N_{u,s}^{\text{tor}}$

Then $\deg N_u' = \underset{c_1(X)d - 1}{\deg N_u} + \sum_s \ell(N_{u,s}^{\text{tor}})$

and $\sum_s \ell(N_{u,s_0}^{\text{tor}})$ = codim of the cuspidal locus

- no cusps but an n-fold pt with $n>3$

$\Rightarrow \quad u(s_1) = \dots = u(s_n) = y \in X$

\# of local condns

$u(s_1) = u(s_2) \qquad 0$

$u(s_1) = u(s_2) = u(s_3) \qquad 1$

$\vdots$

$u(s_1) = \qquad = u(s_n) \qquad n-2$

- triple pt. with ≥ 2 tangent

$u(s_1) = u(s_2) = y \qquad 0$ conditions

$+$

$du(T_{s_1}S) = du(T_{s_2}S) \qquad 1$ condition (ord tacnode)

$+$

$u(s_3) = y \qquad \underline{1 \text{ condition}}$

2 conditions

• higher order tacnode

$\omega(S_1) = \omega(S_2)$ $\qquad$ O

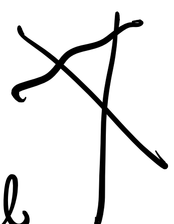
$dir(T_{S_1}, S) = dir(T_{S_2}, S)$ to order > 1 $\qquad$ $\geq 2 \leftarrow \geq 2$ conditions

• A mixture of two type $\to$ sum of conditions

---

triple pts

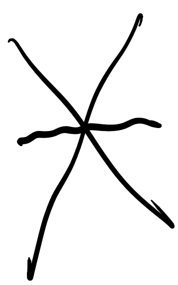
C_λ $\rightsquigarrow$ C_0 $\rightsquigarrow$ C_λ real or conjugate



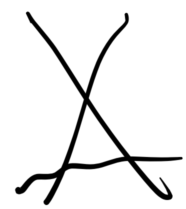
real

real or conjugate

$\lambda \in (-\delta, 0)$



$\lambda = 0$



real

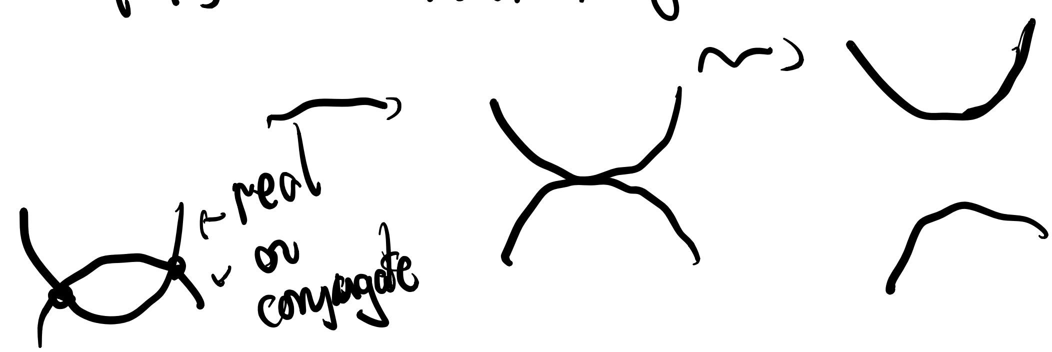
$\lambda \in (0, \delta)$

3 non isolated odp's $\rightsquigarrow$ 3 non-isolated odp's

or

1 real isolated odp $\rightsquigarrow$ 1 isolated odp

so $(-1)^{m(C)}$ is unchanged



$\lambda \in (-\varepsilon, 0)$ $\qquad \lambda = 0 \qquad$ $\lambda \in (0, \varepsilon)$

$2 \begin{pmatrix} \text{non-isolated} \\ \text{isolated} \end{pmatrix}$ odp $\rightsquigarrow$ no real odp's

$(-1)^{m(C)}$ is unchanged

<u>Note</u> you also need to see what happens if the curve $S \xrightarrow{u} X$ degenerates to

$$S_1 \cup S_2 \xrightarrow{u_1 \cup u_2} X$$

"bubbling"

here the singularity type remains constant in the family

At a cusp:

• The map π is ramified to order 2:

$$\ker d\pi = H^0(N_u(-z))^+$$
$$\overset{\mathbb{R}}{=} H^0(\underbrace{N_u'}_{\mathcal{O} \oplus \mathcal{O}(-1)})^+ \oplus \mathbb{R}(s)$$

Showing simple ramification is more involved

Take a path $\bar{\gamma}$ in $\mathbb{R}\mathcal{J}_\omega$ that goes J_0 & transverse to π (ordinary cusp locus). Can assume $J_0 = J_{st}$; $J_0 = J_{st}$ near the cusp $\Rightarrow$ $u_0(t) = (t^2, t^3)$ on B_2

Take the formula

$$u_\lambda(t) = (t^2 - \tfrac{2}{3}\lambda,\ t^3 - \lambda t)$$

Since $du_0 = t(2, 3t)$, $u_\lambda(t) = u_0 + \frac{\lambda}{t} du_0$, $t \neq 0$, so $u_\lambda \sim u_0$ by a diffeo on $B \setminus \{0\}$, so can extend u_λ to S. The corresponding section of $N_u(-z)$ is $\mathcal{O}$ on $N_u'(-z)$ so up to a small change in $\bar{\gamma}$, λ lives over $\bar{\gamma}$.

Compute, $u_\lambda(\pm\sqrt{\lambda}) = (\frac{1}{3}\lambda, 0)$ so C_λ has odp at $(\frac{1}{3}\lambda, 0)$ with equation (for $0<|\lambda|\ll 1$)

$$y^2 + \lambda(x-\tfrac{1}{3}\lambda)^2 - (x-\tfrac{1}{3}\lambda)^3 = 0$$

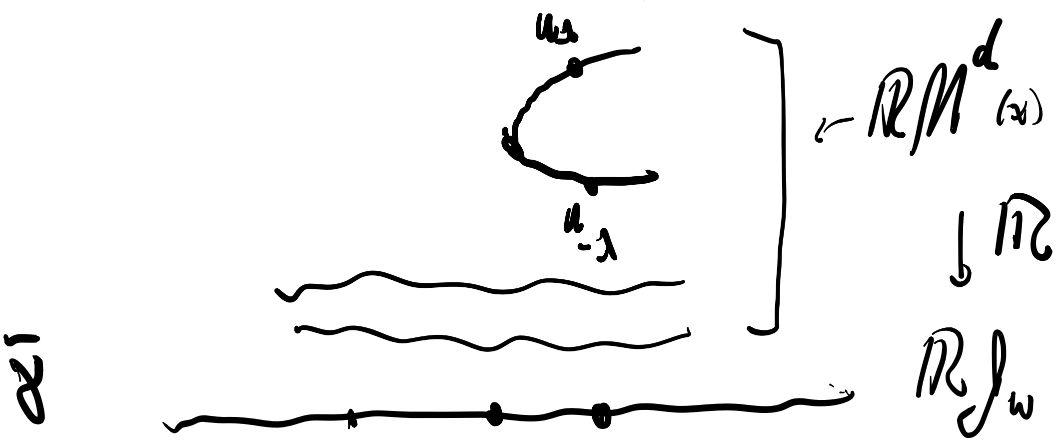
At $\lambda > 0$ $m = +1$.

$\lambda < 0$ $m = 0$

so for $\lambda > 0$ we have

$$m(u_\lambda) = m(u_{-\lambda}) + 1$$

and all other curves in $\pi^{-1}(J_\lambda)$ specialize to distinct curve in $\pi^{-1}(J_0)$ with the same m



$\bar{J}_{-\varepsilon} \quad \bar{J}_0 \quad \bar{J}_{+\varepsilon}$

$\Rightarrow \pi^{-1}(\bar{J}_\varepsilon)$ has the two cones $a_\lambda, a_{-\lambda}$ that "disappear" in going from $\bar{J}_\varepsilon$ to $\bar{J}_{-\varepsilon}$

$\Rightarrow n(\bar{m}) \rightsquigarrow n(\bar{m})-1$

$n(\bar{m}-1) \rightsquigarrow n(\bar{m}-1)-1$

$n(m) \rightsquigarrow n(m)$

for $m \neq \bar{m}, \bar{m}-1$

$$\Rightarrow \chi^d_r(x, \bar{J}_\varepsilon) = \chi^d_r(x, \bar{J}_{-\varepsilon})$$

III Recursion Formulas / Blow-up

Let $y = (y_1, \dots, y_{c_1(X)d-2} = y_{c_1 d-1})$, $y_{c_1 d-2}$ real

Consider the (y, J) pseudo-holo curves C passing thru y and with an odp at $y_{c_1 d-1}$. We assume $\mathbb{R}X$ connected, $s = \#\{\text{real } y_i : i = 1 \dots c_1 d-2\}$. All such C have only odp as singularities, for J generic.

$n := c_1 d - 2$

Def $\hat{n}^+_d(m) = \#\{C,\ m(C) = m \text{ and a non-isol. odp at } y_n\}$

$\hat{n}^-_d(m) = \#\{C,\ m(C) = m \text{ and an isolated odp at } y_n\}$

$$\Theta^d_s(y, J) = \sum (-1)^m \left(\hat{n}^+_d(m) - \hat{n}^-_d(m)\right)$$

Thm 3 Fix d, s. Assume $\mathbb{R}X$ is connected. For (y, J) such that $\Theta^d_s(y, J)$ is defined, $\Theta^d_s(y, J)$ is independent of the choice of (y, J)

Thm 4 $\chi^d_{r+2} = \chi^d_r + 2\Theta^d_{r+1}$

Pf of Thm 3 is similar to that of Theorem 2

On: Let $X' = \mathrm{Bl}_{2n} X$ then

$$n_{d-2E}(m)_{X'} = \hat{n}^+_d(m) + \hat{n}^-_d(m+1)$$

$$\Rightarrow \Theta = \sum (-1)^m (\hat{n}^+_d(m) - \hat{n}^-_d(m))$$

$$= \sum (-1)^m n_{d-2E}(m)_{X'} = \chi^{d-2E}_{s-1}(X')$$

<u>Pf of Thm 4</u> Take y with $r+1$ real pts. Fix a (complex) tangent line at y_r. For general J there are finitely many C passing thru y with tangent line τ at y_r & these all have only odp's as singularities.

Let $\tilde{n}_d(m) = \#$ such C with $m(C) = m$

and
$$\tilde{\chi}^d_r(y, J) = \sum (-1)^m \tilde{n}_d(m)$$

Prop a) $\chi^d_{r+2} = \tilde{\chi}^d_r(y, J) + 2\sum(-1)^m \hat{n}^+(m)$

b) $\chi^d_r = \tilde{\chi}^d_r(y, J) + 2\sum(-1)^m \hat{n}^-(m)$

Pf Take a path $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}X$

with $\gamma(0) = y_n$, $\gamma'(0) \in \tau$ & let

$y_\lambda = (y_1, \ldots, y_n, \gamma(\lambda))$ $\Rightarrow r(y_\lambda) = r+2$

For $-\varepsilon < \lambda < 0$ $(\pi^{\lambda}_{\mathbb{R}} \circ j)^{-1}(J) = \{C_1(\lambda), \ldots, C_j(\lambda)\}$

and as $\lambda \to 0$, $C_i(\lambda) \rightsquigarrow C_i(0)$

with $C_i(0)$ either

i) tangent to τ at y_n

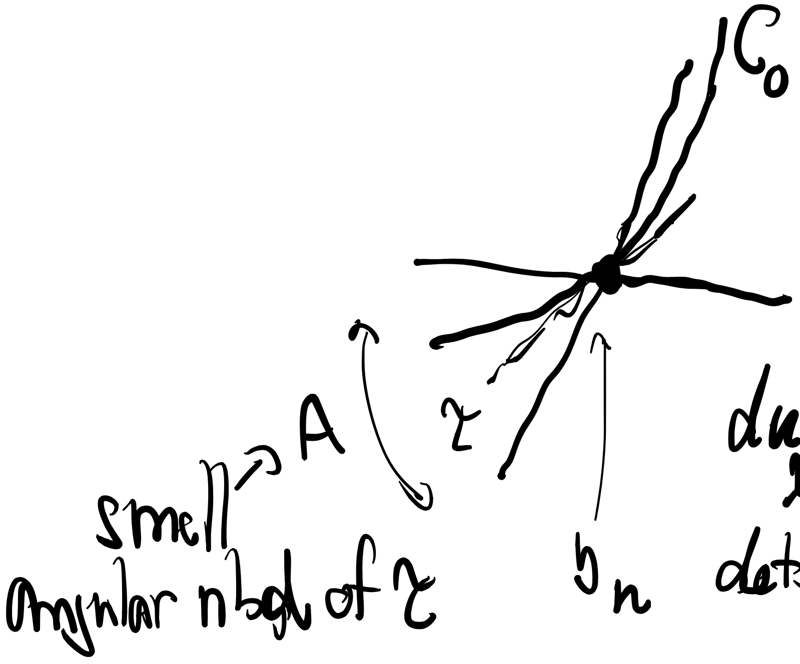
or ii) a non-isolated odp at y_n

Claim for C of type (i), $\exists!$ i with
$$C = \lim_{\lambda \to 0^-} C_i(\lambda)$$

for C of type (ii) $\exists$ exactly two i_1, i_2 with $C = \lim_{\lambda \to 0^-} C_{i_j}(\lambda)$

This implies formula (a), since $m(C_\lambda)$ is const. under smoothings

Pf of claim for $C = C_0$ of type (i)
$= C_{u_0}$



for $|\lambda| \ll \varepsilon$
The condition
$du_\lambda(T_{y_n} S) = \tau \in A$
determines u_λ

so for λ sufficiently small, there is only one $C_i(\lambda) \to C_0$

For C_0 of type (ii). Consider a

map $(S, z) \to (X, y)$

with $z_n, z_{n+1} \to y_n$.

The curve C_0 has two tangent line t_1, t_2 at $y_n \Rightarrow$ there are two m u_1, u_2 with image curve C_0: u_1 with $du_1(T_{z_n}) = t_1$, $du_2(T_{z_{n+1}}) = t_2$, and u_2 the other way around

Deforming gives two curves $C_{i_1}, C_{i_2} \to C_0$

For formula (b) take γ as before but with $c_X(\gamma(\lambda)) = \gamma(-\lambda)$

and let $\tilde{y}_\lambda = (y_1 \ldots y_{c,d-3}, \gamma(\lambda), \gamma(-\lambda))$

r real points

Get the same claim as before except with

(ii)' an isolated odp at y_N

This gives (b): $r(\tilde{y}_1) = r$ so

$$\chi^d_r = \chi^\alpha_r(\tilde{y}_r, J) = \tilde{\chi}^d_r + 2\,\Sigma(-1)^{m_{n-}} n^-(m)$$

3 Generalization and computation

Instead of fixing 1 pt to put an odp, we fix a pts

Let $y = (y_1, \dots, y_{c_1 d - 1 - \sigma})$ be a real conf. with $y_{c_1 d - 2\sigma}, \dots, y_{c_1 d - 1 - \sigma}$ all real

$\tilde{J}$ " structure with s real pts,

$\hat{n}^+_d(m)$ = # of real curves of class m, passing through y with an even number of isolated double pts at $y_{c_1 d - 2\sigma}, \dots$

$\hat{n}^-_d(m)$ = " odd

$$\theta^{d,\sigma}_{s}(g,J) = \sum_{m=d}^{s} (-1)^m \left(\tilde{n}^{+}_{d}(m) - \tilde{n}^{-}_{d}(m)\right) =$$

<u>Theorem 5</u> $\theta^{d,\sigma}_{s}(g,J) = \theta^{d,\sigma}_{s}$ is independent of (g,J) (assuming it is defined)

<u>Theorem 6</u> $\theta^{d,\sigma}_{s+2} = \theta^{d,\sigma}_{s} + 2\theta^{d,\sigma+1}_{s+1}$

<u>Note</u> $\theta^{d,0}_{r} = \chi^{d}_{r}$

<u>Computations</u>

Take $X = \mathbb{CP}^2$ (with standard ω, c)

- $\theta^{3}_{g} = 1$ $\forall$ odd g, $1 \le g \le 7$
- $\theta^{4,3}_{r} = 1$ $\forall$ even r, $4 \le r \le 8$
- $\theta^{6,6}_{s} = 1$ $\forall$ even s, $6 \le s \le 8$

for Θ^3_4 : $\exists!$ $C^{\deg 3}$ passing thru 7 pts, y_7 real w odp at y_7

If C_1, C_2 are such: $C_1 \cdot C_2 = 6 + 4 \cdot 1 = 10 > 9$

C is automatically real

$\Theta^{4,3}_r$: $\exists!$ C deg 4 passing thru 8 pts with y_6, y_7, y_8 real + C having odp there

if two: $C_1 \cdot C_2 = 5 + 4 \cdot 3 = 17 > 16$.

$\Theta^{5,6}_s$: $\exists!$ C deg 5 passing thru 8 pts, odp at $y_3 \cdots y_8$ + all real

if two $C_1 \cdot C_2 = 2 + 4 \cdot 6 = 26 > 25$

Def $\chi^d(T) = \sum_{r=0}^{c,d-1} \chi^d_r T^r$ (for $X = \mathbb{CP}^2$) ($r \equiv c,d-1 \bmod 2$)

$\chi^1 = 1 + T^2$

$\chi^2 = T + T^3 + T^5$

$\chi^3 = 2T^2 + 4T^4 + 6T^6 + 8T^8$

$u(t) = (t^2, t^3)$ $\quad 2t\frac{\partial}{\partial t},\ 3t^2\frac{\partial}{\partial t} = \frac{\partial u}{\partial y}$

$\frac{\partial u}{\partial x}$ " $\quad\quad$ " $\frac{3}{2}t\cdot\frac{\partial u}{\partial x}$

$u_\lambda(t) = \left(t^2 - \frac{2}{3}\lambda,\ t^3 - \lambda t\right)$: tangent to u at $t \neq 0$

$t(t^2-\lambda)$

$u_\lambda(\lambda^{1/2}) = \left(\lambda - \frac{2}{3}\lambda,\ 0\right) = u_\lambda(-\lambda^{1/2})$

" $\frac{1}{3}\lambda$

local equation

$$y^2 = t^6 - 2\lambda t^4 + \lambda^2 t^2$$

$$(x - \tfrac{1}{2}\lambda)^2 = (t^2-\lambda)^2 = t^4 - 2\lambda t^2 + \lambda^2$$

$$(x - \tfrac{1}{3}\lambda)^3 = (t^2-\lambda)^3 = t^6 - 3\lambda t^4 + 3\lambda^2 t^2 - \lambda^3$$

$$y^2 - (x - \tfrac{1}{3}\lambda)^3 = \lambda t^4 - 2\lambda^2 t^2 + \lambda^3$$

$$= \lambda[t^4 - 2\lambda t^2 + \lambda^2)$$

$$y^2 - (x-\tfrac{1}{3}\lambda)^3 - \lambda(x-\tfrac{1}{3}\lambda)^2 = 0$$

$$y^2 = (x - \tfrac{1}{3}\lambda)^3 - \lambda(x - \tfrac{1}{3}\lambda)^2$$

fn $y^2 + \lambda(x - \tfrac{1}{3}\lambda)^2 + \dots = 0$

$\lambda > 0 \quad m = +1$

$\lambda < 0 \quad m = -1$