Mikhalkin correspondence

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§1 Mikhalkin correspondence
§2. Tropicalisation and spines
§3. Complex tropical curves

§1 Mikhalkin correspondence.

$\Delta \subset \mathbb{R}^2$ Newton polygon, fixed.

$s := \#(\partial\Delta \cap \mathbb{Z}^2)$

$g \in \mathbb{Z}$ fixed genus

$\mathcal{P} := \{P_1, \dots P_{s+g-1}\} \subseteq \mathbb{R}^2$ in tropical general position

<u>Defn</u>: $N^{irr}_{trop}(g, \Delta, \mathcal{P}) := \#\{$ irred trop curves C of genus g, deg Δ, passing via $\mathcal{P}$, counted with mult $C\}$

$N_{trop}(g, \Delta, \mathcal{P}) := \#\{$ trop. curves $\dots\}$

Denote these curves by $C_1, C_2, \dots, C_m$.

<u>Thm (Mikhalkin)</u>: If $\mathcal{P}$ is generic, then

$$N^{irr}_{trop}(g, \Delta, \mathcal{P}) = N^{irr}(g, \Delta)$$

$$N_{trop}(g, \Delta, \mathcal{P}) = N(g, \Delta)$$ Gromov-Witten invariant.

Moreover, there exists a configuration $Q \subseteq (\mathbb{C}^*)^2$ of $s+g-1$ points in general position st every trop. curve C of genus g, deg Δ passing via $\mathcal{P}$, there are mult C distinct complex curves of genus g & deg Δ passing via $\mathcal{Q}$. These curves are distinct for distinct C, and are irred if C is irred.

<u>Real version</u>: If $\mathcal{R} \subseteq \mathbb{R}^2$ $s+g-1$ points in tropical general position, then $\exists$ $\mathcal{P} \subset (\mathbb{R}^\times)^2$ of real points in general position st

$$N^{irr}_{trop,\mathbb{R},\omega}(g,\Delta,\mathcal{R}) = N^{irr}_{\mathbb{R},\omega}(g,\Delta,\mathcal{P})$$

real curves of genus g, deg Δ passing via $\mathcal{P}$ counted with elliptic sign

<u>Defn</u>: $t > 1$ $t \in \mathbb{R}$ define

$$H_t := (\mathbb{C}^\times)^2 \longrightarrow (\mathbb{C}^\times)^2$$

$$(z,w) \longmapsto \left(|z|^{\frac{1}{\log t}} \frac{z}{|z|}, |w|^{\frac{1}{\log t}} \frac{w}{|w|} \right)$$

A J_t - holomorphic curve V_t is

$$V_t := H_t(V) \quad V \subsetneq (\mathbb{C}^\times)^2 \text{ a holom curve.}$$

genus $g(V_t) := g(V)$ & $\deg(V_t) := \deg(V)$.

<u>Rk</u>: $\mathrm{Log} \circ H_t = \mathrm{Log}_t : (z,w) \mapsto (\log_t |z|, \log_t |w|)$

$\mathrm{Log}_t(V_t) = \frac{1}{\log t} \mathrm{Log}\, V$

$\log t = \log_e t$

$\log_t x = \frac{\log x}{\log t}$

$\mathrm{Log} : (z, w) \mapsto (\log|z|, \log|w|)$

<u>prop 1</u>: Let $Q \subset (\mathbb{C}^*)^2$ a configuration of points in general position, then for almost $t > 1$ there are $N(g, \Delta)$ (or $N^{irr}(g, \Delta)$) (irred)

J_t - holomorphic curves of deg Δ & genus g passing via Q.

<u>proof</u>

$H_t^{-1}(Q)$ is generic for almost t.

[]

Choose Q config $\subseteq (\mathbb{C}^*)^2$ of $s+g-1$ points in general position st $\mathrm{Log}(Q) = P$. Our correspondence follows from the 2 following theorems:

<u>Thm 1</u>: $\forall \varepsilon > 0$ there exist $T > 1$ st if $t > T$ & V_t is a J_t- holomorphic curve of genus g & deg Δ passing via Q, then

of genus g & deg Δ passing ...
$\mathrm{Log}(V_t) \subseteq \mathcal{N}_\varepsilon(C_j)$ for some $j = 1..m$.

<u>Thm 2</u>: For $\varepsilon > 0$ small, $t >> 1$, then mult C_j is equal to the number of J_t-holomorphic curves V_t of genus g & deg Δ passing via Q and st $\mathrm{Log}(V_t) \subseteq \mathcal{N}_\varepsilon(C_j)$.
Furthermore, if C_j is irred (red), any such J_t-holomorphic curve is irreducible (red).

§2. Tropicalization & Spines
(proof of thm 1).

Let $f = \sum_{(j,k) \in \Delta \cap \mathbb{Z}^2} a_{j,k} z^j w^k$ $\quad a_{j,k} \in \mathbb{C}$.

sometime write $\sum_{I \in \Delta \cap \mathbb{Z}^2} a_I \underline{z}^I$ $\quad \underline{z} = (z_1, z_2)$

$V = V(f) \subseteq (\mathbb{C}^*)^2$
$A_V = \mathrm{Log}(V)$
There are 2 tropical curves associated to V

+ Tropicalization: $V^{trop} = \mathrm{trop}\left(\text{"} \sum_{(j,k) \in \Delta \cap \mathbb{Z}^2} \log|a_{j,k}| x^j y^k \text{"} \right)$

$$\text{"} = \text{"} \max\{ \log|a_{j,k}| + jx + ky \}$$

+ The spine S_V of A_V obtained by $\mathrm{trop}\left(\sum b_{j,k} x^j y^k\right)$

where $$b_{j,k} = \frac{1}{(2\pi i)^2} \int_{\mathrm{Log}^{-1}(r)} \log |f(z,w)| \frac{dz}{z} \frac{dw}{w}$$

where $r \in \mathbb{R}^2 \setminus A_V$ is any point s.t. its <u>order</u> is (j,k). If no such order exist, then $x^j y^k$ is omitted

$$\mathrm{ord} : \mathbb{R}^2 \setminus A_V \longrightarrow \Delta \cap \mathbb{Z}^2$$

$$w \longmapsto (u_1, u_2)$$

$$u_j = \frac{1}{(2\pi i)^2} \int_{\mathrm{Log}|z| = w} \frac{z_j \partial_j f(z)}{f(z)} \frac{dz_1}{z_1} \frac{dz_2}{z_2}$$

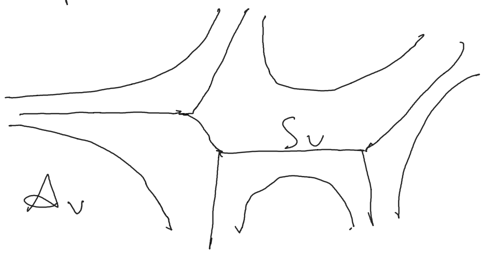
$\left(\text{Classically}, \quad \frac{1}{2\pi i} \int_{\gamma \subset \Omega \subset \mathbb{C}} \frac{f'(z)}{f(z)} dz = \# \text{ zeros of } f - \# \text{ poles of } f\right)$

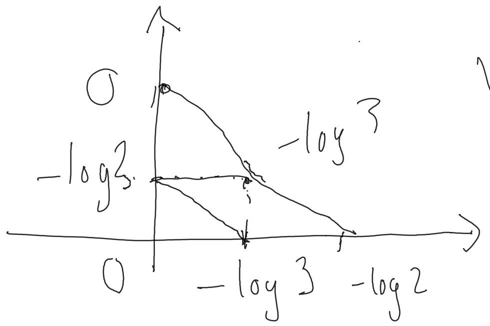
Let $w, w' \in \mathbb{R}^2 \setminus A$, then w, w' belong to the same conn. component of $\mathbb{R}^2 \setminus A$ iff
$$\mathrm{ord}(w) = \mathrm{ord}(w')$$
(Forsberg, Passare, Tsikh)

Moreover, $S_V \subset A_V$ and $\Delta_{S_V} = \Delta$

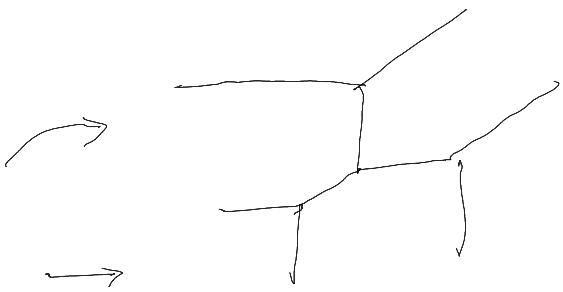
Rk: $S_V \neq V^{trop}$

Example: $f(z,w) = \underline{3}z + \underline{3}zw + 2z^2 + w^2 + 2w + \underline{\underline{1}}$





V^{trop}



lemma 1: If V_t is a J_t - holomorphic curve, then $\mathrm{Log}(V_t) \subset \mathcal{N}_\delta(V^{trop})$ where

$$\delta = \log_t(\#(\Delta \cap \mathbb{Z}^2) - 1)$$

proof

$t = e$

$V = V(f)$ ~~where~~ $f(\underline{z}) = \sum a_I \underline{z}^I$

V^{trop} is defined by $\max\{\log|a_I| + \langle I, \underline{x} \rangle\}$ $x = (x_1, x_2) \in \mathbb{R}^2$

If $\exists\ \underline{x}' \in \mathrm{Log}(V_t)$ but $\underline{x}' \notin \mathcal{N}_\delta(V^{trop})$

$\underline{x}' \notin \mathcal{N}_\delta(V^{trop}_{t}) \Rightarrow \exists\ I_0 \in \Delta \cap \mathbb{Z}^2$ st

$\log|a_{I_0}| + \langle I_0, x' \rangle > \log|a_I| + \langle I, x' \rangle + \delta$

$\log|a_{I_0}| + \langle x, I_0 \rangle > \log|a_I| + \langle x, I \rangle$

$\forall I \neq I_0$

$\underline{x}' \in \mathrm{Log}(V_t) \Rightarrow \exists\, \underline{z} \in V$ st $\mathrm{Log}(\underline{z}') = \underline{x}'$

$f(\underline{z}') = 0 \Rightarrow |a_{I_0} \underline{z}'^{I_0}| \leq \sum_{I \neq I_0} |a_I \underline{z}'^{I}|$

$\leq (\#(\Delta \cap \mathbb{Z}^2) - 1) \cdot \max_I |a_I \underline{z}'^{I}|$

$\log|a_I \underline{z}'^{I_0}| \leq \log (\#(\Delta \cap \mathbb{Z}^2) - 1) + \log|a_I \underline{z}'^{I}|$

$\square$

lemma 2: Let (V_t^k) be a sequence of J_{t_k}-hol. curves of deg Δ & genus g passing via $\mathcal{C}$, $t_k \underset{k \to \infty}{\to} \infty$. There exist a subsequence $(V_{k_\alpha}^t)$ st

$\lim_{k_\alpha \to \infty} \mathcal{A}_{k_\alpha}^t = C_j$ (Hausdorff metric in $\mathbb{R}^2$)

proof:

Claim 1: $\exists$ subsequence $(S_{k_\alpha}^t)$ st $\lim_{k_\alpha \to \infty} \mathcal{A}_{k_\alpha}^t = C$ a tropical curve of deg $\Delta_C \subset \Delta$

Indeed $\mathcal{A}_k^t \supset S_k^t$ its spine

S_k^t is defined by $f^k =$ "$\sum_{I \in \Delta \cap \mathbb{Z}^2} a_I^k x^I$"

Choose a_I^k st $\left.\begin{array}{l} I \mapsto a_I^k \text{ concave} \\ \max a_I^k = 0 \end{array}\right\}$

pass to subsequence $\lim a_I^{k_\alpha} = a_I^\infty \quad \forall I \in \Delta \cap \mathbb{Z}^2$

C is defined by $C = \text{trop}\left(\text{"}\sum a_I^\infty x^I\text{"}\right)$.

$\Rightarrow \lim S^t_{k_\alpha} = C$

lemma 1 $\Rightarrow \lim A^t_k = C$

<u>Claim 2</u>: C is of deg Δ, genus g & pass via $\mathcal{P}$

- $\mathcal{P} \subset C$ because $\mathcal{P} \subset A_{k_\alpha}$
- $\Delta_C = \Delta$

Choose $R >> 1$ s.t.

$\{\text{vertices of } C\} \subset D_R = \{x : \|x\| \le R\}$

and the extended edges of C beyond

D_R don't ~~not~~ intersect.

By claim 1 $S_{k_\alpha} \cap D_R$ is an approximation

of $C \cap D_R$. For $k_\alpha >>$, $S_{k_\alpha} \supset \mathcal{P}'$

where $\mathcal{P}'$ is a configuration of $s+g-2$

points in general position & ~~of~~ $\mathcal{P}'$ is a small

deformation of $\mathcal{P}$

$\Delta_C = \Delta \Leftrightarrow S_{k_\alpha} \setminus D_R$ is disjoint union

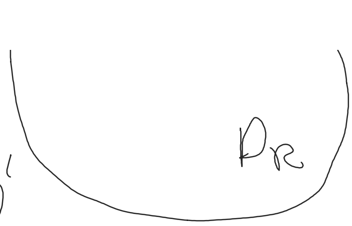
of rays

If $S_{k_\alpha} \setminus D_R$ is not disjoint union of rays

Change the length of E

yields a deformation of

yields a deformation
S_{k_α} s.t all curves
in the family pass via p'



$g(C) \geq g$ by tropical general position

$\Delta_C = \Delta$ s ends

C pass via P

$$g(C) \leq g\left(\lim_{k_\alpha \to \infty} V_{k_\alpha}\right) = g_{\text{complex tropical curve}}$$

s+g-1 points

$S^t_{k_\alpha}$ have more edges than C, but these edges vanish in the limit. All other edge of $S^t_{k_\alpha}$ tend to a parallel edge of C.

Lemma 3: $\exists T > 0$ & a function $\tilde{\delta} : [T, \infty] \to \mathbb{R}$ for every edge E_{k_α} of S_{k_α} whose length is higher than $\tilde{\delta}(t_k)$, there exists an edge E of C parallel to E_k and within $\tilde{\delta}(t_k)$ - distance (in the Hausdorff metric of $\mathbb{R}^2$) from E.

§3 Complex tropical curves:

Defn: Let $(V^t_k)_{k \in \mathbb{N}}$ be a sequence of J_{t_k}-holo. curves st $t_k \to \infty$. If $\lim V^t_k = V_\infty$, then

$k \to \infty$ $\qquad$ $k \to \infty$

V_∞ is called a complex tropical curve.

genus of $g(V_\infty) := g$ if V_∞ is the limit of a sequence of curves of genus g but not smaller

<u>Rk</u>: $\mathrm{Log}(V_\infty)$ is a tropical curve by lemma 2

<u>Another description</u>:

$$\overline{\mathbb{C}((t))} = \left\{ a = a_{q_1} t^{q_1} + a_{q_2} t^{q_2} + \dots \;\middle|\; \begin{array}{l} a_{q_1}, a_{q_2}, \dots \in \mathbb{C}, \\ \{q_1 < q_2 < \dots\} \subseteq \mathbb{Q} \text{ has common dominator} \end{array} \right\}$$

valuation $\mathrm{val}: \overline{\mathbb{C}((t))} \to \mathbb{R} \cup \{-\infty\}$

$$a \mapsto -q_1$$

Field of Puiseux series K = completion of $\overline{\mathbb{C}((t))}$ wrt val

Complexify val to yield

$$w: \overline{\mathbb{C}((t))} \to \mathbb{C}^* \quad (K^* \to \mathbb{C}^*)$$

$$a_{q_1} t^{q_1} + a_{q_2} t^{q_2} \dots = a \mapsto e^{-q_1 + i\arg(a_{q_1})}$$

Set $W := (w, w) : (K^*)^2 \to (\mathbb{C}^*)^2$

$\mathrm{Log} \circ W = \mathrm{Val} = (\mathrm{val}, \mathrm{val}) : (K^*)^2 \to \mathbb{R}^2$

<u>prop 2</u>: V_∞ is a complex tropical curve

$\Leftrightarrow$ $V_\infty = W(V)$ where $V \subset (K^*)^2$ is a curve.

proof :

$\Rightarrow$ If $V_\infty = \lim V_k \Rightarrow \mathrm{Log}(V_\infty)$ is a tropical curve

$$\mathrm{Log}(V_\infty) = \mathrm{trop}\left(\text{"}\sum \alpha_I \underline{x}^I\text{"}\right) \quad \alpha_I \in \mathbb{R}_{trop}$$

To find a presentation of V_∞ as $W(V)$:

we take poly with coeff $\beta_I t^{\alpha_I}$. $\beta_I \in S^1$

to find β_I : since $H_{t_k}^{-1}(V_k)$ is a holomorphic defined by $f^k = \sum_I a_I^{V_k} \underline{z}^I$. st $a_{I_0} = 1$ for one $I_0 \in \Delta \cap \mathbb{Z}^2$

Choose $\beta_I = \lim_{k\to\infty} \arg(a_I^{V_k})$

$\Leftarrow$: $V \subset (K^*)^2$ a curve defined by

$$f = \sum a_I \underline{z}^I$$

$$a_I = a_{q_1}^I t^{q_1} + a_{q_2}^I t^{q_2} + \ldots$$

truncate $a_I^{min} = a_{q_1}^I t^{q_1}$

$$f^{min} := \sum a_I^{min} \underline{z}^I$$

$$V^{min} := V(f^{min} = 0)$$

$$\Rightarrow W(V^{min}) = W(V)$$

Let $f^{t_k} : (\mathbb{C}^*)^2 \to \mathbb{C}$ obtained from f^{min} by plugging $t = t_k$ into coeff

$V(f^{t_k})$

and set ~~V_k~~ $V_k = V($

$\Rightarrow \mathring{V}_\infty = \lim V_k$ $\square$

$\Delta_1 = \mathrm{ConvHull}((0,0)(0,1)(1,0))$

get a ~~tropical~~ complex tropical line $\Lambda \subset (\mathbb{C}^\times)^2$

$\Lambda = W(\{(z,w) \in (K^\times)^2 \mid z + w + 1 = 0\})$

If $\Delta \subset \mathbb{R}^2$ a lattice triangle, $\exists$ affine surj

$\Delta_1 \to\!\!\!\to \Delta$

Let $L_\Delta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear part

$\Rightarrow M_\Delta : (\mathbb{C}^\times)^2 \to (\mathbb{C}^\times)^2$ $\quad \deg M_\Delta = \deg L_\Delta$

same matrix as L_Δ $\quad = 2\,\mathrm{Area}(\Delta)$

$M_\Delta(\Lambda)$ is a complex tropical curve of degree Δ $\quad ((V_\infty = W(V)))$

Defn ~~simple~~: A proper map $h : \mathring{V}_\infty \to (\mathbb{C}^\times)^2$ is called parametrized simple complex tropical curve if it is locally $M_\Delta : \Lambda \to (\mathbb{C}^\times)^2$.

Prop3: Let Δ be a lattice triangle, $\{q_1, q_2\} \subset (\mathbb{C}^\times)^2$ in general position, st

$p_1 = \mathrm{Log}(q_1)$, $q_2 = \mathrm{Log}(q_2)$ are in trop-gen position.

Then:

1) There is unique complex tropical line $\Lambda \subset (\mathbb{C}^*)^2$ passing via q_1 & q_2

2. There is exactly one rational tropical curve C (with only one vertex) dual to $\triangle$ & pass via p_1 & p_2. Assume that ~~there~~ the two edges that pass p_1 & p_2 are of weight w_1 & w_2. Then there are $\frac{2\,\mathrm{Area}(\triangle)}{w_1 w_2}$

rational complex tropical curves passing ~~p_1&p_2~~ q_1 & q_2 and project to C under Log.

proof

1. $(S^1)^2$ complex tropical curves map to C

only $S^1 \times \{\alpha_0\}$ pass q_1 $\alpha_0, \beta_0 \in S^1$

$\{\beta_0\} \times S^1$ pass q_2

$\Rightarrow$ unique.

$((\mathbb{C}^*)^2 \quad ((\mathbb{C}^*)^2$

1) M_Δ: (4) ...

$M_\Delta^{-1}(q_1)$ & $M_\Delta^{-1}(q_2)$

$\Rightarrow$ There are $2\,\mathrm{Area}(\Delta)$ distinct simple parametrized $W_1 W_2$ compl trop curve passing via q_1 & q_2 & of the form

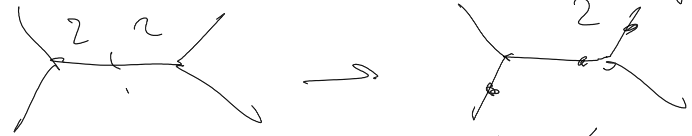
$$M_\Delta \circ h : \Gamma \to (\mathbb{C}^{e})^2$$

& Any simple parametrized complex tropical curve has this form.

<u>Defn</u>: C a simple curve of deg Δ & genus g passing via P. $h : \Gamma \to C$ parametrization. Then the edge mult.

$$\mu_{edge}(C, P) := \prod_{\substack{\text{Edge of } \Gamma \\ \text{disjoint from } P}} \mathrm{weight}(E) \Big(\prod_{\substack{\text{Edges of } \Gamma \\ \text{not disjoint} \\ \text{from } P}} \mathrm{weight}(E) \Big)^2$$

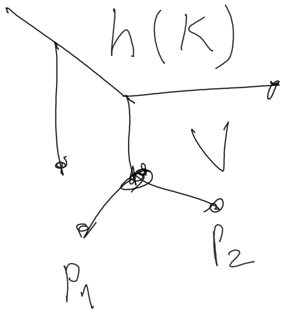
<u>Example</u>



$\mu_{edge}(C, P) = 4$

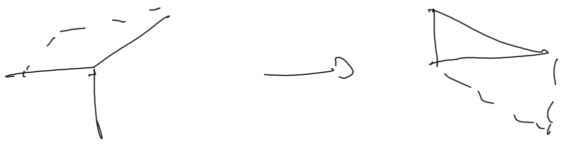
<u>prop 4</u>: There are mult $C / \mu_{edge}(C, P)$ simple tropical curve in $(\mathbb{C}^{e})^2$ of deg Δ & genus g passing via Q & project to C

$K \subset \Gamma \setminus h^{-1}(P)$ conn. component



$V \leftrightarrow \Delta' \Rightarrow$ There are $\frac{2\,\mathrm{Area}(\Delta')}{w_1 w_2}$ curves

$2\,\mathrm{Area}(\Delta') = \mathrm{mult}_V(C)$



<u>prop 5</u>: There are $\mu_{edge}(C, P)$ J_t-holo curve of deg Δ & genus g passing via Q in a nbh of each <u>simple complex tropical curves</u> which map to C_j for each $j = 1..m$

<u>Idea</u>: $f_t^{\zeta} = \sum_{I \in \Delta \cap \mathbb{Z}^2} \arg(\zeta_I)\, t^{\log|\zeta_I|} z^I$

$\subset (\mathbb{C}^*)^2$

$\zeta = (\zeta_I)_{I \in \Delta \cap \mathbb{Z}^2}$

Claim ~~For each~~ If $t \gg 7$ & $V_t \subset (\mathbb{C}^*)^2$ is a J_t-holo curve of genus g s.t. $V_t \supset Q$ & $\mathrm{Log}(V_t) \subset \mathcal{N}_\varepsilon(C) \Rightarrow$

then $V_t = V_t^{\xi}$ for some ξ.

Need V_t^{ξ} to has the right degree genus, pass via $\mathcal{Q}$.

to ensure: cover $\mathbb{R}^2$ by $U(\Delta')$

Δ' are triangles, edges, parallelograms

($\leftrightarrow$ 3-valent vertices, edges, 4-valent vertices)

Use Viro's patchworking principle: Deformation ξ_I with $I \in \Delta'$ has little effect on $V_t^{\xi} \cap \operatorname{Log}_t^{-1}(U(\Delta'))$ for $t \gg 1$