Block-Göttsche refined invariants

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"On Block-Göttsche Mutliplicities for planar Tropical Curves" Itenberg-Mikhalkin

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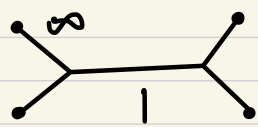
<u>Def</u>: A closed irreducible tropical curve $\bar{C}$ is a connected finite graph without 2-valence vertices whose edges are enhanced with length. Any edge adjacent to a 1-valence vertex has ∞ length. All the Other edges have positive real length



$$\partial \bar{C} = \{ \text{1-valence vertices} \}$$

$$C = \bar{C} \setminus \partial \bar{C}$$

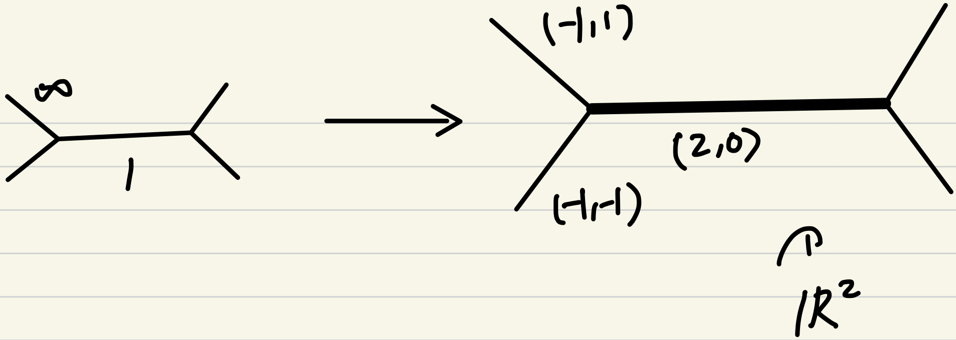
Def: An immersed tropical curve is a smooth map $h: C \to \mathbb{R}^2$ s.t.

① h is a topological immersion

② For every unit vector $u \in T_y(C)$, where y is inside an Edge $E \subset C$ we have $(dh)_y(u) \in \mathbb{Z}^2$. We denote $(dh)_y(u) = u_h(E)$. The GCD of the integer coordinates of $u_h(E)$ is called the weight $w_h(E)$ of the edge E.

③ Balancing condition: for every vertex $v \subset C$, we have

$$\sum_E u_h(E) = 0$$



<u>Rk</u>: An immersed planar tropical is called simple if it is 3-valence, all the intersection points are avoiding from the vertices, and preimage of intersection point consists of 2 points

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$$\Delta = \{ u_h(E) \mid E \text{ unbounded Edge} \}$$ multiset

degree

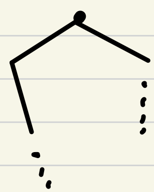
$$g = b_1(C)$$

genus

Let $M_{g,\Delta}^{simple}$ denote the moduli space of all the simple tropical curves with genus g and degree Δ

Mikhalkin proven $M_{g,\Delta}^{simple}$ is stratified by the combinatorial type (i.e. Newton subdivision of Δ^*) and each component has dimension $\#\Delta + g - 1$

$$\#\Delta + g - 1 = 2 + \#\{\text{bounded edges}\} - 2g$$



A configuration $\mathcal{X} = \{p_1 \cdots p_k\}$ is called generic relative to Δ if

① Any tropical curve with degree Δ, genus $\tilde{g}$ s.t. $k = \#\Delta + \tilde{g} - 1$ which passing through $\mathcal{X}$ is simple, the vertices are disjoint from $\mathcal{X}$

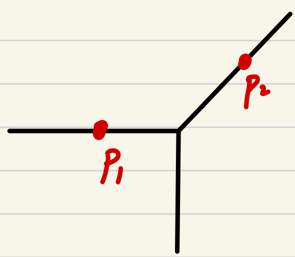
The number of such curves are finite

② $\nexists$ immersed tropical curve with degree Δ genus $\tilde{g}$ s.t $k > \#\Delta + \tilde{g} - 1$ which passing through $\mathcal{X}$

A configuration $\mathcal{X}$ is called generic if it is generic relative to arbitrary degree

Ex: $\Delta = \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

Then $\{p_1, p_2\}$ is generic relative to Δ iff the slope of the line passing through p_1, p_2 is not $0, 1, \infty$



$\{p_1, p_2\}$ is generic iff the slope of the line passing through them is irrational

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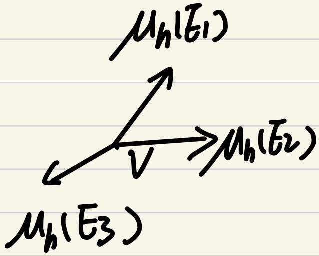
Fix Δ, g. We choose a generic configuration

$\mathcal{X} = \{p_1 \cdots p_k\}$, $k = \#\Delta + g - 1$

Pick $h \in S(g, \Delta, \mathcal{X})$



Recall that

$$\mu_{\mathbb{C}}(h) = \prod_V m_{\mathbb{C}}(V) \qquad m_{\mathbb{C}}(V) = \left|\det(\mu_h(E_1), \mu_h(E_2))\right|$$

$$\mu_{\mathbb{R}}(h) = \prod_V m_{\mathbb{R}}(V) \qquad m_{\mathbb{R}}(V) = \begin{cases} 0 & m_{\mathbb{C}}(V) \text{ even} \\ (-1)^{\frac{m_{\mathbb{C}}(V)-1}{2}} & \text{odd} \end{cases}$$

$$\sum_{h \in S(g, \Delta, \mathcal{X})} \mu_{\mathbb{C}}(h) \qquad \sum_{h \in S(g, \Delta, \mathcal{X})} \mu_{\mathbb{R}}(h)$$

Independent of choice of $\mathcal{X}$

GW invariants $\qquad$ $(g=0)$ Welschinger invariant

Block-Göttsche multiplicity:

$$G_V(y) =$$

$G_V(1) = m_{\mathbb{C}}(V) \qquad G_V(-1) = m_{\mathbb{R}}(V)$

$G_h(y) = \prod_V G_V(y) \rightsquigarrow$ Laurent polynomial of $y^{\pm\frac{1}{2}}$

symmetry: $G_h(y) = G_h(y^{-1})$

<u>Thm</u> (Itenberg-Mikhalkin)

$$\sum_{h \in S(g,0,x)} G_h(y)$$

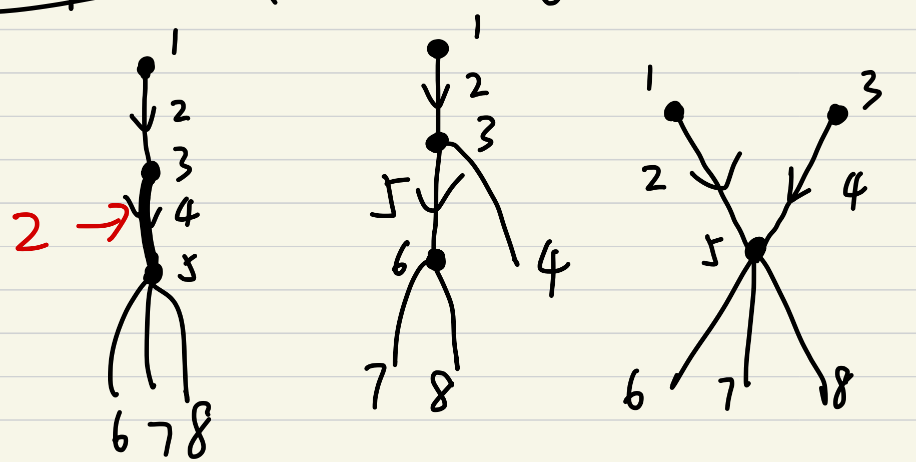
is independent of the choice of x.

We denote the number as $G_{(g,0)}(y)$.

As before, we can use labelled floor diagram or lattic path algorithm to compute $G_{(g,0)}(y)$

For labelled floor diagram, we need to replace multiplicities by $G_v(y)$

Example: $(\mathbb{P}^2, g=0, \deg 3)$. 3 vertices 3 ends



\# labelings = 5 \# labelings = 3

$$\left(\frac{y^{\frac{2}{2}} - y^{-\frac{2}{2}}}{y^{\frac{1}{2}} - y^{-\frac{1}{2}}}\right)^2 \qquad 5 \qquad\qquad 3$$

$$\Rightarrow \quad y + y^{-1} + 10$$

$$y = 1 \rightsquigarrow N^{\mathbb{P}^2}_{g=0,d=3} = 12$$

$$y = -1 \rightsquigarrow W^{\mathbb{RP}^2}_{d=3} = 8$$

Sketch Pf of invariance

Only need to show that for two generic

$$\mathcal{X} = (p_1 \cdots p_{k-1}, p_k)$$

$$\mathcal{X}' = (p_1 \cdots p_{k-1}, p_k')$$

We have

$$\sum_{h \in S(g,\Delta,\mathcal{X})} G_h(y) = \sum_{h \in S(g,\Delta,\mathcal{X}')} G_h(y)$$

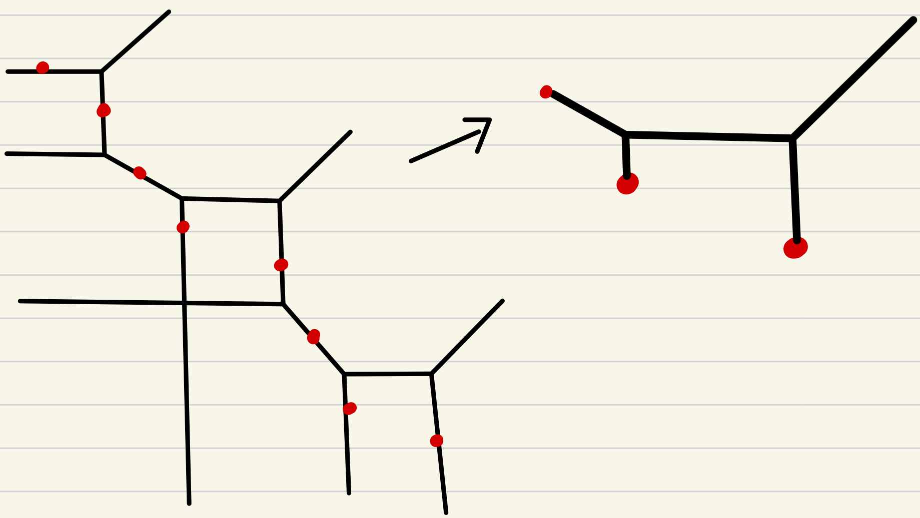
Choose a path $p(t)$ connecting p_k, p_k'

i.e. $p(0) = p_k, \quad p(1) = p_k'$

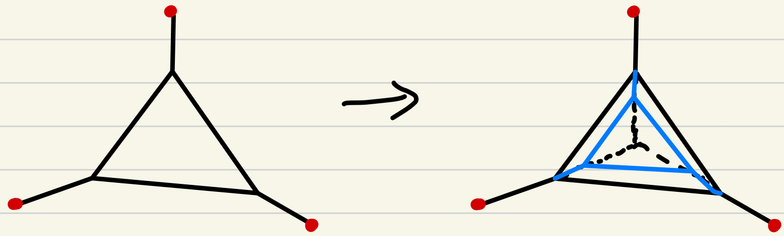
First, let us see how the tropical curve changes when the configuration $(p_1 \cdots p_{k-1}, p(t))$ changes

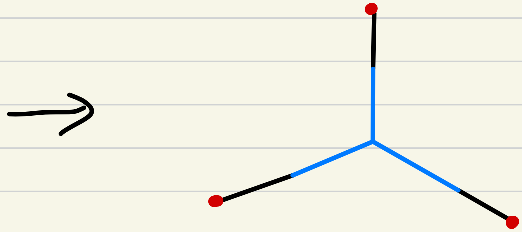
Lemma: (Mikhalkin)

Let $h: C \to \mathbb{R}^2$ be a tropical curve passing through a generic $\mathcal{X}$. Then every connected component of $C \setminus h^{-1}(\mathcal{X})$ is a 3-valence tree with a single leaf going to infinity

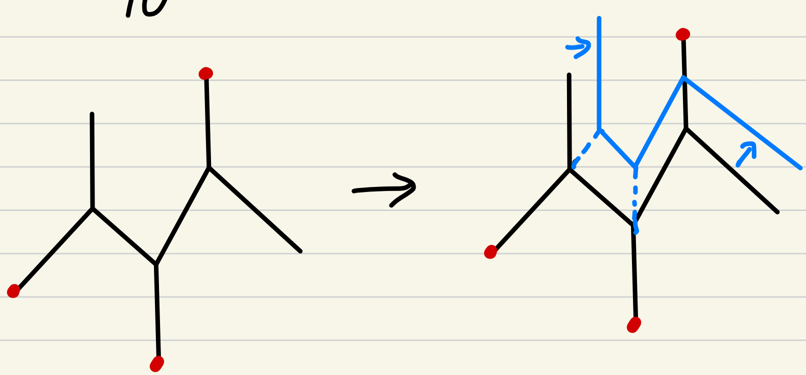


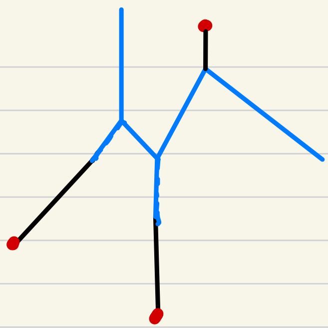
Idea : i) we can always shrink the cycle



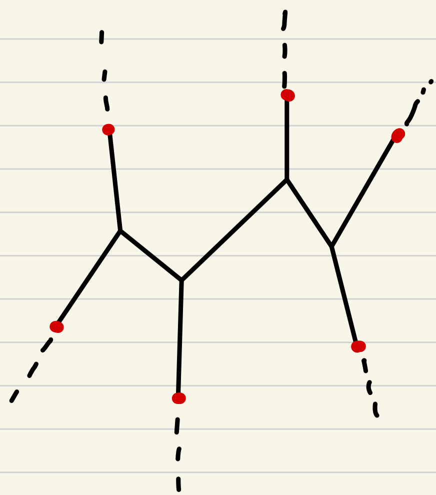


ii) If there are two or more leaves going to infinite, we can perturbe the component which also passing through the same configuration





iii) If there are no leaves going to infinity then a subset of $\mathcal{X}$ is not generic

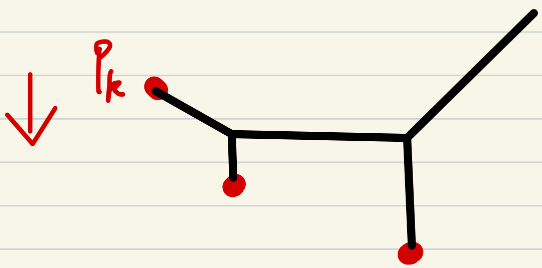


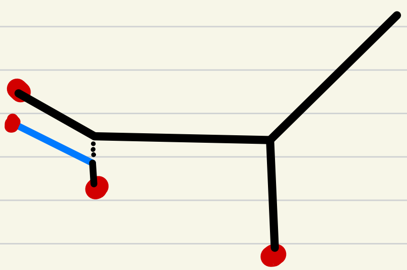
$$k > \#\Delta + g - 1$$

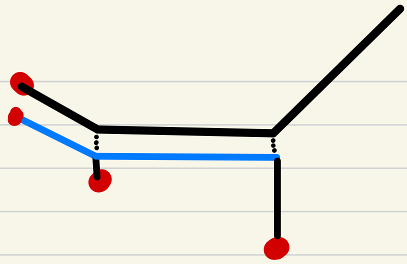
$$6 > 6 + 0 - 1$$

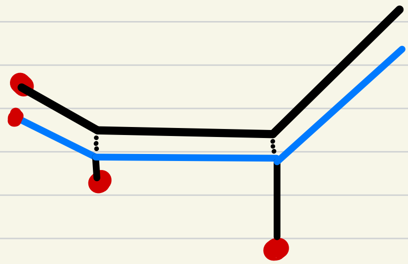
The above lemma gives a unique way to construct a family of tropical curves once we slightly change a generic configuration

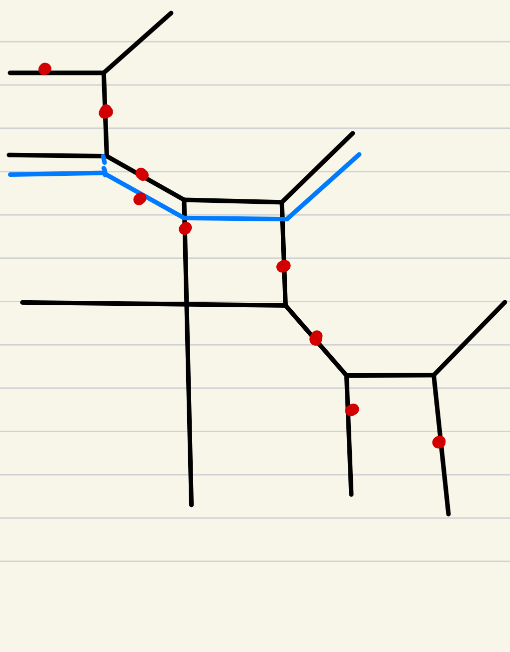
We perturbe each connected component of $C \setminus h^{-1}(X)$ and gluing them together



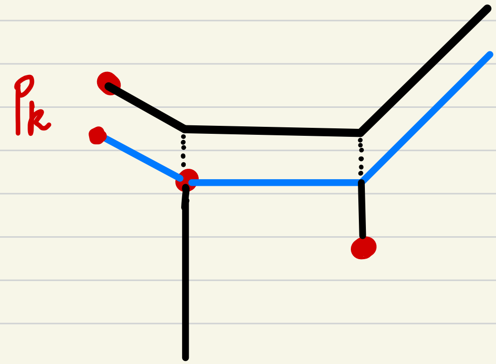








If we continue moving P_k, then we will arrive at a non-generic configuration:

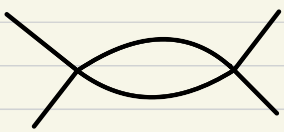


We want to choose a path $P(t)$, so that non-generic configuration can be controlled

$$\mathrm{Def}(C(t_0)) := \sum_{V} (\mathrm{val}(V) - 3) + g - g(C(t_0)) + m$$

where m is the number of vertices map to $x(t_0) = \{p_1 \cdots p_{k-1}, p(t_0)\}$

<u>Lemma</u> There exists a finite set $D \subset \mathbb{R}^2$, s.t. if $p(t_0) \notin D$, then each tropical curve $C(t_0)$ passing through $x(t_0)$ satisfies $\mathrm{Def}(C(t_0)) \leq 1$ or $C(t_0)$ contains

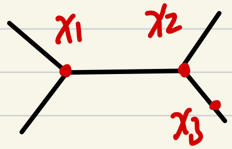


as a subgraph

<u>Idea</u>: Let $M_{g,\Delta}$ be the moduli space of genus $g' \leq g$, degree Δ tropical curves with $k = \#\Delta + g - 1$ markings

$$h : (C, x_1 \cdots x_k) \longrightarrow \mathbb{R}^2$$

$\mathcal{M}_{g,0}$. It has real dim $2k$, which can be further stratified by the slopes of edges and distributions of markings among vertices and edges



Fix such a choice α, we denote the corresponding moduli space as $\mathcal{M}_{g,0}^{\alpha}$. There is a natural evaluation map

$$ev: \mathcal{M}_{g,0}^{\alpha} \to (\mathbb{R}^2)^k$$
$$(h, x_1 \dots x_k) \mapsto (h(x_1) \dots h(x_k))$$

We want $ev^{-1}(p_1 \times \dots p_{k-1} \times \mathbb{R}^2)$ to be non-empty

$$\Rightarrow \dim \mathcal{M}_{g,0}^{\alpha} \geq 2k-2$$

Further more, each α with

$$\dim \mathcal{M}_{g,0}^{\alpha} = 2k-2$$

has at most one (by convenxity) P_k admits a tropical curve h

also the number of such choices α is also finite. so by avoiding those finite P_k. We can always have $\dim \mathcal{M}_{g,\Delta}^{\alpha} > 2k-2$

Such α can be explicit described (by Gathmann and Markwig) $\square$

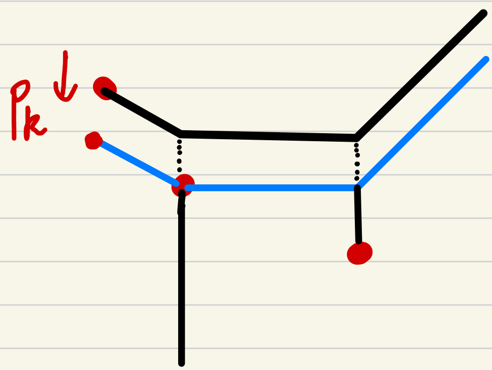
"The number of tropical plane curves through pts in general position"

We now choose a path $p(t)$ connecting $p(0)=P_k$ to $p(1)=P_k'$ avoiding D

For small $0\le t<\varepsilon$, $\chi(t)$ is generic. We can uniquely construct a family of tropical curves

$$h(t): C(t) \longrightarrow \mathbb{R}^2 \quad 0\le t<\varepsilon$$

passing through $\chi(t)$ starting from given $h(0)$

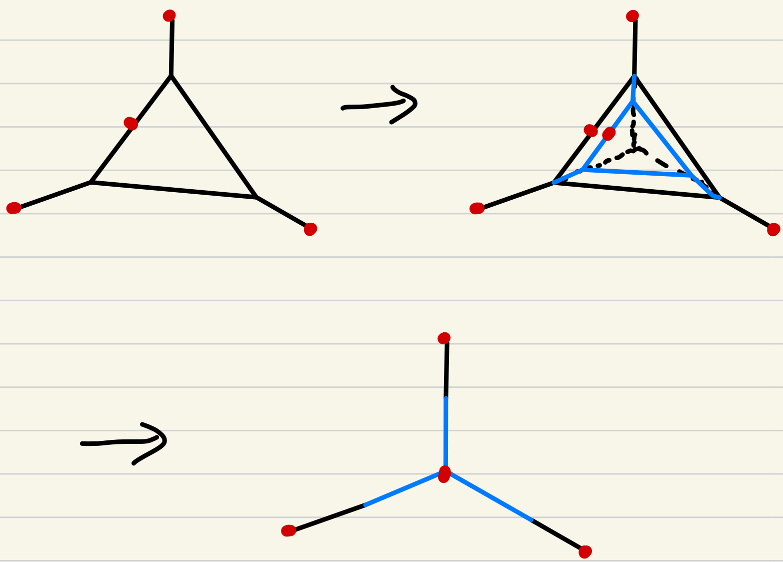


until certain t_0 s.t.

$$Def(C(t_0)) = \sum(val(v)-3) + g - g(C(t_0)) + m$$
$$= 1 \text{ or } 2$$

Claim: $g(C(t_0)) = g$

If $g(C(t_0)) < g$, then $C(t_0)$ must contract certain cycle from $C(t)$ $t < t_0$



So one vertex maps to $X(t_0) \Rightarrow m > 0$

$\Rightarrow Def(C(t_0)) \geq 2$ contrary with the choice of $p(t)$ $\square$

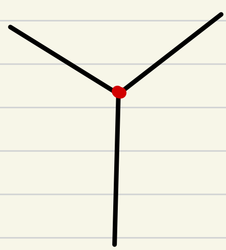
So if

$$Def(C(t_0)) = \sum_{V}(Val(V)-3) + g - g(C(t_0)) + m$$

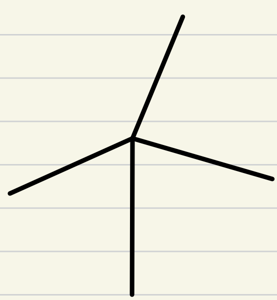
$$= 1$$

$\Rightarrow$ either $Val(V)=3 \ \forall V$

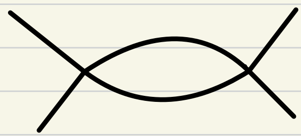
$m=1$

① 

or there exists one 4-valence vertex and $m=0$

② 

if $\mathrm{Def}(C(t_0)) = 2$, it contains

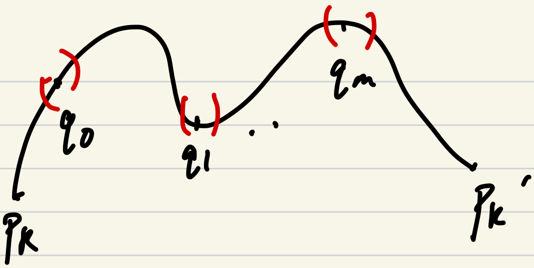
③ 

<u>Lemma</u>: For each tropical curve $h: C \to \mathbb{R}^2$ passing through $x(t_0)$ we have

$$\sum G_{h_j^-}(y) = \sum G_{h_j^+}(y)$$

where h_j^- (resp. h_j^+) runs over all the tropical curves $S(g, 0, x(t_0-\varepsilon))$ (resp. $S(g, 0, x(t_0+\varepsilon))$) such that the limiting curve is h

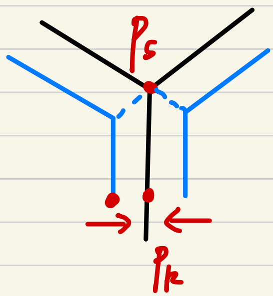
$P(t)$:

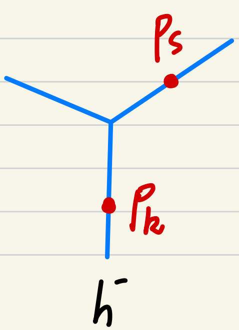


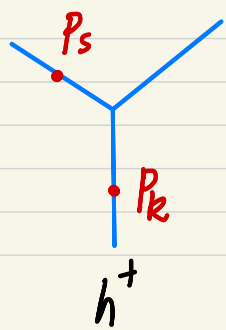
$$\Rightarrow \sum_{h \in S(g,0,x(0))} G_h(y) = \sum_{h' \in S(g,0,x(1))} G_{h'}(y).$$

Pf of Lemma

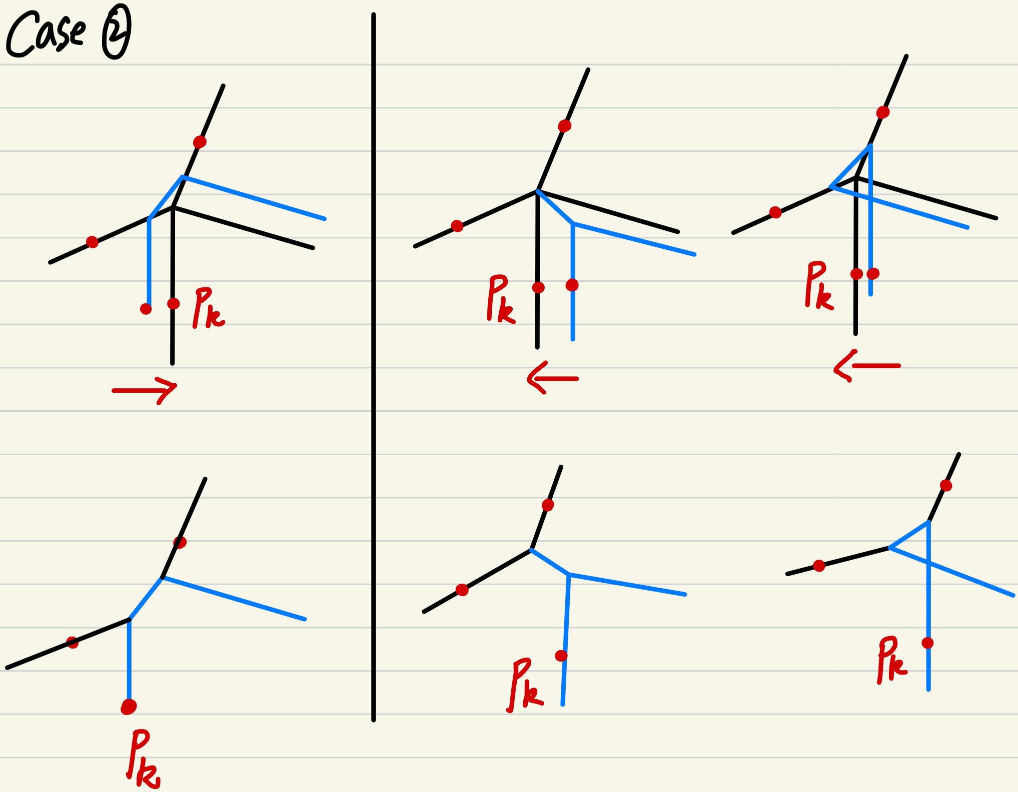
Case ①



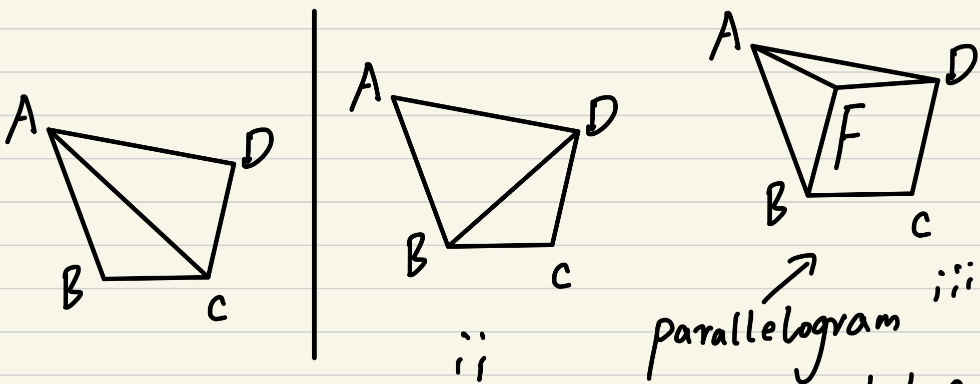




Case ②



dual subdivision



i ii iii

parallelogram with two sides belong to quadrilateral

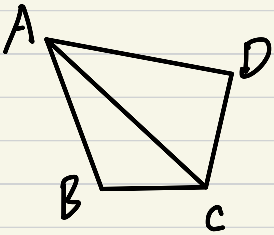
$$\frac{q^{S_{\Delta ABC}}-q^{-S_{\Delta ABC}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\cdot\frac{q^{S_{\Delta ADC}}-q^{-S_{\Delta ADC}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$$

Contribution of ii)

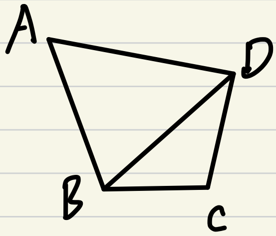
$$\frac{q^{S_{\Delta ABD}}-q^{-S_{\Delta ABD}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\cdot\frac{q^{S_{\Delta BDC}}-q^{-S_{\Delta BDC}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$$

Contribution of iii)

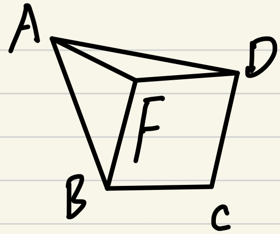
$$\frac{q^{S_{\Delta ABF}}-q^{-S_{\Delta ABF}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}\quad\frac{q^{S_{\Delta ADF}}-q^{-S_{\Delta ADF}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$$



i



ii



iii

i = ii + iii

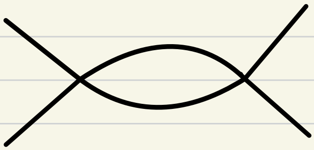
Follows from the area identities

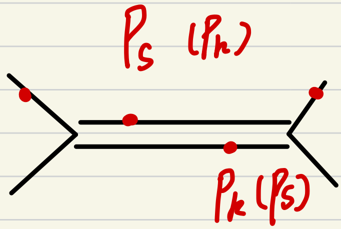
$$S_{\triangle ABC} + S_{\triangle ADC} = S_{\triangle ABD} + S_{\triangle BCD}$$

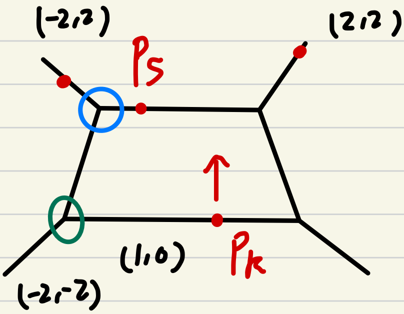
$$S_{\triangle ABF} + S_{\triangle ADF} = S_{\triangle ABD} - S_{\triangle BCD}$$

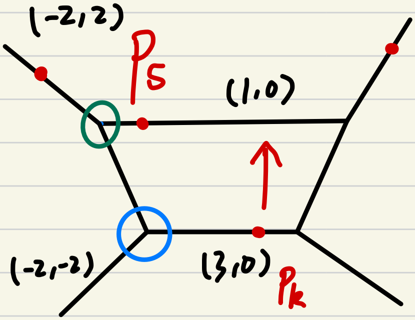
$$S_{\triangle ABF} - S_{\triangle ADF} = S_{\triangle ACD} - S_{\triangle ABC}$$

Case ③

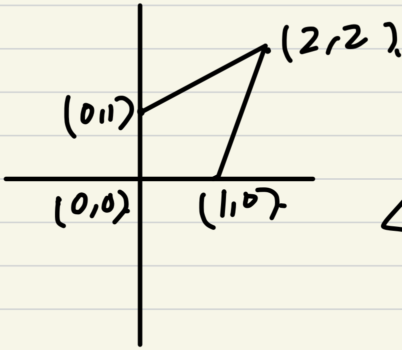








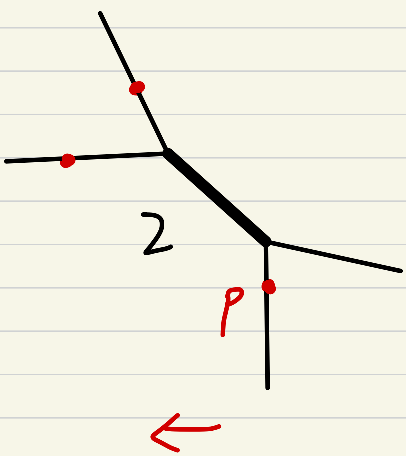
Example (in Heng's talk)

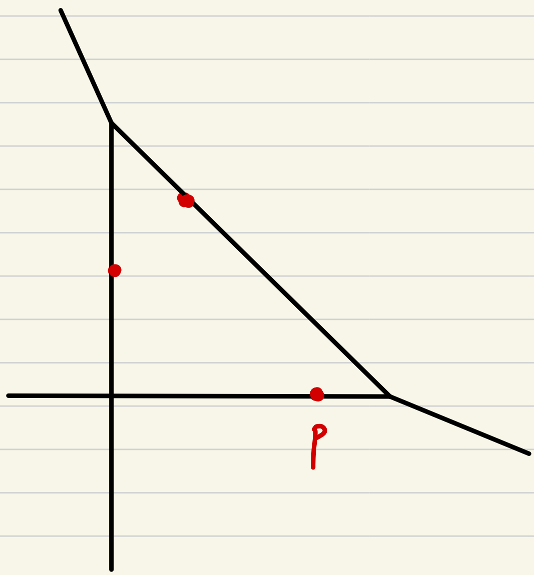


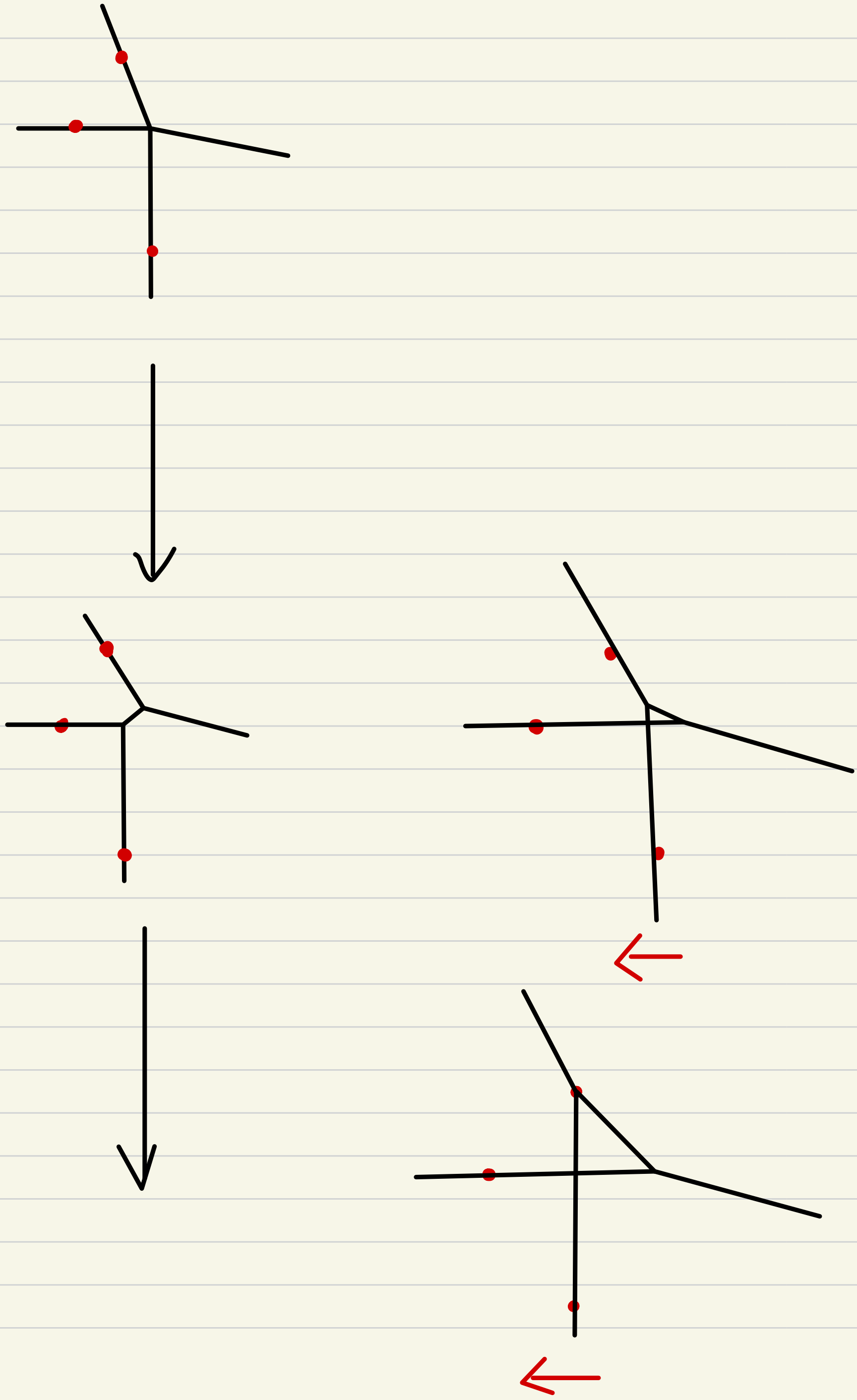
$g = 0$

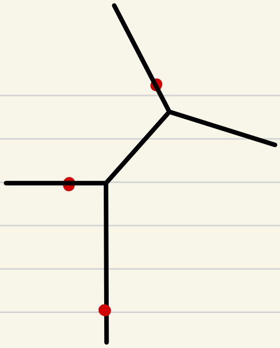
$$\Delta = \left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$$

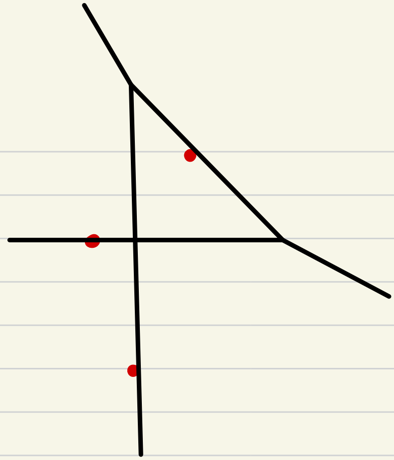
$$k = \#\Delta + g - 1 = 3$$

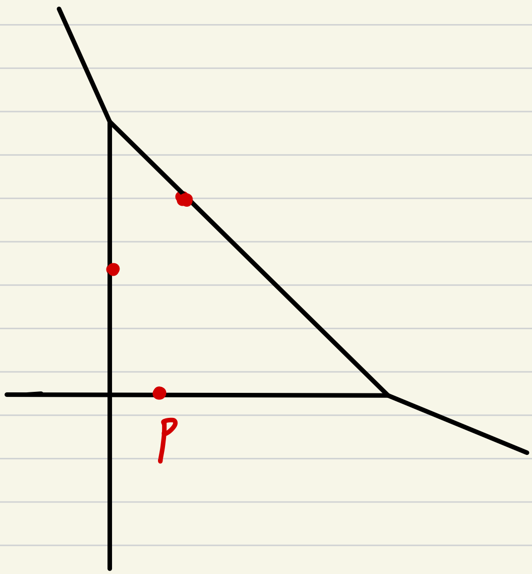








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Relation to log GW thy

- Give Δ, we can get a toric surface X_Δ whose fan is generated by vectors in Δ

Let ∂X_Δ be its toric bdy

- A curve class β_Δ can be determined as follows: for a ray ρ in the fan of X_Δ, it determines a prime toric divisor, then

$$\beta_\Delta \cdot D_\rho = \sum_{v \in D_\rho} GCD(v)$$

Δ_ρ the set of elements $v \in \Delta$

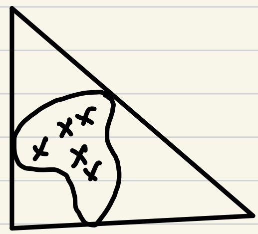
s.t. $\mathbb{R}_{\geq 0} v = \rho$.

- Tangency condition: the curve C intersects D_ρ in $|\Delta_\rho|$ points with multiplicities $GCD(v)$, $v \in \Delta_\rho$

Fix $\Delta, g, \tilde{g} \geq g$

We can construct a moduli space of log stable maps to $(X_\Delta, \partial X_\Delta)$

$\overline{M}_{\tilde{g},\Delta,k}$ with genus $\tilde{g}$, curve class β_Δ and tangency condition determined by Δ with additional k inner markings



To define log GW invariants, we need

i) $ev_i : \overline{\mathcal{M}}_{\tilde{g},\Delta,k} \to X_\Delta$

ii) $[\overline{\mathcal{M}}_{\tilde{g},\Delta,k}]^{vir} \in A_{vdim}(\overline{\mathcal{M}}_{\tilde{g},\Delta,k})$

$vdim = \tilde{g} - 1 + \#\Delta + k$

iii)
$$\begin{array}{c} \mathcal{C} \\ \downarrow \pi \\ \overline{\mathcal{M}}_{\tilde{g},\Delta,k} \end{array} \qquad E = \pi_* \omega_\pi$$

$\lambda_j := c_j(E)$

$$N_{\tilde{g}}^{\Delta,k} := \int_{[\overline{M}_{\tilde{g},0,k}]^{vir}} \left(\pi\, ev_i^*([pt])\right) \cdot \lambda_{\tilde{g}-g}$$

<u>Thm</u> (Bousseau)

Fix g, Δ, $k = \#\Delta + g - 1$

$$\sum_{\tilde{g} \geq g} N_{\tilde{g}}^{\Delta,k}\, u^{2\tilde{g}-2+\#\Delta}$$

$$= G(g,\Delta)(y) \cdot \left((-\sqrt{-1}) \left(y^{\frac{1}{2}} - y^{-\frac{1}{2}} \right) \right)^{2g-2+\#\Delta}$$

where $y = e^{\sqrt{-1}u} = \sum_{n \geq 0} \frac{(\sqrt{-1}u)^n}{n!}$