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Symmetric Obstruction Theories and Behrend's theorem

Let X be D-M stack $/\mathbb{C}$ admitting a symmetric obs. thy.

$$X \hookrightarrow \underset{sm}{M} \qquad \pi : X \to \mathbb{C}$$

$$\underset{\substack{\text{Donaldson}\\\text{Thomas}\\\text{invariant}}}{\#^{vir}(X)} = \pi_* [X]^{vir} \in \mathbb{Q}$$

Intrinsic to X - doesn't depend on M or on the s.o.th

Weighted Euler characteristic.

$$f \in \mathrm{Con}(X) \qquad \chi(X, f) \qquad \in \mathbb{Q}$$
$$\qquad\qquad\qquad \overset{\shortparallel}{\left(\textstyle\sum n\,\chi(f^{-1}(n))\right)}$$

<u>Main thm</u>: For a proper X with a s.o.t.

$$\chi(X, \nu_X) = \pi_* [X]^{vir}$$

defined also for X not proper

a certain weight func.

For X smooth this is Gauss-Bonnet.

Main diagram:

$$\begin{tikzcd}
\mathrm{Con}(X) \arrow[dr, "c_0^{SM}"'] & Z_\cdot(X) \arrow[l, "Eu"', "\sim"] \arrow[r, "L", "\sim"'] \arrow[d, "c_0^M"] & \mathcal{L}_X(\Omega_M) \arrow[dl, "0^!"] \\
& CH_0(X) &
\end{tikzcd}$$

conic Lagrangian cycles on X

$$\nu_X \longleftrightarrow \alpha_X \longleftrightarrow C$$

obs. cone

Now,

$$[X]^{vir} = 0^![C] = c_0^M(\alpha_X) = c_0^{SM}(\nu_X)$$

+ MacPherson / Kasiwara index them

$$\chi(X, f) = \int c_0^M(\mathfrak{C}) \qquad \mathfrak{C} = \mathfrak{d}_X$$
$$\overset{\shortparallel}{Eu(\mathfrak{C})}$$

$\Downarrow$ main thm follows.

A mysterious cycle:

Let X be a scheme $X \hookrightarrow M$ $\quad C_{X|M} = \operatorname{Spec}_{\mathcal{O}_X}\left(\oplus I^n/I^{n+1}\right)$

$\pi : C_{X|M} \longrightarrow X$

$$\mathfrak{d}_{X|M} = \sum_{\substack{C' \text{ components} \\ \text{of } C_{X|M}}} (-1)^{\dim \pi(C')} \operatorname{mult}(C') \cdot \pi(C') \quad \in Z_*(X)$$

<u>Claim</u>: This can be extended for X a DM stack:

$\mathfrak{d}_X$ a cycle on X, s.t $\forall$ $U \xrightarrow{et} X$, $U \hookrightarrow M$

$$\mathfrak{d}_X|_U = \mathfrak{d}_{U|M}$$

<u>Ex</u>: X sm. $\quad \mathfrak{d}_X = (-1)^{\dim X}[X]$

<u>Properties</u>: $X \xrightarrow[sm]{f} Y \quad f^*\mathfrak{d}_Y = (-1)^{\dim X/Y}\mathfrak{d}_X$

$$\mathfrak{d}_{X \times Y} = \mathfrak{d}_X \times \mathfrak{d}_Y$$

§ The Behrend Function $\quad \nu_X := Eu(\mathfrak{d}_X)$

Nash blow-up: X DM-stack of dim n.

$Gr_n(\Omega_X) \leftarrow$ Gr of rk n loc free quotients of Ω_X

Nash blow-up $\rightarrow$ $\tilde{X} \xrightarrow{\mu} X$

$\tilde{T}X :=$ dual of the tautological bundle

$\tilde{V} \xrightarrow{\mu} V$

$$Z_\cdot(X) \xrightarrow{\sim} \mathrm{Con}(X)$$

$$\text{prime cycle } V \longmapsto \left(p \longmapsto \int_{\mu^{-1}(p)} c(\tilde{T}X) \cap s(\mu^{-1}(p), \tilde{V})\right)$$

Properties: $X \xrightarrow[sm]{f} Y \qquad f^* \nu_Y = (-1)^{\dim X/Y} \nu_X$

- If X is smooth, $\nu_X = (-1)^{\dim X}$

Fact: $f : \underset{\text{sm. scheme}}{M} \longrightarrow \mathbb{A}^1 \qquad X = \mathrm{crit}(f) = Z(df)$

$$\nu_X(p) = (-1)^{\dim M}\left(1 - \chi(MF_p(f))\right)$$

(Parusinski-Pragasz)

The weighted Euler characteristic w/ compact supports -

For a scheme $f \in \mathrm{Con}(X)$ $\quad \chi(X,f) = \sum_{n \in \mathbb{Z}} n\, \chi(f^{-1}(n))$

Characterised by:

- for a sm. proper X $\quad \chi(X, 1_X) = \chi(X)$
- $X = \underset{cl}{Z} \sqcup \underset{open}{U}$ $\quad \chi(X,f) = \chi(U, f|_U) + \chi(Z, f|_Z)$
- $\chi(X \times Y, f \times g) = \chi(X,f) \cdot \chi(Y,g)$
- $X \xrightarrow[\text{ét}]{fin} Y$ of dim d, then $\chi(X, f|_X) = d \cdot \chi(Y, f)$

allows to extend the def to (DM) stacks

$\chi(X, f) \in \mathbb{Q}$

$\chi(X, \nu_X)$

Claim: For X sm + proper

$$\chi(X, \nu_X) = \int_{[X]} e(\Omega_X)$$

$$\| \qquad\qquad \|$$

$$(-1)^{\dim X}\chi(X) = (-1)^{\dim X} \int_{[X]} e(T_X)$$

↑ Gauss-Bonnet

<u>Chern-Mather class</u>: $\quad c^M : Z_\cdot(X) \longrightarrow CH_\cdot(X)$

$$\underset{\text{prime cycle}}{V} \longmapsto \nu_* \left(c(\tilde{T}V) \cap [\tilde{V}] \right)$$

For X sm, $c^M(\mathbb{1}_X) = (-1)^{\dim X} c(T_X) \cap [X] = c(\Omega_X) \cap [X]$.

<u>Thm</u>: $\quad \chi(X, Eu(C)) = \int c^M(C)$

(Macpherson / Kasiwara) $\quad C \in Z_\cdot(X)$

↑ schemes.

For $X = [Y/G]$ (↑ finite gp) $\qquad Y \xrightarrow[\text{ét}]{\text{fin}} X$

$$\chi(Y, \nu_Y) = d \cdot \chi(X, \nu_X)$$

$$\int c^M(\mathbb{1}_Y) = d \cdot \int c^M(\mathbb{1}_X)$$

Can also be extended to a general DM-stack.

<u>§ Symmetric obs theories</u>

Perf. o.t: $\quad \phi : E_\cdot \to \tau_{[-1,0]} \mathbb{L}_X \quad \in D(\mathcal{O}_X)$

↑ perfect $\in [-1, 0]$

$h^0(\phi)$ is a iso

$h^{-1}(\phi)$ is a epi.

$E : [E_1 \to E_0]$

$$\mathbb{E} = [E_1^\vee / E_0^\vee]$$
$$\uparrow$$
$$\mathfrak{C}_X$$

$$\begin{array}{ccc} & U & \hookrightarrow M \\ \text{ét} & \downarrow & \\ & X & \end{array}$$

$$\mathfrak{C}_{U/M} = [C_{U/M} / T_M|_U]$$

$$[X]^{vir} := 0^!_{\mathbb{E}} [\mathfrak{C}_X]$$

<u>Proposition</u>: Suppose $\Omega \twoheadrightarrow ob := h^1(E^\vee)$ — The obstruction sheaf

v.b on X

Then $\exists\ C \subset \Omega$ s.t. $\dim C = rk E + rk \Omega$

The obs. cone in Ω.

& $[X]^{vir} = 0^!_\Omega [C] \in CH_{rk E}(X)$

E.g. $\Omega = \Omega_M|_X \supset C = C_{X|M}$

<u>Symmetric Obstruction theories</u>

Let $E \in D^b(\mathcal{O}_X)$

A non-degenerate sym. bil. form, 1-shifted, is

$$\beta : E \overset{L}{\otimes} E \longrightarrow \mathcal{O}_X[1] \quad \text{in } D^b(\mathcal{O}_X)$$

s.t. $\beta(e \otimes e') = (-1)^{\deg(e)\deg(e')} \beta(e' \otimes e)$

$\theta : E \xrightarrow{\sim} E^\vee[1]$ iso

Alternatively $\theta : E \xrightarrow{\sim} E^\vee[1]$ iso, s.t. $\theta = \theta^\vee[1]$

Ex: $E = [\overset{1}{F} \longrightarrow \overset{0}{F^\vee}]$ $\quad F \otimes F \to \mathcal{O}_X$

sym. bil. form.

$E^\vee[1]\ [F \longrightarrow F^\vee]$

Ex: If $f: M \to \mathbb{A}^1$, $X = \mathrm{crit}(f)$

is a sym. complex + p.o.t for X

$$E = \begin{tikzcd} T_M|_X \arrow[r, "\text{Hessian map}"] \arrow[d] & \Omega_M|_X \arrow[d, equal] \\ I/I^2 \arrow[r] & \Omega_M|_X \end{tikzcd}$$

A s.o.t on X is a p.o.t E + Θ - sym. bil form.

Claim: If E is sym, $rk\, E = 0$

$$rk\, E = rk(E^\vee[1]) = -rk(E^\vee) = -rk(E)$$

$\Rightarrow$ If X is proper & has a s.o.t, then

$$\#^{vir}(X) = \deg [X]^{vir} \text{ makes sense.}$$

$[X]^{vir} \in CH_0(X)$

Claim: $ob = \Omega_X$, $ob = h^1(E^\vee)$ For X admitting a sym.o.t E.

pf: $ob = h^1(E^\vee) = h^0(E^\vee[1]) = h^0(E) = h^0(\mathbb{L}_X) = \Omega_X$

Ex: Let M be sm. & ω a 1-form on M

$X = Z(\omega)$ assume ω is "almost closed"

(= satisfying $d\omega \in I \cdot \Omega^2_M$)

then $\omega^\vee : T_M|_X \to I/I^2$

$$H(\omega) = \begin{tikzcd} T_M|_X \arrow[r, "\nabla\omega := d\circ\omega^\vee"] \arrow[d, "\omega^\vee"] & \Omega_M|_X \arrow[d, equal] \\ I/I^2 \arrow[r, "d"] & \Omega_M|_X \end{tikzcd} \longrightarrow \text{a s.o.t for X}$$

Thm: Every sym.o.t. is ét locally of this type.

Proposition: Let $E \to L_X$ a p.o.t.

If X is a reduced lci scheme, then Ob is locally free.

Pf: Locally the p.o.t. looks like

$$\begin{array}{ccccc} & & h^{-1}(E_\cdot) & & \\ & & \downarrow & & \\ & & E_1 & \longrightarrow & \Omega_{M/X} \\ & & \downarrow & & \| \\ 0 & \to & I/I^2 & \longrightarrow & \Omega_{M/X} \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

$h^{-1}(E)$ is loc. free

$ob = h^{-1}(E)^\vee$

Cor: If X has a s.o.t. and is reduced + lci then X is smooth.

Pf: $ob = \Omega_X$ loc. free $\Rightarrow X$ is smooth.

Another example: Let M be a symplectic manifold $/\mathbb{C}$. (has a 2-form σ)

Let $V, W \subset M$ be Lagrangian submanifolds ($\dim V = \frac{1}{2}\dim M$ & σ vanishes on V)

then $X = V \cap W$ has a s.o.t.:

$$\begin{array}{ccc} N_{V/M} & \simeq & \Omega_V \\ \| & & \| \\ T_M/T_V & \longrightarrow & T_V^\vee \\ v & \longmapsto & \sigma(v, \cdot) \end{array}$$

$$\begin{array}{ccccc} E = [\Omega_M & \xrightarrow{res_V \oplus res_W} & \Omega_V \oplus \Omega_W]|_X \\ \theta \quad \uparrow & & \uparrow \theta_1 \quad\quad \theta_0 \uparrow \\ E^\vee[1] = [T_V \oplus T_W & \longrightarrow & T_M \quad]|_X \end{array}$$

$$\theta_0 : T_M \longrightarrow N_{V/M} \oplus N_{W/M}$$

Next time: → Finish Behrend's proof.

+ choose between: → n-shifted sympl. st. in der. geom (PTVV)

$\to$ motivic fundamental class (in $K_0(\mathrm{Var})$)

$\to$ $\nu_X(p) = (-1)^{\dim M}\left(1 - \chi(MF(f))\right)$