# Symmetric Obstruction Theories and Behrend's theorem - talk 2

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**Recall:**

$$\begin{tikzcd}
\mathrm{Con}(X) \arrow[rr, bend left, Rightarrow] \arrow[dr, "c_0^{SM}"] & Z.(X) \arrow[l, "\overset{\mathrm{Eu}}{\sim}"'] \arrow[r, "\sim", "\rho"'] \arrow[d, "c_0^M"] & \mathcal{L}_X(\Omega_M) \arrow[dl, "0^!"] \\
 & CH_0(X) &
\end{tikzcd}$$

$M$ sm$/\mathbb{C}$

## Lagrangian cycles on $\Omega_M$

$\Omega_M = T^\vee_M$ has a cannonical 1-form
symplectic manifold.

$$\alpha : T^\vee M \longrightarrow T^\vee(T^\vee M)$$

$\pi : T^\vee_M \longrightarrow M$

$\pi_* : T(T^\vee M) \rightarrow TM$

$$(\underset{\underset{M}{\cap}}{x}, \underset{\underset{T^\vee_x M}{\cap}}{p}) \longmapsto \Big( \underset{\underset{T(T^\vee(M))}{\cap}}{\xi} \longmapsto p(\pi_* \xi) \Big)$$

$\omega = d\alpha$ - symplectic form

A closed subvariety $V < \Omega_M$ is called Lagrangian if $\dim V = \frac{1}{2}\dim \Omega_M = \dim M$
& $\omega| = 0 \longleftarrow$ sm. locus of $V$.

$L(\Omega_M)$ - Free abelian gp on conic & Lagrangian varieties

$\mathcal{L}_X(\Omega_M)$ - the ones which are supported on $X$ $(\subseteq \Omega_M|_X)$

**Ex:** Given a subvariety $W \subset M$, The conormal bunle $N^\vee_{W/M}$
The closure of the conormal bundle on the smooth locus,
$(=$ diff forms that vanish on it $\simeq I/I^2)$

**Clam:** All conic Lagrangian subvarieties of $\Omega_M$ are of this form.

$$\begin{tikzcd}
 & \text{prime cycle } Z \arrow[rr, mapsto] & & (-1)^{\dim Z} \cdot N^\vee_{Z/M} \\
\text{iso. of gps.} & Z.(X) \arrow[rr, shift left, "L"] \arrow[ddr, "c_0^M"'] & & \mathcal{L}_X(\Omega_M) \arrow[ll, shift left, "\pi"] \arrow[ddl, "0^!"] \\
 & \underbrace{(-1)}^{\dim \pi(V)} \cdot \pi(V) & & V \arrow[ll, mapsto] \\
 & & CH_0(X) &
\end{tikzcd}$$

**Thm:** This commutes

rk d quotients

closure of the section $Z_{sm} \longrightarrow \tilde{M}$

$$\begin{tikzcd}
\tilde{Z} \arrow[r] \arrow[d, "\nu"'] & \tilde{M} = Gr_d(\Omega_M) \arrow[d, "\mu"] \\
Z \arrow[r, hook] & X \arrow[r, hook] & M
\end{tikzcd}$$

$$0 \longrightarrow N|_{\tilde{Z}} \longrightarrow \mu^*\Omega_M|_{\tilde{Z}} \longrightarrow \tilde{T}^\nu Z \longrightarrow 0$$

$$\begin{tikzcd}
\tilde{Z} \arrow[r] \arrow[d] & N|_{\tilde{Z}} \arrow[d] \\
\tilde{M} \arrow[r, "0"] & \mu^*\Omega_M
\end{tikzcd}$$

$$0^![N|_{\tilde{Z}}] = \underbrace{c_d(\tilde{T}^\nu Z) \cap [\tilde{Z}]} \in CH_0(\tilde{Z})$$

$$\begin{tikzcd}
c_d(\tilde{T}^\nu Z) \cap [\tilde{Z}] \arrow[d, "\nu_*"] & CH_0(\tilde{Z}) \arrow[d, "\nu_*"] \\
(-1)^{\dim Z} c_0^M(Z) & CH_0(Z)
\end{tikzcd}$$

$$\nu_* 0^![N|_{\tilde{Z}}]$$

$$\shortparallel$$

$$0^! \eta_*[N|_{\tilde{Z}}]$$

$$\shortparallel$$

$$0^!_{\Omega_M}[C]$$

The conormal bndle of Z

$$\begin{tikzcd}
N|_{\tilde{Z}} \arrow[r, "\eta"] & C \arrow[r] & \Omega_M \\
\tilde{Z} \arrow[u, "0"] \arrow[r, "\nu"] & Z \arrow[u, "0"'] \arrow[r] & M \arrow[u, "0"']
\end{tikzcd}$$

$$0^!_{\Omega_M}[C] = \underbrace{(-1)}^{\dim Z} c_0^M(Z)$$

$$0^![L(Z)] = c_0^M(Z)$$

□

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Let X be endowed with a s.o.t.

Last week:

$$\begin{tikzcd}
C \arrow[r, hook] & \Omega_M|_X \arrow[r, hook] \arrow[d] & \Omega_M \\
 & ob = \Omega_X &
\end{tikzcd}$$

obstruction cone

s.o.t

$\dim C = \dim M = \frac{1}{2} \dim \Omega_M$

$[X]^{vir} = 0^![C]$

**Thm:** The cone C is Lagrangian.

↳ Reduce to the case $X = Z(\omega)$   $\omega$ is an almost closed 1-form on $M$

$$\begin{tikzcd} T_M|_X \arrow[r] \arrow[d, "\omega^\vee"] & \Omega_M|_X \arrow[d, equal] \\ I/I^2 \arrow[r, "d"] & \Omega_M|_X \end{tikzcd}$$

$[dw \in I\,\Omega_M^2]$

$\mathrm{Sym}^\bullet I/I^2 \hookrightarrow \mathrm{Sym}^\bullet T_M|_X$

$C = C_{X/M} \hookrightarrow \Omega_M|_X$

↳ This is Lagrangian.

[If $X$ were smooth then $C = \underset{\text{bundle}}{\text{conormal}}$ - Lagrangian]

**Show:** $\pi[C] = \mathfrak{c}_X$

Locally $C = C_{X|M}$ by definition of $\mathfrak{c}_X$ and of $\pi$.

**Conclusion:** $[X]^{vir} = 0^![C] = c_0^M(\mathfrak{c}_X) = c_0^{SM}(\nu_X)$

$\deg [X]^{vir} = \chi(X, \nu_X)$

**Behrend-Bryan-Szendrői** - Motivic virtual classes for critical loci. $\in K_0(\mathrm{Var}_{\mathbb{C}})$

$K_0(\mathrm{Var}_S)$ is the ab. gp gen by $[X]$ $\quad X/S$ variety, (not nec irr.)

$Z \xrightarrow[\text{imm}]{\text{cl.}} X \qquad [X] = [Z] + [X \setminus Z]$

$[X] \cdot [Y] = [X \times Y]$.

$\mathbb{L} := [\mathbb{A}^1] \quad , \quad [\mathbb{P}^n] = 1 + \mathbb{L} + \ldots + \mathbb{L}^n$

**Modification:** Invert $\mathbb{L}$ ✓, add sqrt $\mathbb{L}$ $\rightsquigarrow$ adjoin $\mathbb{L}^{-\frac{1}{2}}$

Equivariant setting: $\underset{\text{fin. gp}}{G} \hookrightarrow \underset{\text{var.}}{X}$ $\quad$ [good action = every pt is contained is some $G$-eq. aff. nbd]

$K_0^G(\mathrm{Var}_S) :=$ same relation on $[X, G]$

$$+\ [V, G] = [\mathbb{A}^r_S, G].$$

$V \to S$
G-eq. v.b of rk r.

$\hat{\mu} = \varprojlim \mu_n$ ⇝ define $K_0^{\hat{\mu}}(\mathrm{Var}_S)[\mathbb{L}^{-1}] = \boxed{\mathcal{M}_S^{\hat{\mu}}}$

Kontsevich, Denef-Loeser ⇝ motivic integration

Note: For a sing var. $X$, $[X] \in \mathcal{M}_S^{\hat{\mu}}$ won't give any information about sing $X$.

Motivic nearby fibre. $f: \underset{\text{sm, qpr var}}{X} \to \mathbb{C}$

$$[\psi_f] = \sum_{\emptyset \neq I \subset \{1, \dots, r\}} (1-\mathbb{L})^{|I|-1} [\tilde{E}_I^\circ, \mu_{m_I}] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$$

normal crossing divisor →

$$\begin{tikzcd} Y_0 \arrow[r, hook] \arrow[d] & Y \arrow[d, "h"] \\ f^{-1}(0) = X_0 \arrow[r] & X \arrow[d, "f"] \\ & \mathbb{C} \end{tikzcd}$$

log resolution

$$Y_0 = \bigcup_{i=1}^r E_i$$

for $I \subset \{1, \dots, r\}$  $m_I = \gcd_{i \in I}(N_i)$  $N_i$ is the multiplicity of $E_i$ in $Y_0$

$$E_I = \bigcap_{i \in I} E_i \qquad E_I^\circ = E_I \setminus \Big(\bigcup_{j \notin I} E_j\Big)$$

$x_i$

$\tilde{E}_I^\circ$ is a $\mu_{m_I}$-étale cover of $E_I^\circ$

Ex: $[\psi_{x^2}] = (1-\mathbb{L})\cdot[\cdot\,\cdot,\mu_2]$ $\quad x^2 : \mathbb{C} \longrightarrow \mathbb{C}$

$X_0 = \{x^2=0\}$ $\qquad\qquad [\mathbb{G}_m] = \mathbb{L}-1$

$E_1 = [pt] \xleftarrow{\text{2-cover}} \tilde{E}_1$

Def: $[\varphi_f]_{X_0} = [\psi_f]_{X_0} - [X_0]_{X_0} \in \mathcal{M}^{\hat\mu}_{X_0}$

vanishing cycles

Thom-Sebastiani thm: $(-[\varphi_{f+g}] = -[\varphi_f] \star -[\varphi_g])$

$f = x^2, \; g = y^2$

$$\Big(\underbrace{1-[\cdot\,\cdot,\mu_2]}_{\mathbb{L}^{\frac12} := \quad \in \mathcal{M}^{\hat\mu}_{\mathbb{C}}}\Big)^2 = \mathbb{L}$$

The virtual motive of a critical locus:

$f : M \longrightarrow \mathbb{C} \qquad X = Z(df) \subset M$

Define: $[X]_{relvir} = \underbrace{-\mathbb{L}^{-\frac{\dim M}{2}}}\,[\varphi_f]_X \in \mathcal{M}^{\hat\mu}_X$

$[X]_{vir} = p.fd(X \to \mathbb{C}) \in \mathcal{M}^{\hat\mu}_{\mathbb{C}}$

Ex: $f = 0 \qquad X = M \qquad [\varphi_f]_{X_0} = 0$

$[X]_{relvir} = \underbrace{-\mathbb{L}^{-\frac{\dim X}{2}}}\,[X]_X$

$(\mathbb{L}^{\frac12} \longrightarrow -1)$

Thm: $\chi([X]_{vir}) = \sum_{n\in\mathbb{Z}} n\,(\chi(\nu_X^{-1}(n)) = \chi(X,\nu_X))$

use $\mathbb{L}^{\frac12} \to -1$

$$\chi([X]_{relvir}|_p) = \nu_X(P)$$

Follows from $\chi([\varphi_f]_p) = \chi(MF_p(f)) - 1$

$$-\mathbb{L}^{-\frac{\dim M}{2}} \longrightarrow (-1)^{\dim M}$$

$$\chi([X]_{vir}) = (-1)^{\dim M}\underbrace{(1 - \chi(\underline{MF_f(p)}))}_{}{}_p = \nu_X(P)$$

Applications

$$\underbrace{Hilb^n(\mathbb{A}^3)} = crit(f)$$

$$f : M_{n\times n}^3 \times \mathbb{A}^n / GL_n \xrightarrow{f} \mathbb{A}^1$$

$$(A, B, C, v) \longmapsto tr(A[B, C])$$

$$[\varphi_f] = [f^{-1}(1)] - [f^{-1}(0)]$$

$$Z_{\mathbb{C}^3}(t) = \sum_{n=0}^{\infty} [Hilb^n(\mathbb{C}^3)]_{vir}\, t^n = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} (1 - \mathbb{L}^{2+k-m/2} t^m)^{-1}$$