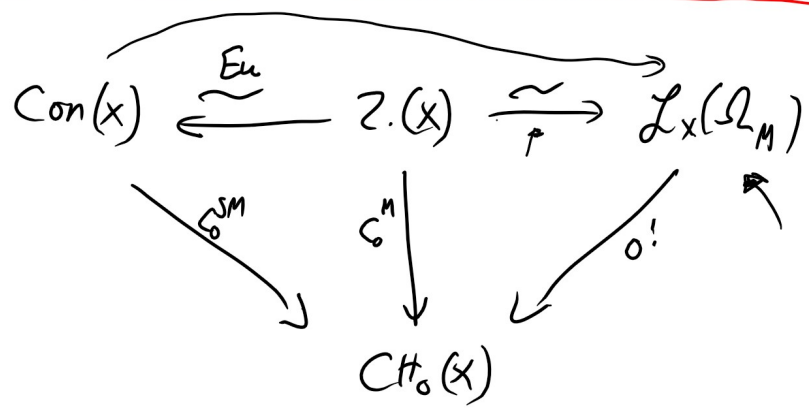


Symmetric Obstruction Theories and Behrend's theorem - talk 2

Recall:



M_{sm}/\mathbb{C}

Lagrangian cycles on Ω_M

$\Omega_M = T^*_M$ has a canonical 1-form
symplectic manifold.

$\alpha: T^*M \rightarrow T^*(T^*M)$

$\pi: T^*_M \rightarrow M$ $(x, p) \mapsto \left(\{ \cdot \} \mapsto p(\pi_* \{ \cdot \}) \right)$
 $\pi_*: T(T^*M) \rightarrow TM$ $\begin{matrix} \cong \\ M \end{matrix} \xrightarrow{\begin{matrix} \cong \\ T^*_x M \end{matrix}} \begin{matrix} \cong \\ T(T^*(M)) \end{matrix}$

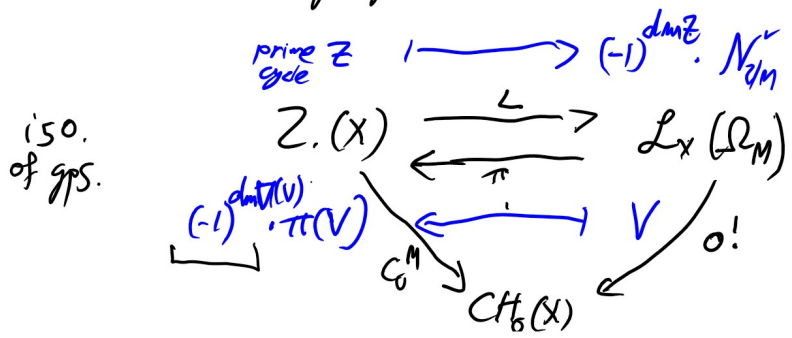
$\omega = d\alpha$ - symplectic form

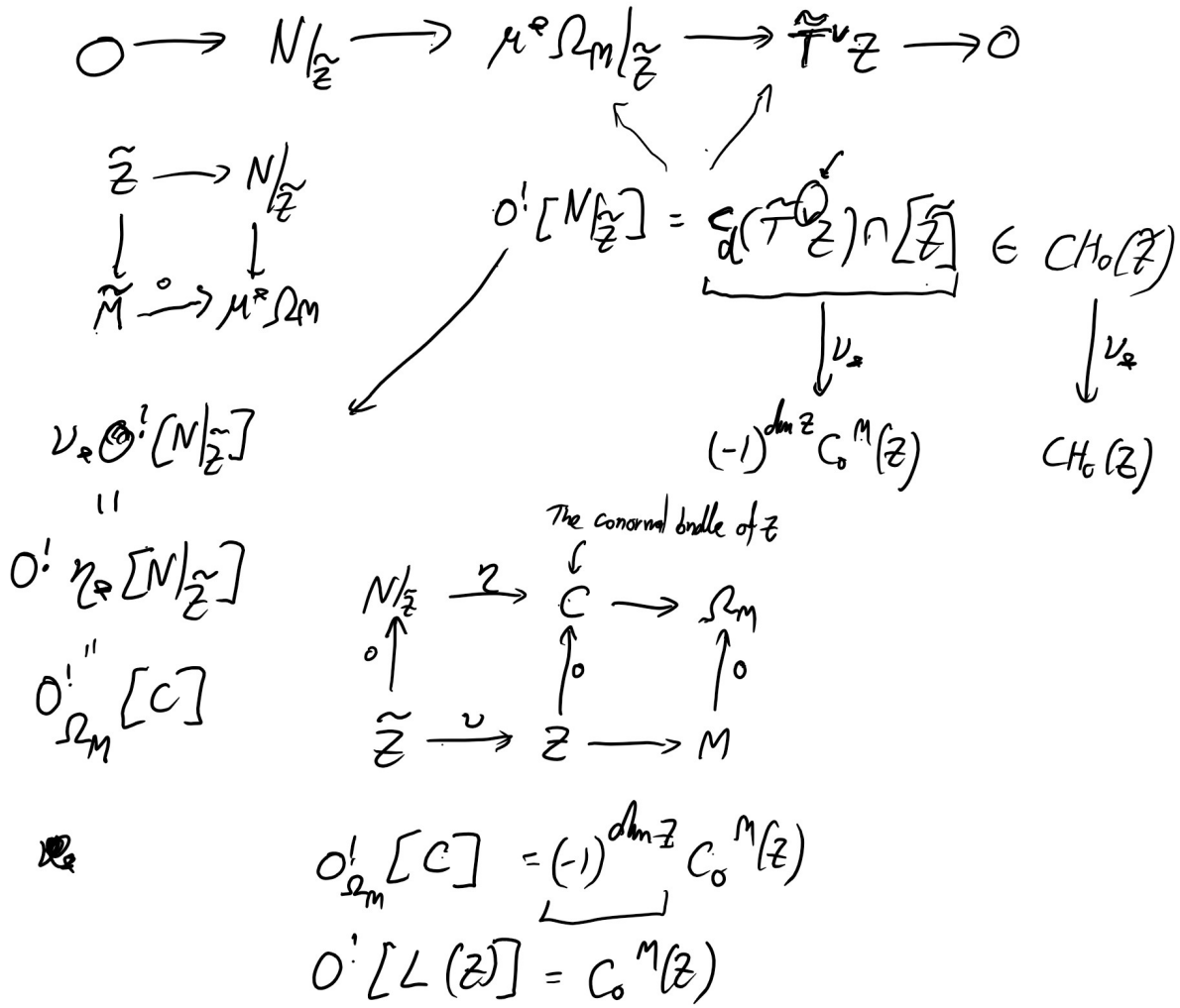
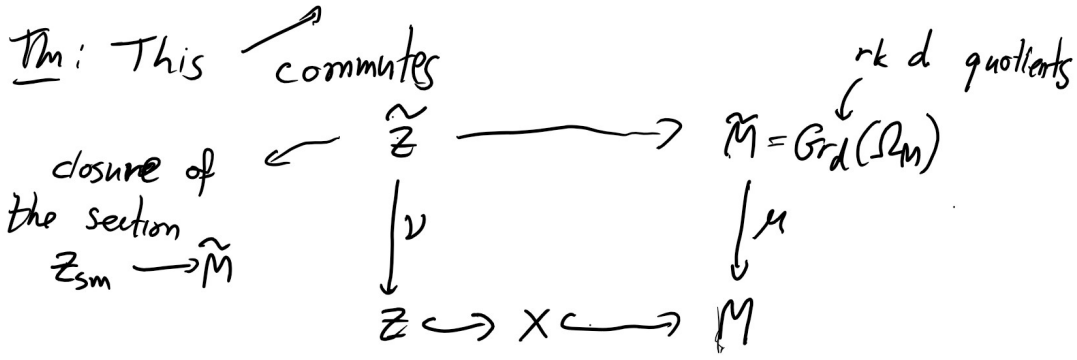
A closed subvariety $V \subset \Omega_M$ is called Lagrangian if $dim V = \frac{1}{2} dim \Omega_M = dim M$
 & $\omega|_V = 0$ sm. locus of V .

$Z(\Omega_M)$ - Free abelian gp on conic & Lagrangian varieties
 $L_x(\Omega_M)$ - the ones which are supported on x ($\subseteq \Omega_M(x)$)

EX: Given a subvariety $W \subset M$, The conormal bundle $N^*_{W/M}$
 The closure of the conormal bundle on the smooth locus,
 (= diff forms that vanish on it $\cong \mathbb{P}^{1/2}$)

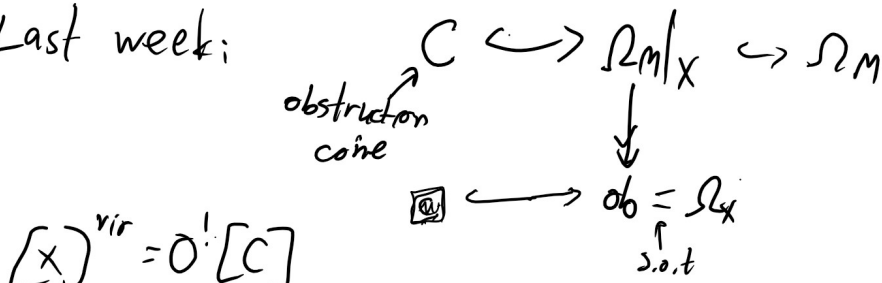
Claim: All conic Lagrangian subvarieties of Ω_M are of this form.





Let X be endowed with a s.o.t.

Last week:



$$\dim C = \dim M = \frac{1}{2} \dim \Omega_M$$

$$[X]^{vir} = O^1[C]$$

Thm: The cone C is Lagrangian.

\hookrightarrow Reduce to the case $X = Z(w)$

w is an almost closed 1-form on M

$$[dw \in \Omega_m^2]$$

$$T_m/x \rightarrow \Omega_m/x$$

$$\text{Sym } T_{\mathbb{R}^2} \hookrightarrow \text{Sym } T_m/x$$

$$\begin{array}{ccc} \downarrow \omega^\vee & & \downarrow \\ \mathbb{R}^2 & \xrightarrow{d} & \Omega_m/x \end{array}$$

$$C = C_x/\mathbb{R} \hookrightarrow \Omega_m/x$$

This is Lagrangian.

[If X were smooth then
 $C = \text{conormal bundle} - \text{Lagrangian}$]

Show: $\pi[C] = dx$

Locally $C = C_x/\mathbb{R}$ by definition of dx and of π .

Conclusion: $[X]^{\text{vir}} = 0 \cdot [C] = C_0^m(dx) = C_0^{SM}(V_x)$

$$\text{deg}[X]^{\text{vir}} = \chi(X, V_x)$$

Behrend-Brylun-Szendroi - Motivic virtual classes for critical loci:
 $\cong K_0(\text{Var}_{\mathbb{C}})$

$K_0(\text{Var}_S)$ is the ab. gp gen by $[X]$ X/S variety,
 (not nec irr.)

$$\begin{array}{ccc} Z & \xrightarrow[\text{imm}]{d_i} & X \\ & & [X] = [Z] + [X \setminus Z] \end{array}$$

$$[X] \cdot [Y] = [X \times Y]$$

$$\mathbb{L} := [A^1], \quad [P^n] = 1 + \mathbb{L} + \dots + \mathbb{L}^n$$

Modification: Invert \mathbb{L} , add sqrt $\mathbb{L} \rightsquigarrow$ adjoin $\mathbb{L}^{-1/2}$

Equivariant setting:

$$\begin{array}{ccc} G & \hookrightarrow & X \\ \text{fm. gp} & & \text{var.} \end{array}$$

[good action = energy pt is contained
 is some G -eq. aff. rd]

$$K_0^G(\text{Vars}) := \text{same relation on } [X, G] \\ + [V, G] = [A_S^r, G].$$

$V \rightarrow S$
 G -eq. r/o of r/r.

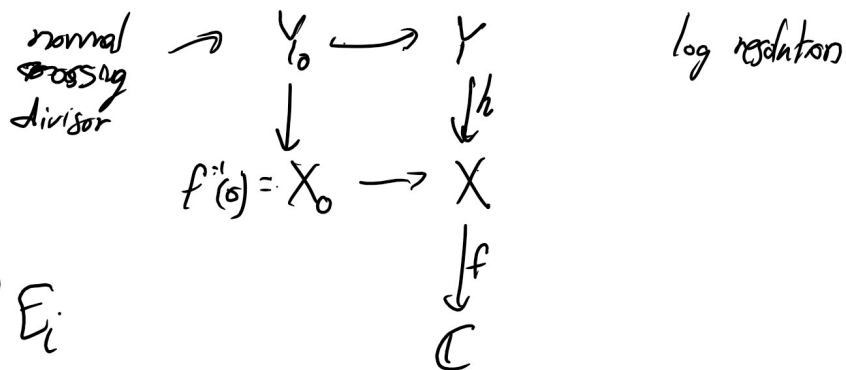
$$\hat{\mu} = \varprojlim \mu_n \rightarrow \text{define } K_0^{\hat{\mu}}(\text{Vars})[\mathbb{L}^{-1}] = \boxed{M_S^{\hat{\mu}}}$$

Kontsevich, Denef - Loeser \rightarrow motivic integration

Note: For a sing var. X , $[X] \in M_S^{\hat{\mu}}$
 won't give any information
 about $\text{sing} X$.

Motivic nearby fibre. $f: X \rightarrow \mathbb{C}$
sm, gpr var

$$[\Psi_f] = \sum_{\emptyset \neq I \subset \{1, \dots, r\}} (1 - \mathbb{L})^{|\mathbb{L}|-1} [\tilde{E}_I^{\circ}, \mu_{m_I}] \in M_{\mathbb{C}}^{\hat{\mu}}$$



$$Y_0 = \bigcup_{i=1}^r E_i$$

for $I \subset \{1, \dots, r\}$ $m_I = \gcd_{i \in I} (N_i)$ N_i is the multiplicity of E_i in Y_0

$$E_I = \bigcap_{i \in I} E_i \quad E_I^{\circ} = E_I - \bigcup_{j \notin I} E_j$$

\tilde{E}_I° is a μ_{m_I} -étale cover of E_I°

x_i

Ex: $[\Psi_{x^2}] = (1 - \mathbb{L}) \cdot [\dots, \mu_2] \quad x^2: \mathbb{C} \rightarrow \mathbb{C}$

$X_0 = \{x^2=0\}$

$[G_m] = \mathbb{L} - 1$

$E_1 = [\text{pt}] \xleftarrow{2 \cdot \text{cov}} \hat{E}_1$

Def: $[\Psi_f]_{X_0} = [\Psi_f]_{X_0} - [X_0]_{X_0} \in \mathcal{M}_{X_0}^{\hat{A}}$

vanishing cycles

Thm - Seibastiani thm: $(-[\Psi_{f+g}] = -[\Psi_f] * -[\Psi_g])$

$f = x^2, g = y^2$

$(1 - [\dots, \mu_2])^2 = \mathbb{L}$
 $\mathbb{L}^{\frac{1}{2}} \in \mathcal{M}_{\mathbb{C}}^{\hat{A}}$

The virtual motive of a critical locus:

$f: M \rightarrow \mathbb{C} \quad X = Z(df) \subset M$

Define: $[X]_{\text{rel vir}} = -\mathbb{L}^{\frac{-\dim M}{2}} [\Psi_f]_X \in \mathcal{M}_X^{\hat{A}}$

$[X]_{\text{vir}} = \text{p.f.d.}(x \rightarrow \mathbb{C}) \in \mathcal{M}_{\mathbb{C}}^{\hat{A}}$

Ex: $f=0 \quad X=M \quad [\Psi_f]_{X_0}=0$

$[X]_{\text{rel vir}} = -\mathbb{L}^{\frac{-\dim X}{2}} [X]_X$

$(\mathbb{L}^{\frac{1}{2}} \rightarrow -1)$

Thm: $\chi([X]_{\text{vir}}) = \sum_{n \in \mathbb{Z}} n \cdot \chi(\nu_x^{-1}(n)) = \chi(X, \nu_X)$
 use $\mathbb{L}^{\frac{1}{2}} \rightarrow -1$

$$\chi([X]_{\text{vir}} | p) = \nu_x(p)$$

Follows from $\chi([\psi_f]_p) = \chi(M_{F_p}(f)) - 1$
 $- \mathbb{L}^{-\frac{\dim M}{2}} \rightarrow (-1)^{\dim M}$

$$\chi([X]_{\text{vir}}) = (-1)^{\dim M} \left(1 - \chi(M_{F_p}(f)) \right) = \nu_x(p)$$

Application

$$\text{Hilb}^n(\mathbb{A}^3) = \text{crit}(f)$$

$$f: M_{n \times \mathbb{A}^3} \times \mathbb{A}^n \xrightarrow{f} \mathbb{A}^4$$

GL_n

$$(A, B, C, v) \mapsto t(A[B, C])$$

$$[\psi_f] = [f^{-1}(1)] - [f^{-1}(0)]$$

$$Z_{\mathbb{C}^3}(t) = \sum_{n=0}^{\infty} [\text{Hilb}^n(\mathbb{C}^3)]_{\text{vir}} t^n = \prod_{m=1}^{\infty} \prod_{k=0}^{m-1} \left(1 - \mathbb{L}^{2+k-m/2} t^m \right)^{-1}$$