The intrinsic normal cone for (higher) Artin Stacks.

§1. Introduction.

Recall: the Behrend-Fantechi construction of the virtual fund. class for $\mathcal{X}$ Deligne-Mumford stack, equipped with P.O.T. (extra data on $\mathcal{X}$)

$$\begin{tikzcd}
\text{cone stack} \arrow[d] & \mathcal{X} \arrow[d, "\text{vertex}"] \\
C_{\mathcal{X}} \arrow[r, hook] \arrow[d] & \mathcal{E}_{\mathcal{X}} \\
\mathcal{X} & \text{vector bundle stack} \arrow[u]
\end{tikzcd}$$

virtual fund. class := intersection of $C_{\mathcal{X}}$ and $\mathcal{X}$ in $\begin{array}{c} CH_*(\mathcal{X}) \\ \wr\wr \\ CH_*(\mathcal{E}_{\mathcal{X}}) \end{array}$

Rmk: (1) $\mathcal{X}$ DM stack, $H_i(L_{\mathcal{X}/\mathrm{Spec}(k)}) = 0 \quad i<0$

(2) Normal bundle stack, $\eta_{\mathcal{X}} = h^1/h^0(\tau_{[0,1]} L_{\mathcal{X}})$

locally:

$$\begin{tikzcd}
U = \mathrm{Spec}(A) \arrow[d, "f"', "\text{ét}"] \arrow[r, hook, "i"] & M \\
\mathcal{X} & \text{smooth}
\end{tikzcd}$$

$$f^*\eta_{\mathcal{X}} = \eta_U = \left[\frac{\mathrm{Spec}(\mathrm{Sym}\, I/I^2)}{\mathrm{Spec}(\mathrm{Sym}\, i^*\Omega_M)}\right]$$

$C_{\mathcal{X}} \doteq$ intrinsic Normal cone, (locally) $[C_{U/M}/i^*T_M] \hookrightarrow = [N_{U/M}/i^*T_M]$

(3) $[\mathcal{E} \to L_{\mathcal{X}}]$ P.O.T. for $\mathcal{X}$, $\mathcal{E} \in$ Tor-amplitude $[0,1]$

$$\begin{tikzcd}
V(\mathcal{E}[-1]) := h^1/h^0(\mathcal{E}) \arrow[d, "\pi"] & \eta_{\mathcal{X}} \arrow[l, hook'] & C_{\mathcal{X}} \arrow[l, hook'] \\
\mathcal{X} & &
\end{tikzcd}$$

$\| \text{ locally}$

$$\left[\frac{\mathrm{Spec}(\mathrm{Sym}\, \mathcal{E}_1)}{\mathrm{Spec}(\mathrm{Sym}\, \mathcal{E}_0)}\right]$$

$\rightsquigarrow [\mathcal{X}, \mathcal{E}] := (\pi^*)^{-1}([C_{\mathcal{X}}]) \in CH_*(\mathcal{X})$.

Today: we view an extension of the B-F construction, works for (relative) higher Artin stacks in the sense of Lurie, Simpson, Toen-Vezzosi.

More precisely:

(1) Construction/Characterization of the Normal Sheaf functor

$$N : \mathrm{RelArt} \longrightarrow \mathrm{Art}$$
$$(\mathcal{X} \to \mathcal{Y}) \longmapsto N_{\mathcal{X}}(\mathcal{Y})$$

(2) Construction of the Normal Cone functor ($\overset{\text{check that indeed}}{\downarrow}$= B.F. for DM stacks)

$$C : \mathrm{RelArt} \longrightarrow \mathrm{Art}, \quad (\mathcal{X} \to \mathcal{Y}) \longmapsto C_{\mathcal{X}}(\mathcal{Y})$$

(3) Virtual fundamental classes (but see also next week).

§ 2. Some recollections: Artin stacks after Lurie et al.

2.1. Definitions

Aff = Affine k-schemes (k fixed ground field)

$= (\mathrm{CAlg}_k)^{op}$

∞-cat of spaces

$$\begin{tikzcd} \mathrm{PStk} = \mathrm{PSh}(\mathrm{Aff}, S) \arrow[d, shift left] \\ \mathrm{Stk} = \mathrm{Shv}_{\acute{e}t}(\mathrm{Aff}, S) \arrow[u, shift left] \end{tikzcd}$$

simply étale sheaves of spaces.

sheaf = satisfies hyperdescent

Def: (Relative n-Artin stack) $f : \mathcal{X} \to \mathcal{Y} \in \mathrm{Stk}$. Inductively: Lurie

(1) 0-Artin if $\forall\ \mathrm{Spec}(A) \to \mathcal{Y}$, $\mathcal{X} \times_{\mathcal{Y}} A$ is an alg. space (Rep. diagonal + atlas)

(2) 0-Artin and smooth if $\mathcal{X} \times_{\mathcal{Y}} A \to \mathrm{Spec}(A)$ smooth + (1).

(3) n-Artin if

$$\begin{tikzcd} \mathcal{X} \arrow[d] & \mathcal{X} \times_{\mathcal{Y}} A \arrow[l] \arrow[d] & \exists\, U \arrow[l, two heads, "sm."'] \arrow[dl, bend left] \\ \mathcal{Y} & \mathrm{Spec}(A) \arrow[l] & \end{tikzcd}$$

, $U \to \mathrm{Spec}(A) \times_{\mathcal{Y}} \mathcal{X}$ $(n-1)$-Artin.

(4) n-Artin and smooth if (3) + smooth, $U = \coprod \mathrm{Spec}(B_i)$

(Rmk: a) we can also introduce (−1) Artin = Affine)

b) for the derived minds: $\mathcal{X} \in d\mathrm{Stk} = \mathrm{Shv}_{\acute{e}t}(d\mathrm{CAlg}_{k, \geq 0}, S)$

simpl. comm. k-Alg. $\cong$ ($d\mathrm{CAlg}_{k,\geq 0}$)

satisfies hyperdescent (Shv)

$\mathcal{X}$ is n-Artin if: *) n-stack

*) m-geometric for some m

$\mathcal{X}(A) \in S_{\leq n}$ $\forall$ A discrete. (Note: HAG. 2.1.1.2, n-Artin stacks according to previous def. are $n+1$ truncated)

Prop: (1) n-Artin $\Rightarrow$ m-Artin $m \geq n$.

(2) pull back of (smooth) rel. n-Artin is (smooth) rel. n-Artin.

(3) Rel. n-Artin are closed under composition.

(4) $n > 0$. We say that $f : \mathcal{X} \to \mathcal{Y}$ is n-submersion if smooth, surj. and relative n-stack (say. Artin).

$$\begin{tikzcd} \mathcal{X} \arrow[r, "{(n-1 \text{ submersion})}"'] \arrow[rr, bend left] & \mathcal{Y} \arrow[r] & \mathcal{Z} \end{tikzcd}$$

n-Artin $\Rightarrow$ $\mathcal{Y} \to \mathcal{Z}$ is n-Artin.

In particular, any morphism $\mathcal{X} \to \mathcal{Y}$ between Artin stacks (= rel. to Spec k) is automatically a relative n-Artin stack.

<u>Rmk</u> (Toën)

(1) (−1) Artin = Affine scheme.

$f : \mathcal{X} \to \mathcal{Y}$ in Stk is (−1) representable (or Affine)

if $\forall\ \operatorname{Spec}(A) \to \mathcal{Y}$, $\mathcal{X} \times_{\mathcal{Y}} A$ is (−1) Artin ($\Leftrightarrow$ Affine)

(2) Assume we have defined (n−1) Artin & (n−1) representable maps & smooth (n−1) rep. maps.

Then:

* $\mathcal{X}$ is n-Artin if $\exists$ $\overset{\text{"smooth n-Atlas"}}{\overset{\downarrow}{\coprod \operatorname{Spec}(B_i)}} \to \mathcal{X}$ smooth, (n−1) rep., surj.

[Surjective: such that $\forall\ \operatorname{Spec}(A) \xrightarrow[f]{} \mathcal{X}$, f factors fpqc-locally on $\mathcal{X}$ through $\coprod \operatorname{Spec}(B_i)$]

* $f : \mathcal{X} \to \mathcal{Y}$ is n-representable or geometric if

$\forall\ \operatorname{Spec}(A) \to \mathcal{Y}$, $\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec}(A)$ is n-Artin.

* $f : \mathcal{X} \to \mathcal{Y}$ n-rep. is smooth if

$$\begin{tikzcd}
\mathcal{X} \times_{\mathcal{Y}} \operatorname{Spec}(A) \arrow[d] & \overset{\exists}{U} \arrow[l, "\text{smooth n-atlas}"'] \arrow[dl, bend left, "\text{smooth map of schemes}"] \\
\operatorname{Spec}(A) &
\end{tikzcd}$$

<u>Ex</u>: $B\mathbb{G}_m = [\operatorname{Spec}(k)/\mathbb{G}_m]$ $\qquad$ $\mathrm{Aff}^{op} \longrightarrow S$

$= $ sheafification of the presheaf $R \longmapsto K(R^{\times}, 1)$

$B^n\mathbb{G}_m =$ " " " $R \longmapsto K(R^{\times}, n)$.

↑ n-Artin stack.

<u>Rmk</u>; $B^n\mathbb{G}_m = \Omega B^{n+1}\mathbb{G}_m = \left(\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec}(k) \arrow[d] \\ \operatorname{Spec}(k) \arrow[r] & B^{n+1}\mathbb{G}_m \end{tikzcd} \right)$

⇒ Can prove by induction using ↗ + proposition (4).

2.2. Quasi coherent sheaves on higher stacks.

Notation: R (discrete) ring. Mod_R = ∞-Cat of R-Modules
(= ∞-Cat enhancement of $D(R)$)

Mod_R has the standard t-structure (from $Ch(R)$)

$\mathrm{Mod}_R^{\heartsuit} = (\mathrm{Mod}_R)_{\geq 0} \cap (\mathrm{Mod}_R)_{\leq 0}$ = usual ab. cat. of R-Modules.

$\mathcal{X} \in \mathrm{Stk}$, define $$QCoh(\mathcal{X}) = \lim_{\mathrm{Spec}(A) \to \mathcal{X}} \mathrm{Mod}_A$$

Thus: $\mathcal{F} \in QCoh(\mathcal{X})$, ⇝ $\forall\ f: \mathrm{Spec}(A) \to \mathcal{X}$
$f^*\mathcal{F} = "\mathcal{F}(\mathrm{Spec}(A))" \in \mathrm{Mod}_A$.

Rmk: * $QCoh(-)$ satisfies étale descent (actually flat descent)

* $QCoh(\mathcal{X})$ is stable presentable ∞-Cat, with t-structure induced by t-structure on Mod_A $\forall\ A \xrightarrow{f} \mathcal{X}$: $\mathcal{F} \in QCoh(\mathcal{X})$ is connective iff $f^*\mathcal{F} \in (\mathrm{Mod}_A)_{\geq 0}$.

If $\mathcal{X}$ is Artin, then on $QCoh(\mathcal{X})$ the t-structure moreover satisfies:

Prop: $\mathcal{F} \in QCoh(\mathcal{X})$ is (co)connective iff $\forall\ \mathrm{Spec}(A) \xrightarrow{p} \mathcal{X}$ smooth, $p^*\mathcal{F} \in \mathrm{Mod}_A$ is (co)connective.

Def: $\mathcal{F} \in QCoh(\mathcal{X})$ is perfect / perfect of amplitude $[a,b]$ / perfect to order n if $\forall\ f: \mathrm{Spec}(A) \to \mathcal{X}$,
$\mathrm{Mod}_A \ni f^*\mathcal{F}$ is perfect / perfect of amplitude $[a,b]$ / …

Thm (Lurie) $\mathcal{X} \xrightarrow[f]{} \mathcal{Y}$ relative n-Artin stack, then f admits a cotangent complex $L_{\mathcal{X}/\mathcal{Y}}$, $(-n)$-connective and perfect to order -1.

* If f is locally of finite type (def. again in terms of atlas), then $L_{\mathcal{X}/\mathcal{Y}}$ is perfect to order 0.

* If f is smooth, $L_{\mathcal{X}/\mathcal{Y}}$ is perfect of non-positive amplitude.

* $\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$ ⇝ $f^*L_{\mathcal{Y}/\mathcal{Z}} \to L_{\mathcal{X}/\mathcal{Z}} \to L_{\mathcal{X}/\mathcal{Y}}$ cofib. seq. in $QCoh(\mathcal{X})$.

Rmk: the cotangent cpx is in fact defined for derived stacks (correct univ. property makes sense only if we evaluate at non discrete rings)

Computational tool:

$X \in PSh(SCRng_{/A}, S)$, fix $x : Spec(A) \to X$

set $\Omega X := \left(\begin{tikzcd} \cdot \arrow[r] \arrow[d] & x \arrow[d] \\ x \arrow[r] & X \end{tikzcd} \right)$, Assume $\forall\ M \in Mod_{A \geq 0}$, the diagram

$$\left\{ \begin{tikzcd} X(A \oplus M) \arrow[r] \arrow[d] & X(A) \arrow[d, "d_0"] \\ X(A) \arrow[r, "d_0"'] & X(A \oplus M[1]) \end{tikzcd} \text{ is pull back} \right\}$$

Then

$$L_X(x) \text{ exists} \iff L_{\Omega X}(\delta_x) \text{ exists}$$

$$\begin{tikzcd} \delta_x : x \arrow[r] \arrow[dr, equal] & \Omega X \arrow[d] \\ & x \end{tikzcd}$$

if both exist,

$$L_X(x) = L_{\Omega X}(\delta_x)[-1]$$

Ex: $\Omega(B\mathbb{G}_m) = \mathbb{G}_m$, $i^* L_{B\mathbb{G}_m} = \delta^* L_{\Omega B\mathbb{G}_m}[-1]$

$i : Spec(k) \to B\mathbb{G}_m$ $\quad = \delta^* L_{\mathbb{G}_m}[-1]$

$\delta : Spec(k) \to \mathbb{G}_m$ $\quad = k[-1]$.

More generally, G comm. alg. gp, smooth $/k$

$$\Omega BG = G, \quad L_{BG} = \mathfrak{g}^\vee[-1]$$

↑ Lie Algebra of G.

§3 Abelian cone construction

3.1 Definitions

$\mathcal{X} \in \mathrm{Stk}$, $\mathcal{E} \in \mathrm{QCoh}(\mathcal{X})$.

$$C_{\mathcal{X}}(\mathcal{E}) \in \mathrm{PSh}(\mathrm{Aff}^{op}_{/\mathcal{X}}, \mathcal{S}) \overset{}{=} \mathrm{PSh}(\mathrm{Aff}^{op}_{k}, \mathcal{S})_{/\mathcal{X}}, \quad \begin{pmatrix} \mathrm{Spec}(A) \\ \downarrow \\ \mathcal{X} \end{pmatrix} \mapsto \underset{\mathrm{Mod}_A}{\overset{\text{mapping space}}{\overset{\downarrow}{\mathrm{Map}}}}(f^*\mathcal{E}, A)$$

Rmk: this clearly recovers the Abelian cone for schemes introduced by Charanya:

$X \in \mathrm{Sch}_{/k}$, $\mathcal{E} \in \mathrm{QCoh}(X)^{\heartsuit}$ (classical)

$$\begin{tikzcd} X \\ \mathrm{Spec}(A) \arrow[u, "f"] \end{tikzcd}$$

$$\mathrm{Map}_{\mathrm{Mod}_A}(Lf^*\mathcal{E}, A) \overset{\mathcal{E}\text{ connective, } A \in \mathrm{Mod}_A^{\heartsuit}}{\overset{\downarrow}{=}} \mathrm{Hom}_{\mathrm{Mod}_A^{\heartsuit}}(f^*\mathcal{E}, A) = \mathrm{Hom}(\mathrm{Spec}\, A, \underline{\mathrm{Spec}}_X(\mathrm{Sym}\, f^*\mathcal{E}))$$

$(Lf^* : \mathrm{QCoh}(X) \to \mathrm{Mod}_A)$
derived pullback

Prop: if $\mathcal{X} \in \mathrm{Stk}$, $C_{\mathcal{X}}(\mathcal{E})$ is also a stack.

(proof: it has étale hyperdescent: follows from classical flat descent for modules after reduction to affine)

In fact, if $\mathcal{X}$ is Artin ~~[illegible]~~ and $\mathcal{E}$ is perfect to order -1, then $C_{\mathcal{X}}(\mathcal{E})$ is Artin. We prove this now:

Some properties first:

(1) Base change: $\varphi : \mathcal{X} \to \mathcal{Y}$, $\mathcal{E} \in \mathrm{QCoh}(\mathcal{Y})$, $C_{\mathcal{X}}(\varphi^*\mathcal{E}) = \mathcal{X} \times_{\mathcal{Y}} C_{\mathcal{Y}}(\mathcal{E})$

$$\begin{tikzcd} f : \mathrm{Spec}(A) \arrow[r] & \mathcal{X} \arrow[r, "\varphi"] & \mathcal{Y} \\ & \mathcal{X} \times_{\mathcal{Y}} C \arrow[u] \arrow[r] & C_{\mathcal{Y}}(\mathcal{E}) \arrow[u] \end{tikzcd}$$

$$\mathrm{Map}((\varphi \circ f)^*\mathcal{E}, A) = C_{\mathcal{Y}}(\mathcal{E})(\varphi \circ f)$$
$$\shortparallel$$
$$\mathrm{Map}(f^*\varphi^*\mathcal{E}, A) = C_{\mathcal{X}}(\varphi^*\mathcal{E})(f)$$

(2) $C_{\mathcal{X}}(\mathcal{E}) = C_{\mathcal{X}}(\mathcal{E}_{\leq 0})$

$$\mathrm{Map}_A(f^*\mathcal{E}, \underset{\text{discrete}}{\underset{\uparrow}{A}}) = \mathrm{Map}_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{E}, \underset{\text{coconnective}}{\underset{\uparrow}{f_*A}})$$
$$\shortparallel$$
$$\mathrm{Map}_{\mathrm{QCoh}(\mathcal{X})}(\mathcal{E}_{\leq 0}, f_*A)$$
$$\shortparallel$$
$$\mathrm{Map}(f^*\mathcal{E}_{\leq 0}, A) = C_{\mathcal{X}}(\mathcal{E}_{\leq 0})(A)$$

(3) if $\mathcal{E}_{\geq 0}$, by (1) & (2) $C_{\mathcal{X}}(\mathcal{E}) \to \mathcal{X}$ is affine.

Partial converse: if $C_{\mathcal{X}}(\mathcal{E}) \to \mathcal{X}$ affine and $\mathcal{E}$ bounded below, then $\mathcal{E}$ is in fact connective.

(4) $\mathcal{E} \to \mathcal{E}' \to P$ in $QCoh(\mathcal{X})$, cofiber sequence with P perfect of non-positive amplitude,

$\Rightarrow$ $C_{\mathcal{X}}(P) \to C_{\mathcal{X}}(\mathcal{E}') \to C_{\mathcal{X}}(\mathcal{E})$ cof. seq. of stacks.

proof: $\mathcal{X} = Spec(A)$ (WLOG)

$f: Spec(B) \to Spec(A)$

$Map_B(-,B)\left((f^*P)[-1] \to f^*\mathcal{E} \to f^*\mathcal{E}' \right)$ $\quad \underset{\shortparallel}{Map_B}(f^*P, B[1])$

$= Map_B(f^*\mathcal{E}', B) \to Map_B(f^*\mathcal{E}, B) \to Map_B(f^*P[-1], B)$

Take π_0 $\Rightarrow$ $\pi_0(Map_B(f^*P, B[1])) = Ext^1(f^*P, B) = 0$

Thus $C_{\mathcal{X}}(\mathcal{E}') \to C_{\mathcal{X}}(\mathcal{E})$ surjective on π_0.

Since $C_{\mathcal{X}}(-): QCoh(\mathcal{X})^{op} \to St/_{\mathcal{X}}$ (Actually $Ab(St/_{\mathcal{X}})$) takes colimits to limits,

$$\begin{tikzcd} P\oplus\mathcal{E}' \arrow[r, phantom, "\simeq"] & \mathcal{E}'\oplus_{\mathcal{E}}\mathcal{E}' & \mathcal{E}' \arrow[l] \arrow[r] & C_{\mathcal{X}}(\mathcal{E}') \times_{C_{\mathcal{X}}(\mathcal{E})} C_{\mathcal{X}}(\mathcal{E}') = C_{\mathcal{X}}(\mathcal{E}') \times_{\mathcal{X}} C_{\mathcal{X}}(P) \\ P \arrow[u] & \mathcal{E}' \arrow[l] \arrow[u] & \mathcal{E} \arrow[l] \arrow[u] & \end{tikzcd}$$

3.2 Proof of Thm: $\mathcal{X} \in St$, $\mathcal{E} \in QCoh(\mathcal{X})$ perfect to order -1 and $(-n)$ connective, $n \geq 0$

$\Rightarrow$ $C_{\mathcal{X}}(\mathcal{E}) \to \mathcal{X}$ is n-Artin stack

if $\mathcal{E}$ is $^{(*)}$ perfect of non-positive amplitude, $C_{\mathcal{X}}(\mathcal{E})$ is smooth.

proof: $\mathcal{X} = Spec(A)$, $C_A(\mathcal{E}) = C_A(\mathcal{E}_{\leq 0})$,

$\rightsquigarrow$ Can assume $\mathcal{E}_{\leq 0} := (0 \to E_0 \to \underbrace{E_{-1} \to \cdots \to E_{-n}}_{\text{f.gen + projective.}} \to 0)$

$\uparrow$ Also proj. if $\mathcal{E}$ satisfies $(*)$

induction on n.

$n=0$, $\mathcal{E}_{\leq 0} = E_0$ in degree $=0$ $\Rightarrow$ $C_{\mathcal{X}}(\mathcal{E}) \to \mathcal{X}$ is (0)- Artin

$n>0$ $\mathcal{E} \to E_0 \to P$ with P perfect coconnective,

$\Rightarrow$ $C(E_0) \times_{C(\mathcal{E})} C(E_0) = C(E_0) \times_{\mathcal{X}} C(P)$, $C(P) \to \mathcal{X}$ $(n-1)$ Artin by induction.

Rmk on comparison with Behrend-Fantechi.

Consider the prestack$/\mathcal{X}$

$$\Gamma(\mathcal{E}) : (\mathrm{Spec}(A) \to \mathcal{X}) \mapsto \mathrm{map}_{\mathrm{Mod}_A}(A, f^*\mathcal{E})$$

If $\mathcal{X}$ is DM, one can check that $\Gamma|_{\mathrm{QCoh}_{\leq 1}} = h^1/h^0$

Next: consider

$$\begin{tikzcd} \mathrm{QCoh}(\mathcal{X}) \arrow[d, "\Gamma"'] & \mathrm{APerf}(\mathcal{X})^{op} \arrow[l] \arrow[dl, bend left, "\underline{\mathrm{map}}(-, \mathcal{O}_{\mathcal{X}})"] \\ \mathrm{St}/\mathcal{X} & \end{tikzcd}$$

$\mathrm{APerf}(\mathcal{X})^{op}$ $\|$ $\leftarrow$ $\mathcal{X}$ Noetherian

substack spanned by $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})$ such that $H^i(\mathcal{F}) = 0$ $i \gg 0$. $H^i(\mathcal{F})$ coherent $\forall i$.

(More general notion also available)

$$\Omega^\infty \underline{\mathrm{map}}(\mathcal{F}, \mathcal{O}_{\mathcal{X}})(\mathrm{Spec}(A) \to \mathcal{X})$$

$\|$

$$\mathrm{map}_{\mathrm{Mod}_A}(f^*\mathcal{F}, A) = C_{\mathcal{X}}(\mathcal{F})(f).$$

One can check that $C_{\mathcal{X}}(\mathcal{E}[-1]) = h^1/h^0(\mathcal{E}^\vee)$.

§4 Normal sheaf

$U \hookrightarrow V$ closed emb. of schemes, I ideal sheaf

$L_{U/V}$ is 1-connective and $H_1(L_{U/V}) = I/I^2$

$N_{U/V} = C_U(I/I^2)$ classically.

If $\mathcal{X} \to \mathcal{Y}$ map of Artin stacks, $N_{\mathcal{X}}(\mathcal{Y}) = C_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{Y}}[-1])$.

(clearly if $U \hookrightarrow V$ as above, $N_U(V) = C_U(L_{U/V}[-1])$

$$= C_U((L_{U/V}[-1])_{\geq 0})$$
$$= C_U(I/I^2)\ )$$

Prop: if $\mathcal{X} \to \mathcal{Y}$ locally of finite type, then $N_{\mathcal{X}}\mathcal{Y}$ is Artin.

(1) If $\mathcal{X} \to \mathcal{Y}$ is n-Artin, $N_{\mathcal{X}}\mathcal{Y}$ is $n+1$-Artin

(2) $\mathcal{X} \to \mathcal{Y}$ smooth, $N_{\mathcal{X}}(\mathcal{Y}) \to \mathcal{X}$ also smooth.

Def: $\mathrm{RelArt} = \mathrm{Fun}_{\mathrm{loc\,f.t.}}(\Delta^1, \mathrm{Art})$

Thm: $N: \mathrm{RelArt} \longrightarrow \mathrm{Art}$ is uniquely det (up to canonical nat. equiv.) by the following 4 properties:

(1) $U \hookrightarrow V$ closed emb. of schemes, $N_U V \cong C_U(I/I^2)$

(2) N preserves coproducts $N_{\mathcal{X} \amalg \mathcal{X}'}(\mathcal{Y} \amalg \mathcal{Y}') \simeq N_{\mathcal{X}}(\mathcal{Y}) \amalg N_{\mathcal{X}'}(\mathcal{Y}')$

(3) N preserves smooth & smooth + surjective maps

(4) N commutes with smooth pullback.

Note that (1), (2), (4) are clearly satisfied by $N_{\mathcal{X}}(\mathcal{Y}) = C_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{Y}}[-1])$
(follows immediatly by standard properties of cotangent cpx)
Trust that everything indeed works as stated.

$$\begin{tikzcd}
(\mathcal{X}' \to \mathcal{Y}') \arrow[r] & (\mathcal{X} \to \mathcal{Y}) \\
 & (W \to Z) \arrow[u]
\end{tikzcd}$$

smooth map in RelArt (both maps smooth)

$(W \to Z)$ arbitrary

$$\rightsquigarrow N_{\mathcal{X}' \times_{\mathcal{X}} W}(\mathcal{Y}' \times_{\mathcal{Y}} Z) = N_{\mathcal{X}'}(\mathcal{Y}') \times_{N_{\mathcal{X}}(\mathcal{Y})} N_W(Z)$$

§5 Normal Cone of a morphism of Artin Stacks

$U \hookrightarrow V$, I id. sheaf,

$$\begin{tikzcd} \mathcal{C}_U V = \underline{\mathrm{Spec}}_U\left(\bigoplus I^n/I^{n+1}\right) \arrow[d, hook] \\ \mathcal{N}_U V \end{tikzcd}$$

Thm: $\exists$ unique functor $\mathcal{C}$: RelArt $\to$ Art "Normal cone" s.t.

(1) if $U \hookrightarrow V$ closed emb. of schemes, $\mathcal{C}_U(V) = \underline{\mathrm{Spec}}_U\left(\bigoplus I^n/I^{n+1}\right)$

(2) $\mathcal{C}$ preserves coproducts

(3) $\mathcal{C}$ preserves smooth & smoothly surjective maps

(4) $\mathcal{C}$ commutes with pullbacks along smooth morph. of relative Artin stacks.

Moreover, $\exists$ unique natural transformation $\mathcal{C} \to \mathcal{N}$ such that $\forall$ $U \hookrightarrow V$ schemes, coincides with $\mathcal{C}_U(V) \to \mathcal{N}_U V$

Also: (1) $\mathcal{C}_{\mathcal{X}} \mathcal{Y} \hookrightarrow \mathcal{N}_{\mathcal{X}} \mathcal{Y}$

(2) if $(\mathcal{X}' \to \mathcal{Y}') \to (\mathcal{X} \to \mathcal{Y})$ smooth, then

$$\begin{tikzcd} \mathcal{C}_{\mathcal{X}'} \mathcal{Y}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{N}_{\mathcal{X}'}(\mathcal{Y}') \arrow[d] & (**) \\ \mathcal{C}_{\mathcal{X}} \mathcal{Y} \arrow[r] & \mathcal{N}_{\mathcal{X}}(\mathcal{Y}) & \end{tikzcd}$$

Preserves pullbacks along smooth maps

"Idea of the proof": $\mathcal{C}$ is a cosheaf on RelArt "adapted" to the class of smooth maps. (cosheaf for smooth topology on Artin stacks)
One can show that an adapted cosheaf uniquely extends from a generating subcategory: in this case the subcategory of morphisms $U \to V$ of (Affine) schemes.

left Kan extension

More properties:

(1) $\mathcal{C}_{\mathcal{X}} \mathcal{Y} \to \mathcal{X}$ canonical projection admits a section
(locally this is the vertex of the cone stack)

(2) Smooth base change

(3) Étale invariance

(4) if $\mathcal{X} \to \mathcal{Y}$ morphism of 1-stacks of finite type which is a relative DM stack, then $\mathcal{C}_{\mathcal{X}} \mathcal{Y}$ = intrinsic normal cone of BF
(use (2)+(3) to reduce to the case of schemes + use (**) for local embedding)

Key / Interesting Lemma:

$$\begin{tikzcd} \text{Pair} \arrow[d] \\ \text{RelArt} \end{tikzcd}$$

Pair = cat of $U \hookrightarrow V$ closed embeddings of schemes.

Lemma:

Any relative Artin stack admits a smooth surjection from a pair of schemes in the above sense.
(⇒ Pair is a generating subcategory of RelArt)

Proof:

$$\begin{tikzcd} \mathcal{X} \quad \text{n-Artin} \\ \coprod_i \mathrm{Spec}(B_i) \arrow[u, two heads, "\text{smooth and (n-1) representable}"'] \end{tikzcd}$$

(i.e. iterated intersections are (n-1)-Artin)

⇝ by induction, RelArt is generated by maps of schemes.

ETS that closed embeddings are enough.

$f : X \longrightarrow Y$ rel. Artin stack where X, Y are disj. union of aff.

f is loc. of finite type ⇝

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, equal] & Y' \arrow[d, two heads] \\ X \arrow[r] & Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} (X \hookrightarrow Y') \arrow[d, two heads] \\ (X \longrightarrow Y) \end{tikzcd}$$

as required

Same story works for iterated intersections.

We can use the same trick (define the functor on closed embeddings of schemes + check that it is a cosheaf on Pair for the smooth topology then extend to RelArt) to construct the "deformation space"

§ 6. Deformation space and fundamental classes

How do we use $\mathcal{C}_X(Y)$ if $X \hookrightarrow Y$ closed emb. of schemes?

Fulton: deform $X \hookrightarrow Y$ into the zero section embedding AKA deformation to the normal cone.

From intersection theory (Fulton, 5.1)

$$\begin{tikzcd} X \times \mathbb{P}^1 \arrow[r, hook] \arrow[dr] & M^\circ \arrow[d, "\rho"] \\ & \mathbb{P}^1 \end{tikzcd}$$

ρ flat, *) $\rho^{-1}(\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}) = Y \times \mathbb{A}^1 \hookleftarrow X \times \mathbb{A}^1$

*) $\rho^{-1}(\infty) = \mathbb{P}(C \oplus 1) \hookleftarrow X$

zero sect. of $X \hookrightarrow C = C_X(Y)$

Explicitly, $M = Bl_{\{X \times \infty\}}(Y \times \mathbb{P}^1)$

$\quad\;\, \uparrow$

M° = complement of $Bl_X(Y) \hookrightarrow Bl_{X \times \{\infty\}}(Y \times \mathbb{P}^1)$.

Rmk: if $Y = \mathrm{Spec}(A) \hookleftarrow X = \mathrm{Spec}(A/I)$, $M^\circ \times_{\mathbb{P}^1} (\mathbb{P}^1 - \{0\}) \simeq \mathrm{Spec}(R)$

$$R = R(A, I) := \bigoplus_{k \in \mathbb{Z}} I^k t^{-k} \subset A[t, t^{-1}]$$

Can define a functor:

$$\mathrm{Pair} \longrightarrow \mathrm{Art}_{/\mathbb{P}^1}, \quad (X \hookrightarrow Y) \mapsto M^\circ = M^\circ_X Y$$

it preserves coproducts, smooth morphisms, smooth surj. and commutes with pullbacks along smooth maps.

↳ It is a cosheaf ~~on the~~ "adapted" to the class of smooth maps. (i.e. commutes with pullbacks along the class T = smooth maps)

⇒ Extends uniquely with nat. transf:

$$\begin{tikzcd} M^\circ : \mathrm{RelArt} \arrow[r] & \mathrm{Art}_{/\mathbb{P}^1} \\ (-) \times \mathbb{P}^1 \arrow[u, Rightarrow] & \end{tikzcd}$$

Moreover, $M^\circ_{\mathcal{X}}(\mathcal{Y}) \to \mathbb{P}^1$ is flat $\forall$ $(\mathcal{X} \xrightarrow{f} \mathcal{Y}) \in \mathrm{RelArt}$

& over $\mathbb{P}^1 - \{\infty\}$ looks like $\mathcal{X} \times \mathbb{A}^1 \xrightarrow[f \times id]{} \mathcal{Y} \times \mathbb{A}^1$

∞ looks like $\mathcal{X} \hookrightarrow \mathcal{C}_{\mathcal{X}}(\mathcal{Y})$.

6.1 Obstruction theory

$(\mathcal{X} \to \mathcal{Y}) \in \mathrm{RelArt}$

$\varphi : \mathcal{E} \longrightarrow L_{\mathcal{X}/\mathcal{Y}}[-1]$ in $\mathrm{QCoh}(\mathcal{X})$

obstr. theory if $H_0(\varphi)$ surjective, $H_i(\varphi)$ iso $\forall\, i \leq -1$

Perfect if $\mathcal{E}$ is perfect of non positive amplitude.

⚠ this should rather be seen as a shadow of derived scheme

Fact: $\varphi : \mathcal{E} \to L_{\mathcal{X}/\mathcal{Y}}[-1]$ is obstruction theory iff $\mathcal{E}$ is bounded below & $N_{\mathcal{X}}\mathcal{Y} \longrightarrow C_{\mathcal{X}}(\mathcal{E})$ is a closed immersion.

Def: If $\mathcal{E}$ is P.O.T. we say that $\mathcal{E}$ has global resolution if $\exists$ $\mathcal{E} \to E$ injective on H_0 with $E \in QCoh(\mathcal{X})^{\heartsuit}$ locally free of finite rank

If $\mathcal{X} \to \mathcal{Y}$ 1-Artin Stack with target of pure dimension

$$[\mathcal{X} \to \mathcal{Y}, \mathcal{E}]^{vir} := 0^! \Big[\underset{\substack{\uparrow \\ \text{smooth} \\ \text{atlas} \\ \text{for } C_{\mathcal{X}}(\mathcal{E})}}{C_{\mathcal{X}}(E)} \times_{C_{\mathcal{X}}(\mathcal{E})} \mathfrak{C}_{\mathcal{X}}(\mathcal{Y})\Big] \in \underset{\substack{\uparrow \\ \text{Kresh' Chow} \\ \text{group}}}{CH_{r-\chi(\mathcal{E})}}(\mathcal{X})$$

We use the map

$$\mathfrak{C}_{\mathcal{X}}(\mathcal{Y}) \longrightarrow C_{\mathcal{X}}(\mathcal{E})$$

given by

$$\begin{tikzcd} \mathfrak{C}_{\mathcal{X}}(\mathcal{Y}) \arrow[r, hook] & N_{\mathcal{X}}(\mathcal{Y}) \simeq C_{\mathcal{X}}(L_{\mathcal{X}/\mathcal{Y}}[-1]) \arrow[d, hook] \\ & C_{\mathcal{X}}(\mathcal{E}) \end{tikzcd}$$

Rmk: this in fact does not depend on the choice of a global resolution.