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I Duality transfer morphisms

Noetherian of finite dim. (or at least quasi-compact and quasi-separated)
(resp. Sm_X with X a scheme)

Let k be a perfect field and Sm_k be the category of smooth k-varieties, i.e. smooth finite-type (separated) k-schemes (resp. smooth finite-type separated X-schemes), with all morphisms of schemes.

We denote by $Shv_{Nis_{(\bullet)}}(Sm_k^{(X)})$ the category of Nisnevich simplicial sheaves on $Sm_k^{(X)}$ (pointed) and by $Spc_{mot_{(\bullet)}}^{(X)}(k)$ the category of motivic spaces, i.e. the full subcategory (pointed) of $Shv_{Nis}(Sm_k^{(X)})$ (Sm_X) whose objects are the $\mathbb{A}^1$-invariant (pointed) Nis. simplicial sheaves $\mathcal{X}$ (i.e. $\forall U \in Sm_k^{(X)}$ $\mathcal{X}(U) \to \mathcal{X}(U \times \mathbb{A}^1_k{}^{(X)})$ induced by the projection is a weak equivalence).

We denote by $SH(k)$ (Sm_X) the stable motivic homotopy category, i.e. $Spc_{mot_\bullet}^{(X)}(k)[(\mathbb{P}^1)^{-1}]$. (i.e. $(\mathbb{A}\mathbb{P}^1)$; pointed by ∞)

We denote for $Y \in Sm_k$ by $Y_+ := Y \amalg Spec(k)$ ($Y \amalg X$) pointed by $Spec(k)$ (X), by $\Sigma^\infty Y_+ := ((\mathbb{P}^1)^{\wedge n} \wedge Y_+, \mathrm{Id})_{n \in \mathbb{N}_0}$ and by $\mathbb{1}_k := \Sigma^\infty(Spec(k))_+$ ($\mathbb{1}_X := \Sigma^\infty(X_+)$).

Let $f: Y \to X$ be a smooth morphism in Sm_k. We have the following adjunctions:

$$f_\#: Spc_{mot_\bullet}(Y) \to Spc_{mot_\bullet}(X) \dashv f^*: Spc_{mot_\bullet}(X) \to Spc_{mot_\bullet}(Y) \dashv f_*: Spc_{mot_\bullet}(Y) \to Spc_{mot_\bullet}(X)$$

For each $\mathcal{G} \in Spc_{mot_\bullet}(Y)$, $f_*(\mathcal{G})$ is the sheaf: $(g: Z \to X) \mapsto \mathcal{G}(Z \times_X Y)$.

For each $\mathcal{F} \in Spc_{mot_\bullet}(X)$, $f^*(\mathcal{F})$ is the sheaf: $(h: T \to Y) \mapsto \mathcal{F}(T)$. ($X$-scheme via $f \circ h$)

$f_\#(\mathcal{G})$ is more complicated to define, except when $\mathcal{G} = \mathrm{Hom}(-, h: T \to Y)$ is the sheaf associated (Yoneda embedding) to a Y-scheme $T \in Sm_Y$: $f_\#(\mathcal{G})$ is then the sheaf associated to the X-scheme T, i.e.

$f_\#(\mathcal{G}) = \mathrm{Hom}(-, f \circ h: T \to X)$.

These adjunctions descend to SH:

$$f_\#: SH(Y) \to SH(X) \dashv f^*: SH(X) \to SH(Y) \dashv f_*: SH(Y) \to SH(X)$$

Recall the six-functor formalism for SH (see for instance "An introduction to six-functor formalisms" by Martin Gallauer) with $f^* \dashv f_*$ (as above), $f_! \dashv f^!$ ("exceptional pullback") and $\otimes \dashv \underline{Hom}$ ("internal Hom") ("compactly supported pushforward"/"exceptional pushforward") and the following properties (see "Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique" by Joseph Ayoub for proofs):

- If $p: Y \to X$ is a proper morphism then we have the canonical isomorphism $\alpha_p: p_! \to p_*$. (1.5.1 in Gallauer's paper)
- If $f: Y \to X$ is an étale morphism then we have the relative purity isomorphism

  $P_f: f_! \to f_\#$

  (1.5.3 Ex. 1.39 in Gallauer's paper + $f_!$ left adjoint to $f^!$ + $f_\#$ left adjoint to f^*)

Note that a morphism of schemes is finite étale (i.e. étale and finite) iff it is étale and proper.

Def: The duality transfer (morphism) tr_f ass. to a finite étale morphism of schemes $f: Y \to X$ is the following composite morphism in $SH(X)$:

$$\mathbb{1}_X \underset{\substack{\text{unit of the}\\ \text{adjunction } (f^*, f_*)\\ (\text{on } \mathbb{1}_X)}}{\longrightarrow} f_* f^* \mathbb{1}_X \underset{\text{ind. by } \alpha_f^{-1}}{\xrightarrow{\sim}} f_! f^* \mathbb{1}_X \underset{\text{ind. by } \mathfrak{P}_f}{\xrightarrow{\sim}} f_\# f^* \mathbb{1}_X \underset{\text{ind. by } f^*\mathbb{1}_X \simeq \mathbb{1}_Y}{\xrightarrow{\sim}} f_\# \mathbb{1}_Y$$

II) Duality transfers and homotopy modules

Def: A Nisnevich sheaf $M: Sm_k^{op} \to Ab$ (abelian groups) is strictly $\mathbb{A}^1$-invariant if:

$\forall X \in Sm_k \ \forall i \in \mathbb{N}_0 \quad H^i_{Nis}(X, M) \xrightarrow{\sim} H^i_{Nis}(X \times_k \mathbb{A}^1_k, M)$.

The contraction of M is the Nisnevich sheaf $M_{-1}: \begin{cases} Sm_k^{op} \to Ab \\ X \in Sm_k \mapsto \mathrm{Ker}\,(M(X \times_k \mathbb{G}_{m,k}) \xrightarrow{ev_1} M(X)) \\ f: Y \to X \mapsto M(f \times_k \mathrm{Id}) \end{cases}$

and if $\varphi: M \to N$ then $\varphi_{-1}: M_{-1} \to N_{-1}$ is $\varphi(- \times_k \mathbb{G}_{m,k})$.

Def: A homotopy module is a family $(M_n, \mu_n)_{n \in \mathbb{Z}}$ of couples such that:

- $\forall n \in \mathbb{Z}$ M_n is a strictly $\mathbb{A}^1$-invariant Nisnevich sheaf;
- $\forall n \in \mathbb{Z}$ $\mu_n: M_n \to (M_{n+1})_{-1}$ is a natural isomorphism.

Def: A morphism of homotopy modules $(\varphi_n)_{n \in \mathbb{Z}}: (M_n, \mu_n) \to (N_n, \rho_n)$ is: $\varphi_n: M_n \to N_n$ s.t. $(\varphi_{n+1})_{-1} \circ \mu_n = \rho_n \circ \varphi_n$ (for $n \in \mathbb{Z}$).

Ex: For all $m \in \mathbb{Z}$ and $E \in SH(k)$, $\left(\pi_m(E)_n := a_{Nis}(U \mapsto [\Sigma^\infty U_+ \wedge S^m, E \wedge \mathbb{G}_m^{\wedge n}]_{SH(k)}), \wedge \mathrm{Id}_{\mathbb{G}_m}\right)_{n \in \mathbb{Z}}$ (associated Nis. sheaf) is a homotopy module.

Def: The homotopy t-structure on $SH(k)$ is the following couple of full subcategories of $SH(k)$:

- $SH(k)_{\geq 0}$: its objects are the $E \in SH(k)$ such that: $\forall m < 0 \ \pi_m(E)_* = 0$;
- $SH(k)_{\leq 0}$: its objects are the $E \in SH(k)$ such that: $\forall m > 0 \ \pi_m(E)_* = 0$.

Prop: It is a t-structure. In particular, its heart $SH(k)^\heartsuit = SH(k)_{\geq 0} \cap SH(k)_{\leq 0}$ is an abelian category.

Prop: $\begin{cases} SH(k)^\heartsuit \to \mathrm{HomMod} \\ E \mapsto \pi_0(E)_* \end{cases}$ (homotopy modules) is an equivalence of categories. ← See Ch. 3 in Fabien Morel's book "$\mathbb{A}^1$ Alg. Top. over a Field"

Not: $\underline{K}^{MW} \in SH(k)^\heartsuit$ is Milnor-Witt K-theory. (already a Nis. sheaf for π_0)

Not: For $E \in SH(k)^\heartsuit$ and $n \in \mathbb{Z}$, we denote by $E_n := \pi_0(E)_n : U \mapsto [\Sigma^\infty U_+, E \wedge \mathbb{G}_m^{\wedge n}]_{SH(k)} \simeq [\mathbb{1}_U, s_U^* E \wedge \mathbb{G}_{m,U}^{\wedge n}]_{SH(U)}$ (structure m. $s_U: U \to \mathrm{Spec}(k)$)

Ex: $\underline{K}_n^{MW}(U) \simeq [\Sigma^\infty U_+, \mathbb{G}_m^{\wedge n}]_{SH(k)}$.

Rk. If $f: Y \to X$ is finite étale in Sm_k then for each $F \in SH(k)^\heartsuit$ and $n \in \mathbb{Z}$:

$$F_n(Y) \simeq [\mathbb{1}_Y, s_Y^* F \wedge \mathbb{G}_{m,Y}^{\wedge n}]_{SH(Y)} \simeq [\mathbb{1}_Y, f^* s_X^* F \wedge f^* \mathbb{G}_{m,X}^{\wedge n}]_{SH(Y)}$$
$$\underset{f_\# \dashv f^*}{\simeq} [f_\# \mathbb{1}_Y, s_X^* F \wedge \mathbb{G}_{m,X}^{\wedge n}]_{SH(X)}.$$

Def: The duality transfer associated to $f: Y \to X$ finite étale in Sm_k, $F \in SH(k)^\heartsuit$ and $n \in \mathbb{Z}$ is:

$$tr_f(F): \begin{cases} F_n(Y) \to F_n(X) \\ \varphi \mapsto \varphi \circ tr_f \end{cases}$$

Not: For $p, q \in \mathbb{Z}$, $a \in \underline{K}_p^{MW}(U)$, $F \in SH(k)^\heartsuit$, $b \in F_q(V)$, ($U, V \in Sm_k$), $a.b := p_U^*(a) \wedge p_V^*(b) \in F_{p+q}(U \times_k V)$ (with $\begin{cases} p_U: U \times_k V \to U \\ p_V: U \times_k V \to V \end{cases}$ the projections).

$\overbrace{p_U^*(\mathbb{1}_U) \wedge p_V^*(\mathbb{1}_V)}$ $\overbrace{p_U^*(\mathbb{G}_{m,U}^{\wedge p}) \wedge p_V^*(s_V^* F \wedge \mathbb{G}_{m,V}^{\wedge q})}$

$[\mathbb{1}_{U \times_k V}, s_{U \times_k V}^* F \wedge \mathbb{G}_{m, U \times_k V}^{\wedge (p+q)}]_{SH(U \times_k V)}$

If $U = V$ then $a \bullet b := \delta_U^*(a.b)$ with $\delta_U: U \to U \times_k U$ the diagonal.

Motives WS24-25
Talk 4 2/2

[Bachmann]: "Motivic and real étale stable homotopy theory"

III Properties of duality transfers

same afterwards (we can omit "in Sm_k" → Y and W smooth; ~ by continuity of F, we may assume that X and V are

Prop. 10 (in [Bachmann]) (Base change): Let $g: V \to X$ be a morphism between (essentially) smooth k-schemes, $f: Y \to X$ be finite étale in Sm_k (or ess. smooth) and $F \in SH(k)^{\heartsuit}$ ↔ hom. module

$$\begin{tikzcd} W := V \times_X Y \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\ V \arrow[r, "g"'] & X \end{tikzcd}$$

We have the base change formula: $g^* \circ \mathrm{tr}_f(F) = \mathrm{tr}_p(F) \circ q^* : F_n(Y) \to F_n(V)$. (for any $n \in \mathbb{Z}$)

Proof: First note that in $SH(V)$: $g^*(\mathrm{tr}_f) = \mathrm{tr}_p$ by naturality of $^* \dashv {}_*$, α, β and can.:

$$\begin{cases} \mathrm{tr}_p : \mathbb{1}_V \xrightarrow{\text{unit adj.}} p_* p^* \mathbb{1}_V \xrightarrow{\alpha_p^{-1}} p_! p^* \mathbb{1}_V \xrightarrow{\beta_p} p_\# p^* \mathbb{1}_V \xrightarrow{\text{can.}} p_\# \mathbb{1}_W \\ g^*(\mathrm{tr}_f) : g^*(\mathbb{1}_X) \xrightarrow{g^*(\text{unit})} g^* f_* f^* \mathbb{1}_X \xrightarrow{g^*(\alpha_f^{-1})} g^* f_! f^* \mathbb{1}_X \xrightarrow{g^*(\beta_f)} g^* f_\# f^* \mathbb{1}_X \xrightarrow{\text{can.}} g^* f_\# \mathbb{1}_Y \end{cases}$$

It is thus enough to show that the following diagram is commutative for any morphism $t: A \to B$ in $SH(X)$, then apply to tr_f:
($s: X \to \mathrm{Spec}(k)$ structure map) (note that $g^*: F_n(Y) \to F_n(W)$ is q^*)

$$\begin{tikzcd} F_n(B) = [B, s^*F \wedge \mathbb{G}_{m,X}^{\wedge n}]_{SH(X)} \arrow[r, "\circ t"] \arrow[d, "g^*"'] & {[A, s^*F \wedge \mathbb{G}_{m,X}^{\wedge n}]_{SH(X)} = F_n(A)} \arrow[d, "g^*"] \\ F_n(g^*B) = [g^*B, g^*s^*F \wedge \mathbb{G}_{m,V}^{\wedge n}]_{SH(V)} \arrow[r, "\circ g^*(t)"'] & {[g^*A, g^*s^*F \wedge \mathbb{G}_{m,V}^{\wedge n}]_{SH(V)} = F_n(g^*A)} \end{tikzcd}$$

This follows from the fact that g^* is a functor ($g^*(r) \circ g^*(t) = g^*(r \circ t)$).

My version of Lemma 11 in [Bachmann] (which is only used by Bachmann to prove Corollary 12):

(New) Lemma 11: Let $f: X' \to X$ and $g: Y' \to Y$ be finite étale in Sm_k and $F \in SH(k)^{\heartsuit}$ ↔ hom. mod.

$\forall p, q \in \mathbb{Z}$ $\forall a \in \underline{K}_p^{MW}(X')$ $\forall b \in F_q(Y')$ $\quad \mathrm{tr}_{f \times_k g}(F)(a.b) = \mathrm{tr}_f(\underline{K}^{MW})(a).\mathrm{tr}_g(F)(b)$.

Proof: Note that $a.b = p_{X'}^*(a) \wedge p_{Y'}^*(b) \in [p_{X'}^*(\mathbb{1}_{X'}) \wedge p_{Y'}^*(\mathbb{1}_{Y'}), \underbrace{p_{X'}^*(\mathbb{G}_{m,X'}^{\wedge p}) \wedge p_{Y'}^*(s_{Y'}^*F \wedge \mathbb{G}_{m,Y'}^{\wedge q})}_{s_{X' \times_k Y'}^* F \wedge \mathbb{G}_{m,X' \times_k Y'}^{\wedge (p+q)} = (f \times_k g)^*(s_{X \times_k Y}^* F \wedge \mathbb{G}_{m,X \times_k Y}^{\wedge (p+q)})}]_{SH(X' \times_k Y')}$

$$\begin{tikzcd} X^{(\prime)} \times_k Y^{(\prime)} \arrow[r, "p_{Y^{(\prime)}}"] \arrow[d, "p_{X^{(\prime)}}"'] \arrow[dr, "s_{X^{(\prime)} \times_k Y^{(\prime)}}"] & Y^{(\prime)} \arrow[d, "s_{Y^{(\prime)}}"] \\ X^{(\prime)} \arrow[r, "s_{X^{(\prime)}}"'] & \mathrm{Spec}(k) \end{tikzcd}$$

$\overset{(f \times_k g)_\# \dashv (f \times_k g)^*}{\simeq} [(f \times_k g)_\#(p_{X'}^*(\mathbb{1}_{X'}) \wedge p_{Y'}^*(\mathbb{1}_{Y'})), \mathbb{G}_{m,X \times_k Y}^{\wedge p} \wedge s_{X \times_k Y}^* F \wedge \mathbb{G}_{m,X \times_k Y}^{\wedge q}]_{SH(X \times_k Y)}$ $\quad \mathbb{G}_{m,X \times_k Y}^{\wedge (p+q)}$

$\simeq [p_X^*(f_\#(\mathbb{1}_{X'})) \wedge p_Y^*(g_\#(\mathbb{1}_{Y'})), \underbrace{\mathbb{G}_{m,X \times Y}^{\wedge p}}_{p_X^*(\mathbb{G}_{m,X}^{\wedge p})} \wedge \underbrace{s_{X \times_k Y}^* F \wedge \mathbb{G}_{m,X \times_k Y}^{\wedge q}}_{p_Y^*(s_Y^* F \wedge \mathbb{G}_{m,Y}^{\wedge q})}]_{SH(X \times_k Y)}$ (see proof below)

$[\mathbb{1}_{X \times_k Y}, \underbrace{(f \times_k g)_\# \mathbb{1}_{X' \times_k Y'}}_{p_X^*(\mathbb{1}_X) \wedge p_Y^*(\mathbb{1}_Y)}]_{SH(X \times_k Y)}$ should follow from naturality prop. of $^* \dashv {}_*$, α, β and can.

so that when composing $\mathrm{tr}_{f \times_k g} = p_X^*(\mathrm{tr}_f) \wedge p_Y^*(\mathrm{tr}_g)$ with this we get, by functoriality of p_X^* and p_Y^*,

$p_X^*(\mathrm{tr}_f(\underline{K}^{MW})(a)) \wedge p_Y^*(\mathrm{tr}_g(F)(b))$, i.e. $\mathrm{tr}_f(\underline{K}^{MW})(a).\mathrm{tr}_g(F)(b)$.

Now let us prove this as above.

Note that we have the natural transformation $p_{X'}^*(\text{unit of } f_\# \dashv f^*) \wedge p_{Y'}^*(\text{unit of } g_\# \dashv g^*)$:

$$p_{X'}^* \wedge p_{Y'}^* \longrightarrow (\underbrace{\underbrace{p_{X'}^* \circ f^*}_{(f\circ p_{X'})^*}}_{(p_X \circ (f \times_k g))^*} \circ f_\#) \wedge (\underbrace{\underbrace{p_{Y'}^* \circ g^*}_{(g \circ p_{Y'})^*}}_{(p_Y \circ (f\times_k g))^*} \circ g_\#) \cong (f \times_k g)^* \circ (p_X^* \wedge p_Y^*) \circ (f_\#, g_\#)$$

which corresponds, via $(f \times_k g)_\# \dashv (f \times_k g)^*$, to the natural transformation $\varphi : (f\times_k g)_\# \circ (p_{X'}^* \wedge p_{Y'}^*) \to (p_X^* \wedge p_Y^*) \circ (f_\#, g_\#)$.

Let us show that this natural transformation is a natural isomorphism.

Since all the functors involved preserve colimits (being left adjoints) and $SH(X')$ (resp. $SH(Y')$) is generated under colimits by the $\Sigma^\infty(\underbrace{X'' \to X'}_{\in Sm_{X'}})_+$ (resp. $\Sigma^\infty(\underbrace{Y'' \to Y'}_{\in Sm_{Y'}})_+$), it is enough to show that:

$\forall X'' \xrightarrow{f'} X' \in Sm_{X'}, \forall Y'' \xrightarrow{g'} Y' \in Sm_{Y'}$ $\varphi(\underbrace{\Sigma^\infty(f')_+}_{\text{denoted } X'' \text{ for short}}, \underbrace{\Sigma^\infty(g')_+}_{\text{denoted } Y'' \text{ for short}}) : (f \times_k g)_\# (\underbrace{p_{X'}^*(X'') \wedge p_{Y'}^*(Y'')}_{(X'' \times_k Y') \times_{X' \times_k Y'} (X' \times_k Y'')}) \xrightarrow{\sim} \underbrace{p_X^*(f_\#(X'')) \wedge p_Y^*(g_\#(Y''))}_{(X'' \times_k Y) \times_{X \times_k Y} (X \times_k Y'')}$

$(f\times_k g)_\#$: seeing as an $X \times_k Y$-scheme

can. iso. between these (which are can. iso. to $X'' \times_k Y''$ seen as an $(X \times_k Y)$-scheme) ✓

Ex. 12 (in [Bachmann]): Let $f : Y \to X$ be finite étale in Sm_k (or essentially smooth) and $F \in SH(k)^\heartsuit \leftrightarrow$ homotopy module.
(Projection formula)

① $\forall p, q \in \mathbb{Z}\ \forall a \in \underline{K}_p^{MW}(Y)\ \forall b \in F_q(X)\ tr_f(F)(a \cdot f^*(b)) = tr_f(\underline{K}^{MW})(a) \cdot b.$

② $\forall p, q \in \mathbb{Z}\ \forall a \in \underline{K}_p^{MW}(X)\ \forall b \in F_q(Y)\ tr_f(F)(f^*(a) \cdot b) = a \cdot tr_f(F)(b).$

Proof:

① $tr_f(F)(a \cdot f^*(b)) = tr_f(F)(\delta_Y^*(\overset{Id_Y^*(a)}{\overset{\|}{a}} . f^*(b))) = tr_f(F)(((Id_Y \times_k f) \circ \delta_Y)^*(a . b))$

$= \delta_X^*(tr_{f \times_k Id_X}(F)(a . b))$ by Prop. 10 on

$$\begin{tikzcd} Y \arrow[r, "(Id_Y \times_k f) \circ \delta_Y"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner"] & Y \times_k X \arrow[d, "f \times_k Id_X"] \\ X \arrow[r, "\delta_X"'] & X \times_k X \end{tikzcd}$$

$= \delta_X^*(tr_f(\underline{K}^{MW})(a) . \underbrace{tr_{Id_X}(b)}_{b})$ by (New) Lemma 11

$= tr_f(\underline{K}^{MW})(a) \cdot b.$

② $tr_f(F)(f^*(a) \cdot b) = tr_f(F)(\delta_Y^*(f^*(a) . \overset{Id_Y^*(b)}{\overset{\|}{b}})) = tr_f(F)(((f \times_k Id_Y) \circ \delta_Y)^*(a . b))$

$= \delta_X^*(\overbrace{tr_{Id_X \times_k f}(F)(a . b)}^{a})$ by Prop. 10 on

$$\begin{tikzcd} Y \arrow[r, "(f \times_k Id_Y) \circ \delta_Y"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner"] & X \times_k Y \arrow[d, "Id_X \times_k f"] \\ X \arrow[r, "\delta_X"'] & X \times_k X \end{tikzcd}$$

$= \delta_X^*(tr_{Id_X}(\underline{K}^{MW})(a) . tr_f(F)(b))$

$= a \cdot tr_f(F)(b).$