TALK 7: PRELIMINARY OBSERVATIONS [Bac18, §8]

References:

[Bac18] Bachmann, "Motivic and real-étale stable homotopy theory"

[Bac21] Bachmann, "Remarks on étale motivic stable homotopy theory"

[CD] Cisinski, Déglise "Triangulated categories of mixed motives"

[Dug] Dugger, "Coherence for invertible objects and multi-graded homotopy rings"

[Hov] Hovey, "Spectra and symmetric spectra in general model categories"

[Hoy] Hoyois, "The six operations in equivariant motivic homotopy theory"

[Joy] Joyal, "The theory of quasi-categories and its applications"

Recall the setting and notations:

- S is a Noetherian scheme of finite Krull dimension.
- $SH(S) := Ho(\mathcal{SH}(S))$ MOTIVIC STABLE HOMOTOPY CATEGORY.

  It is the symmetric monoidal triangulated category obtained taking the homotopy category of

  $$\mathcal{SH}(S) := Spt(Spc_{Mot}(S), \mathbb{P}^1)$$

  the symmetric monoidal stable model category of $\mathbb{P}^1$-spectra with the stable model structure induced by

  $$Spc_{Mot}(S) := L_{\mathbb{A}^1_+} Sh_{Nis}(Sm/S; Spc) \quad \text{CATEGORY OF MOTIVIC SPACES}$$

  the symmetric monoidal model category given by the category of pointed simplicial Nisnevich sheaves over Sm/S with the local model structure, left localized with respect to the set of pointed $\mathbb{A}^1$-homotopies.

  The tensor product in all these categories is denoted by $\wedge$.
- We will be also interested in considering a version of the motivic stable homotopy category where we don't invert $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$ but only S^1, namely

  $$SH^{S^1}(S) := Ho(\mathcal{SH}^{S^1}(S)) \quad \text{where} \quad \mathcal{SH}^{S^1}(S) := Spt(Spc_{Mot}(S), S^1)$$

Also $SH^{S^1}(S)$ is a symmetric monoidal triangulated category and $\mathcal{SH}^{S^1}(S)$ is a symmetric monoidal stable model category. The tensor product is still denoted by $\wedge$.

- We consider all the analogous constructions above obtained replacing Nisnevich topology over Sm/S with real-étale topology:

  $SH(S)^{rét} := Ho(\mathcal{SH}(S)^{rét})$ REAL-ÉTALE STABLE HOMOTOPY CATEGORY

  where $\mathcal{SH}(S)^{rét} := Spt(Spc_{rét}(S), \mathbb{P}^1)$

  $Spc_{rét}(S) := L_{\mathbb{A}^1_+} Sh_{rét}(Sm/S; Spc)$

  $SH^{S^1}(S)^{rét} := Ho(\mathcal{SH}^{S^1}(S)^{rét})$

  where $\mathcal{SH}^{S^1}(S)^{rét} := Spt(Spc_{rét}(S), S^1)$

  Also $SH(S)^{rét}$ and $SH^{S^1}(S)^{rét}$ are symmetric monoidal triangulated categories and $\mathcal{SH}(S)^{rét}$ and $\mathcal{SH}^{S^1}(S)^{rét}$ are symmetric monoidal stable model categories. The tensor product is still denoted by $\wedge$.

- We consider the morphism in $Spc_{mot}(S)$

$$p: S^0 \longrightarrow \mathbb{G}_m$$

where - $S^0 := S \amalg S$, S the Nisnevich sheaf over Sm/S represented by S

It is the constant sheaf over Sm/S associated to the pointed simplicial set S^0.

$$S^0 : U \longmapsto (\{\bullet, *\}, \bullet)$$ constant pointed simplicial set

It is the unit of $\wedge$.

- $\mathbb{G}_m$ is the Nisnevich sheaf over Sm/S represented by $\mathbb{G}_m$ pointed at 1

$$\mathbb{G}_m : U \longmapsto (\mathcal{O}^\times(U), 1)$$ constant pointed simplicial set

p is defined as the morphism of sheaves over Sm/S given by

$$p(U) : S^0(U) \longrightarrow \mathbb{G}_m(U)$$
$$\bullet \longmapsto 1$$ the base point must be preserved!
$$* \longmapsto -1$$

Analogously, replacing Nisnevich topology with real-étale topology, we have the morphism $p: S^0 \longrightarrow \mathbb{G}_m$ in $Spc_{rét}(S)$.

We still denote by $\rho: \mathbb{S}^0 \to \mathbb{G}_m$ its image inside the stable categories $\mathcal{SH}(S)$, $\mathcal{SH}^{\mathbb{S}^1}(S)$, $\mathcal{SH}(S)^{\text{ét}}$ and $\mathcal{SH}^{\mathbb{S}^1}(S)^{\text{ét}}$.

- We are interested in considering monoidal ρ-localizations of the above stable model categories.

Monoidal localization of model categories with respect to a morphism has been discussed in TALK 5 [Bac18, §6].

We recall that, since the above model categories are all left proper combinatorial, hence admit left Bousfield localizations, and since $\mathbb{S}^0$ and $\mathbb{G}_m$ are both cofibrant object, then the monoidal ρ-localizations are left Bousfield localizations with respect to the set of morphisms

$$H := \{ \mathrm{id}_E \wedge \rho : \underset{\overset{\parallel}{E}}{E \wedge \mathbb{S}^0} \to E \wedge \mathbb{G}_m \mid E \in G \text{ set of cofibrant homotopy generators} \}$$

it always exists in a combinatorial model cat.

b/c in $Sh_{Nis}(Sm/S, Spc)$ every object is cofibrant, since cofibrations are monomorphisms, hence also the images via left Quillen functors (which preserve cofibrations)

$$\begin{tikzcd}[row sep=small] & & \mathcal{SH}(S) \\ Sh_{Nis}(Sm/S, Spc) \arrow[r] & Sp_{mot}(S) \arrow[ur, "\Sigma^\infty_{\mathbb{P}^1}"] \arrow[dr, "\Sigma^\infty_{\mathbb{S}^1}"'] & \\ & & \mathcal{SH}^{\mathbb{S}^1}(S) \end{tikzcd}$$

are cofibrant. Analogously for the real-étale version.

So we obtain the symmetric monoidal stable model categories

$$\mathcal{SH}(S)[\rho^{-1}] \quad \mathcal{SH}^{\mathbb{S}^1}(S)[\rho^{-1}] \quad \mathcal{SH}(S)^{\text{rét}}[\rho^{-1}] \quad \mathcal{SH}^{\mathbb{S}^1}(S)^{\text{rét}}[\rho^{-1}]$$

and we denote their homotopy categories, which are symmetric monoidal triangulated,

$$SH(S)[\rho^{-1}] \quad SH^{\mathbb{S}^1}(S)[\rho^{-1}] \quad SH(S)^{\text{rét}}[\rho^{-1}] \quad SH^{\mathbb{S}^1}(S)^{\text{rét}}[\rho^{-1}].$$

The tensor product is still always denoted by $\wedge$.

The goal of this talk is to prove the existence of some canonical equivalences between the above categories. They are summarized in the following diagram:

$$\begin{tikzcd}
SH^{S^1}(S)^{\text{rét}} \arrow[r, "\text{Prop C}"] \arrow[d, dotted] & SH^{S^1}(S)^{\text{rét}}[p^{-1}] \arrow[d] & SH^{S^1}(S)[p^{-1}] \arrow[l, dotted] \arrow[d] \\
SH(S)^{\text{rét}} \arrow[r, "\text{Prop B}"] & SH(S)^{\text{rét}}[p^{-1}] & SH(S)[p^{-1}] \arrow[l, dotted]
\end{tikzcd}$$

Cor A

In the next talk will be proved also the existence of other canonical equivalences of categories (the dotted arrows in the above diagram).

We will also prove that $SH(\cdot)[p^{-1}]$ satisfies the full 6 functors formalism. This will also be useful in the next talk to reduce some proofs to the case of a base field.

⚠ In [Bac18] we take $-p$ instead of p. But since monoidal localization of p or $-p$ is the same, taking p or $-p$ won't make any difference in this talk.

We start with the following general lemma:

Lemma: [Bac18, Lemma 26]

these properties guarantees the existance of left Bousfield localizations

Let $\mathcal{M}$ be a left proper, combinatorial, monoidal model category, $T \in \mathcal{M}$ a cofibrant object, $H \subset \mathcal{M}$ a set of morphisms. Then, there is a canonical isomorphism of model categories

left Bousfield localizations

$$Spt(L_H \mathcal{M}, T) = L_{H'} Spt(\mathcal{M}, T)$$

where $H' := \bigcup_{n \geq 0} \Sigma_T^{\infty-n} H$ and categories of spectra carry the projective stable model structure.

Recall:

$\forall\, n \geq 0 \quad \Sigma_T^{\infty-n} : \mathcal{M} \longrightarrow Spt(\mathcal{M}, T)$

$X \longmapsto \{\overset{0}{0}, \dots, \overset{n-1}{0}, \overset{n}{X}, \underset{X \otimes T}{\overset{n+1}{\Sigma_T X}}, \underset{X \otimes T^2}{\overset{n+2}{\Sigma_T^2 X}}, \dots\}$

is left adjoint to

$$\mathrm{Ev}_n : \mathrm{Spt}(\mathcal{M}, T) \longrightarrow \mathcal{M}$$
$$E = \{E_n\}_{n \geq 0} \longmapsto E_N$$

and they also form a Quillen pair both with respect to projective unstable and stable model structures

Notation:
WE = weak equivalences
(TRIV) FIB = trivial fibrations
(TRIV) COF = trivial cofibrations
llp/rlp = left/right lifting property

Proof:

Step 1: (Reduction to the projective unstable model structure):

Recall from [Hov] that, given $\mathcal{V}$ a left proper combinatorial monoidal model category, $T \in \mathcal{V}$ a cofibrant object, we can endow the category of sequential T-spectra $\mathrm{Spt}(\mathcal{V}, T)$ with:

⚠ In [Hov] they are called global and local instead of unstable and stable. That's also the reason for the notation $-_{gl}$

- the projective unstable model structure on categories of spectra is the one s.t. WE and FIB are level-wise:

  $f : E \to E'$ is a WE (FIB) $\Leftrightarrow$ $f_n : E_n \to E'_n$ is a WE (FIB) $\forall n \geq 0$

  (COF = llp(TRIV FIB))

  We denote it by $\mathrm{Spt}(\mathcal{V}, T)_{gl}$.

Recall:
By definition, $\mathcal{V}$ combinatorial model cat. is cofibrantly generated. A set I of generating cofibrations is s.t.
COF = llp(rlp(I))

- the projective stable model structure on categories of spectra is obtained from the unstable one by left Bousfield localization with respect to a set $\mathcal{S}$ of morphisms which only depends on a set I of generating cofibrations.

  Explicitly, it is the set of morphisms

  $$\mathcal{S} := \{ \Sigma_T^{\infty-(n+1)} \Sigma_T QC \longrightarrow \Sigma_T^{\infty-n} QC \mid \bullet\ C \text{ domain or codomain of morphisms in } I,\ \bullet\ n \geq 0 \}$$

  $Q : \mathcal{V} \to \mathcal{V}$ cofibrant replacement functor

Now consider $\mathcal{V} = \mathcal{M}, L_H\mathcal{M}$.

Since $\mathcal{M}$ is left proper, combinatorial, monoidal model category, then also $L_H\mathcal{M}$ is and they have a same set of generating cofibrations.

Hence we can take the same set of morphisms $\mathcal{S}$ in $\mathrm{Spt}(\mathcal{M}, T)$ and $\mathrm{Spt}(L_H\mathcal{M}, T)$ to obtain the projective stable model structure, that is ① $\mathrm{Spt}(\mathcal{M}, T) = L_{\mathcal{S}}(\mathrm{Spt}(\mathcal{M}, T)_{gl})$

② $\mathrm{Spt}(L_H\mathcal{M}, T) = L_{\mathcal{S}}(\mathrm{Spt}(L_H\mathcal{M}, T)_{gl})$.

So we can reduce to prove the lemma for the projective unstable model structure because then

$$\begin{aligned} Spt(L_H \mathcal{M}, T) &\overset{②}{=} L_{\mathcal{G}}(Spt(L_H \mathcal{M}, T)_{ge}) = L_{\mathcal{G}}(L_{H'} Spt(\mathcal{M}, T)_{ge}) \\ &= L_{H'}(L_{\mathcal{G}} Spt(\mathcal{M}, T)_{ge}) \overset{①}{=} L_{H'} Spt(\mathcal{M}, T) \end{aligned}$$

$L_{\mathcal{G}} L_{H'} = L_{\mathcal{G} \cup H'} = L_{H'} L_{\mathcal{G}}$

<u>Step 2</u>: (Proof of $Spt(L_H \mathcal{M}, T)_{ge} = L_{H'}(Spt(\mathcal{M}, T)_{ge})$):

By [Hov, Prop E.1.10] it is sufficient to prove that:

i) They have the same underlying category.

ii) They have the same COF.

iii) They have the same fibrant objects.

i) is true because left Bousfield localization doesn't changes the underlying category.

ii) Since $L_{H'} Spt(\mathcal{M}, T)_{ge}$ have the same COF of $Spt(\mathcal{M}, T)_{ge}$, because it is a left Bousfield localization, then we have to prove that $Spt(\mathcal{M}, T)_{ge}$ and $Spt(L_H \mathcal{M}, T)_{ge}$ have the same COF.

Since $COF = llp(\text{TRIV FIB})$, then it is equivalent to show that they have the same TRIV FIB. This is true because

TRIV FIB of $Spt(\mathcal{M}, T)_{ge}$ = level-wise TRIV FIB of $\mathcal{M}$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ||$
TRIV FIB of $Spt(L_{H'} \mathcal{M}, T)_{ge}$ = " " " " of $L_H \mathcal{M}$

because $\text{TRIV FIB} = rlp(COF)$ and COF of $L_H \mathcal{M}$ are the same of $\mathcal{M}$, since it is a left Bousfield localization

iii) $E = \{E_n\}_{n \geq 0}$ is fibrant in $L_{H'}(Spt(\mathcal{M}, T)_{ge})$, i.e. is an H'-local object

$\iff$ • E is fibrant in $Spt(\mathcal{M}, T)_{ge}$, i.e. E_n fibrant $\forall n \geq 0$ because fibrations are level-wise

• $\forall f: X \to Y$ in H, $\forall n \geq 0$

Quillen adjunction $\Sigma_T^{\infty - n} \dashv Ev_n$ $\longrightarrow$

$$\begin{array}{ccc} Map^d(\Sigma_T^{\infty-n} Y, E) & \xrightarrow{\sim} & Map^d(\Sigma_T^{\infty+T} X, E) \\ \cong & & \cong \\ Map^d(Y, E_n) & & Map^d(X, E_n) \end{array}$$

is a WE of simplicial sets

$\iff$ $\forall n \geq 0$ E_n are H-local objects, i.e. fibrant in $L_H\mathcal{M}$

$\iff$ E is fibrant in $Spt(L_H\mathcal{M}, T)_{st}$ because fibrations are level-wise.

∎

From this lemma directly follow two of the equivalences of categories we have to prove:

<u>Cor A</u>: [Bac18, Proposition 27]

There are canonical equivalences of categories

$$SH^{S^1}(S)[p^{-1}] \xrightarrow{\sim} SH(S)[p^{-1}]$$

$$SH^{S^1}(S)^{r\acute{e}t}[p^{-1}] \xrightarrow{\sim} SH(S)^{r\acute{e}t}[p^{-1}].$$

Proof:

We prove the first. The real-étale version is the same since we will not use anything specific of Nisnevich topology.

We work at the level of model categories. We have the Quillen adjunctions

$$\begin{array}{ccc} & & \mathcal{SH}(S) \\ & & \downarrow\uparrow \\ \mathcal{SH}^{S^1}(S) & \underset{}{\overset{\Sigma^\infty_{\mathbb{G}_m}}{\rightleftarrows}} & Spt(\mathcal{SH}^{S^1}(S), \mathbb{G}_m) \end{array}$$

where this is a Quillen equivalence.

In general, given $\mathcal{M}$ a monoidal model category, $X, Y \in \mathcal{M}$ objects, we have the Quillen equivalence

$$Spt(\mathcal{M}, X\otimes Y) \rightleftarrows Spt(Spt(\mathcal{M}, X), Y)$$

$$\{E_n\} \longmapsto \{F_{m,n}\} \qquad F_{m,n} := \begin{cases} E_n \otimes X^{m-n} & m \geq n \\ E_m \otimes Y^{n-m} & n \geq m \end{cases}$$

$$\{F_{n,n}\} \longleftarrow \{F_{m,n}\}$$

We want to apply the previous lemma to

$\mathcal{M} = \mathcal{SH}^{S^1}(S)$

$T = \mathbb{G}_m$

$H = \{ id_E \wedge p : E \wedge S^0 \to E \wedge \mathbb{G}_m \mid E \in G$ set of cofibrant homotopy generators of $\mathcal{SH}^{S^1}(S) \}$

<u>Recall</u>: A set of homotopy generators of a <u>stable</u> model category $\mathcal{M}$ is a set of objects G s.t. the smallest localizing Δ-subcategory of $Ho(\mathcal{M})$ containing G is the whole $Ho(\mathcal{M})$

By the lemma we have that

$$\mathrm{Spt}(L_H \mathcal{SH}^{S^1}(S), \mathbb{G}_m) = L_{H'}\, \mathrm{Spt}(\mathcal{SH}^{S^1}(S), \mathbb{G}_m) \quad \circledast$$

where H' is the set of morphisms of $\mathrm{Spt}(\mathcal{SH}^{S^1}(S), \mathbb{G}_m)$

$$H' = \bigcup_{n\geq 0} \Sigma^{\infty-n}_{\mathbb{G}_m} H = \{ \mathrm{id}_{\mathcal{E}'} \wedge p : \mathcal{E}' \wedge \mathbb{S}^0 \to \mathcal{E}' \wedge \mathbb{G}_m \mid \mathcal{E}' \in G' \text{ set of cofibrant homotopy generators of } \mathrm{Spt}(\mathcal{SH}^{S^1}(S), \mathbb{G}_m)\}$$

In general, if a stable monoidal model category $\mathcal{M} \ni T$ has a set G of cofibrant homotopy generators, then $G' := \bigcup_{n\geq 0} \Sigma^{\infty-n}_T G$ is a set of cofibrant homotopy generators of $\mathrm{Spt}(\mathcal{M}, T)$.

RHS of $\circledast$:

Consider also the set of morphisms of $\mathcal{SH}(S)$

$$H'' := \{ \mathrm{id}_{\mathcal{E}''} \wedge p : \mathcal{E}'' \wedge \mathbb{S}^0 \to \mathcal{E}'' \wedge \mathbb{G}_m \mid \mathcal{E}'' \in G'' \text{ set of cofibrant homotopy generators of } \mathcal{SH}(S)\}$$

It can be checked that H'' and H' satisfy sufficient conditions (see [Hov, Prop 2.3]) s.t. we have an induced Quillen equivalence from the one $\mathcal{SH}(S) \rightleftarrows \mathcal{SH}^{S^1}(S), \mathbb{G}_m)$

that's monoidal p-localization is computed!

$$\begin{array}{ccc} L_{H''}\, \mathcal{SH}(S) & \rightleftarrows & L_{H'}\, \mathrm{Spt}(\mathcal{SH}^{S^1}(S), \mathbb{G}_m) \\ \| & & \| \\ \mathcal{SH}(S)[p^{-1}] & & \mathrm{Spt}(\mathcal{SH}^{S^1}(S), \mathbb{G}_m)[p^{-1}] \end{array}$$

LHS of $\circledast$:

Notice that the Quillen adjunction

$$\begin{array}{ccc} L_H \mathcal{SH}^{S^1}(S) & \overset{\Sigma^\infty_{\mathbb{G}_m}}{\rightleftarrows} & \mathrm{Spt}(\underbrace{L_H \mathcal{SH}^{S^1}(S)}_{\mathcal{SH}^{S^1}(S)[p^{-1}]}, \mathbb{G}_m) \\ \| & & \\ \mathcal{SH}^{S^1}(S)[p^{-1}] & & \end{array}$$

is a Quillen equivalence b/c $\mathbb{G}_m \in \mathcal{SH}^{S^1}(S)[p^{-1}]$ is weak equivalent to the unit $\mathbb{S}^0$, hence the endofunctor $\Sigma_{\mathbb{G}_m} : - \wedge \mathbb{G}_m : \mathcal{SH}^{S^1}(S) \circlearrowleft$ is a Quillen equivalence, and then conclude by [Hov, Theorem 5.1].

Putting together, we have the canonical Quillen equivalences

$$\begin{array}{c} \mathcal{SH}(S)[\rho^{-1}] \\ \downarrow\uparrow \\ \mathcal{SH}^{S^1}(S)[\rho^{-1}] \rightleftarrows Spt(\mathcal{SH}^{S^1}(S)[\rho^{-1}], \mathbb{G}_m) = Spt(\mathcal{SH}^{S^1}(S), \mathbb{G}_m)[\rho^{-1}] \end{array}$$

Taking the homotopy categories, we get the wanted canonical equivalences of categories. ■

To prove the other two canonical equivalences, we need the following result.

Prop: Inside proof of [Bac18, Proposition 29]

> The morphism $\rho: \mathbb{S}^0 \to \mathbb{G}_m$ in $Shv_{ét}(Sm/S; Spc)$ admits a retraction, that is, there exists a morphism
> $$\eta: \mathbb{G}_m \to \mathbb{S}^0$$
> s.t. $\mathbb{S}^0 \xrightarrow{\rho} \mathbb{G}_m \xrightarrow{\eta} \mathbb{S}^0$ is the identity.

Proof:

We need the following:

$(r, \geq 0)$ ordered field s.t.
- $\forall x \in r \ \exists \sqrt{x} \in r$
- every polynomial in $r[T]$ of odd deg has a root in r

$\leq$ is determined by the ring structure: $x \leq y \iff \exists z$ s.t. $x + z^2 = y$

$x^2 \geq 0 \ \forall x \in r$

Def: Let R be a ring. An element $a \in R^\times$ is called TOTALLY POSITIVE if $\forall r$ real closed field and $\forall f: R \to r$ ring homomorphism
$$f(a) > 0.$$
Otherwise $a \in R^\times$ is called TOTALLY NEGATIVE.

Notice it can't happen that $f(a) = 0$ because $a \in R^\times$.

Rmk:
- $\forall a \in R^\times$, a^2 is TOTALLY POSITIVE

  Indeed $f(a^2) = f(a)^2 > 0$
- $a \in R^\times$ is TOTALLY POSITIVE $\iff$ $-a \in R^\times$ is TOTALLY NEGATIVE.

  Indeed $f(-a) = -f(a)$ and in an ordered field it holds that $-x \leq 0 \leq x$ or $x \leq 0 \leq -x$

Now we go back to the proof of the proposition. We define the subpresheaves of $\mathbb{G}_m$

$G_+ : U \mapsto G_+(U) := \{\text{TOTALLY POSITIVE elements of } \mathcal{O}^\times(U)\} \subset \mathcal{O}^\times(U) = \mathbb{G}_m(U)$

$G_- : U \mapsto G_+(U) := \{$ " NEGATIVE " " " $\} \subset \mathcal{O}^\times(U) = \mathbb{G}_m(U)$

By universal property of $\amalg$ and applying real-étale sheafification we get the morphism in $Sh_{rét}(Sm/S; Spc)$

$$\varphi : a_{rét} G_+ \amalg a_{rét} G_- \to \mathbb{G}_m$$

We want to prove that this is an isomorphism.

Since real-étale topology has enough points, we can check it on real-étale stalks. So, let (A, m) be an Henselian ring with A/m real closed field. We denote $A \to A/m$, $a \mapsto \bar{a}$.

Then $A_+ := \{a \in A^\times \mid \bar{a} > 0\} \subset A^\times$

$A_- := \{a \in A^\times \mid \bar{a} < 0\} \subset A^\times$

are TOTALLY POSITIVE and NEGATIVE ELEMENTS of $A^\times$

↳ indeed $\forall r$ real closed field and $\forall f : A \to r$ ring hom, the induced field extension $\bar{f} : A/m \to r$ is ordered because: $\bar{a} \leq \bar{b} \rightsquigarrow \exists\, \bar{c} \in A/m$ s.t. $\bar{a} + \bar{c}^2 = \bar{b}$

$\rightsquigarrow \bar{a} + T^2 = \bar{b}$ has a root in A/m

$\overset{A \text{ Henselian}}{\rightsquigarrow} a + T^2 = b$ has a root d in A

$\rightsquigarrow \underset{\bar{f}(\bar{a}) + \bar{f}(\bar{b})^2}{\underset{\|}{f(a+d^2)}} = f(b) = \bar{f}(\bar{b}) \rightsquigarrow \bar{f}(\bar{a}) \leq \bar{f}(\bar{b})$

Hence real étale stalks of φ are

$$A_+ \amalg A_- \to A^\times$$

which is an isomorphism since any element of A is either TOTALLY POSITIVE or TOTALLY NEGATIVE.

Now, we define

$$\eta : \mathbb{G}_m \overset{\varphi}{\cong} a_{rét} G_+ \amalg a_{rét} G_- \to S^0$$

induced by universal property of $\amalg$ from

$a_{rét}(G_+ \to \cdot \to \cdot \amalg *)$ $\qquad \cdot \amalg * = S^0$

$a_{rét}(G_- \to * \to \cdot \amalg *)$

It is s.t. the composition

$$\begin{array}{ccccc} S^0 & \xrightarrow{p} & \mathbb{G}_m & \xrightarrow{q} & S^0 \\ \bullet & \longmapsto & 1 \in G_+ & \longmapsto & \bullet \\ * & \longmapsto & -1 \in G_- & \longmapsto & * \end{array}$$

because, by the above remarks, $1 = 1^2 \in G_+$ and hence $-1 \in G_-$

is the identity. ■

Now we prove the two remaining equivalences.

Prop B: [Bac18, Proposition 29]

> There is a canonical equivalence of categories
> $$SH(S)^{r\acute{e}t} \xrightarrow{\sim} SH(S)^{r\acute{e}t}[p^{-1}].$$

Proof:

The canonical functor is the one induced on homotopy categories by the Quillen adjunction given by left localisation

$$\mathcal{S}\mathcal{S}\mathcal{H}(S)^{r\acute{e}t} \longrightarrow \mathcal{S}\mathcal{S}\mathcal{H}(S)^{r\acute{e}t}(p^{-1})$$

To prove it is an equivalence, we need to prove that the Quillen adjunction is in fact a Quillen equivalence. It suffices to prove that p is already a WE in $\mathcal{S}\mathcal{S}\mathcal{H}(S)^{r\acute{e}t}$.

This follows from the following:

⚠ This strengthens [Bac18, Lemma 30], which is sufficient for Prop B, but not for the next Prop C!

Prop: [Bac21, Proposition 2.1]

> Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal model category, $I \in \mathcal{C}$ invertible object with morphisms
> $$I \xrightarrow{e} X \xrightarrow{f} I$$
> for some object $X \in \mathcal{C}$, s.t. $fe = id$.
> If X is symmetric, then e and f are homotopy equivalences one the inverse of the other.

(See below ✱ for definitions and proof)

because is the unit of $\wedge$

Indeed, we apply it to $\mathcal{C} = \mathcal{SH}(S)^{\text{ét}}$, $I = \mathbb{S}^0$ which is invertible, $X = \mathbb{G}_m$ which is symmetric because is invertible [Dug, Lemma 4.17], $e = p$, $f = \eta$. We conclude that p and η are inverse homotopy equivalences, hence p is a we. ■

Prop C: [Boc21, Thm 4.2]

There is a canonical equivalence of categories
$$SH^{\mathbb{S}^1}(S)^{\text{ét}} \xrightarrow{\sim} SH^{\mathbb{S}^1}(S)^{\text{ét}}[p^{-1}].$$

Proof:

The proof is exactly the same of the one above, but we apply the proposition to $\mathcal{C} = \mathcal{SH}^{\mathbb{S}^1}(S)^{\text{ét}}$, $I = \mathbb{S}^0$ which is invertible, $X = \mathbb{G}_m = \mathbb{S}^{-1} \wedge \mathbb{S}^1 \wedge \mathbb{G}_m = \mathbb{S}^{-1} \wedge \underbrace{\mathbb{S}^{2,1}}_{}$ which is symmetric because

$\mathbb{S}^{p+q,q} := (\mathbb{S}^1)^{\wedge p} \wedge (\mathbb{G}_m)^{\wedge q}$ motivic spheres

- $\mathbb{S}^{-1}$ is invertible $\rightsquigarrow$ $\mathbb{S}^{-1}$ is 3-symmetric by [Dug, Lemma 4.17]
- $\mathbb{S}^{2,1}$ 3-symmetric because motivic spheres are 3-symmetric by [Hoy, Lemma 6.3]
- $\wedge$ of 3-symmetric objects is 3-symmetric, hence symmetric,

$e = p$, $f = \eta$. ■

Finally, we prove that $SH(\cdot)[p^{-1}]$ satisfies the full 6 functors formalism and some properties.
Recall that in [CD] is proved that $SH(\cdot)$ satisfies the full 6 functors formalism and satisfies also the compact generation and continuity properties. The ones for $SH(\cdot)[p^{-1}]$ are deduced from $SH(\cdot)$.

Prop: [Bac18, Proposition 28]

There exists a pseudofunctor

$$SH(\cdot)[\rho^{-1}] : \{\text{Noetherian schemes of finite Krull dim.}\}^{op} \longrightarrow \{\text{closed symmetric monoidal triangulated categories}\}$$

$$\begin{array}{ccc} S & \longmapsto & SH(S)[\rho_S^{-1}] \\ f\uparrow & & \downarrow f^* \\ T & \longmapsto & SH(T)[\rho_T^{-1}] \end{array}$$

satisfying the full 6 functors formalism, compact generation and continuity properties

- $SH(\cdot)[\rho^{-1}]$ satisfies the full 6 functors formalism:

By [CD, Theorem 2.4.50] it suffices to prove that it is a pre-motivic category satisfying homotopy, stability and localization properties.

① $SH(\cdot)[\rho^{-1}]$ is a pseudo-functor:

Given $f: T \to S$ a morphism of Noetherian schemes of finite Krull dimension, denote $\rho_S : \mathbb{S}^0 \to \mathbb{G}_m \in Spc_{mot}(S)$

$\rho_T : \mathbb{S}^0 \to \mathbb{G}_m \in Spc_{mot}(T)$.

Notice that the pullback functor on motivic spaces

$$f^* : Spc_{mot}(S) \to Spc_{mot}(T)$$

is s.t. $f^*\rho_S = \rho_T$.

Hence also the functor induced on stabilizations

$$f^* : \mathcal{SH}(S) \longrightarrow \mathcal{SH}(T)$$

is s.t. $f^*\rho_S = \rho_T$.

By universal property of left localization, we get a functor

$$f^* : \mathcal{SH}(S)[\rho_S^{-1}] \longrightarrow \mathcal{SH}(T)[\rho_T^{-1}]$$

and taking homotopy categories we get

$$f^* : SH(S)[\rho_S^{-1}] \longrightarrow SH(T)[\rho_T^{-1}].$$

Functorial properties follow from the ones of f^* on motivic spaces.

② f^* has a triangulated right adjoint f_*:

The Quillen adjunction
$$f^*: Spc_{mot}(S) \rightleftarrows Spc_{mot}(T): f_*$$
induces the Quillen adjunction
$$f^*: \mathcal{SSH}(S)[p_S^{-1}] \rightleftarrows \mathcal{SSH}(T)[p_T^{-1}]: f_* \qquad \text{because } f^*p_S = p_T.$$
Taking homotopy categories, we get the triangulated adjunction
$$f^*: SH(S)[p_S^{-1}] \rightleftarrows SH(T)[p_T^{-1}]: f_*$$

③ If f is smooth, f^* has a left adjoint $f_\#$

Recall the explicit description of the p-localization functor [Bac18, Lemma 15]
$$SH(T) \longrightarrow SH(T)[p^{-1}]$$
$$\mathcal{E} \longmapsto \operatorname{hocolim}_{n\geq 0} \mathcal{E}\wedge \mathbb{G}_m^{\wedge n} \qquad \mathcal{E}\cong\mathcal{E}\wedge S^0 \xrightarrow{id\wedge p} \mathcal{E}\wedge\mathbb{G}_m \cong \mathcal{E}\wedge\mathbb{G}_m\wedge S^0 \xrightarrow{id\wedge p^2} \mathcal{E}\wedge\mathbb{G}_m^{\wedge 2}\dots$$

Notice that since $f_\#: SH(T)\to SH(S)$ is left adjoint to f^*, then it commutes with homotopy colimits, hence $f_\#(\mathcal{E}[p^{-1}]) = (f_\#\mathcal{E})[p^{-1}]$, so it induces a functor $f_\#: SH(T)[p^{-1}] \longrightarrow SH(S)[p^{-1}]$ which is left adjoint to f^* by the adjunction $f_\# \dashv f^*$ for SH: $\forall \mathcal{E}\in SH(T)$, $\mathcal{F}\in SH(S)$
$$\mathrm{Hom}_{SH(S)[p^{-1}]}(f_\#(\mathcal{E}[p^{-1}]), \mathcal{F}[p^{-1}]) = \mathrm{Hom}_{SH(S)[p^{-1}]}((f_\#\mathcal{E})[p^{-1}], \mathcal{F}[p^{-1}]) = \mathrm{Hom}_{SH(S)}(f_\#\mathcal{E}, \mathcal{F}) =$$
$$= \mathrm{Hom}_{SH(T)}(\mathcal{E}, f^*\mathcal{F}) = \mathrm{Hom}_{SH(T)[p^{-1}]}(\mathcal{E}[p^{-1}], f^*\mathcal{F}[p^{-1}]) = \mathrm{Hom}_{SH(T)[p^{-1}]}(\mathcal{E}[p^{-1}], f^*(\mathcal{F}[p^{-1}]))$$

④ Smooth base change holds:

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ g\downarrow & & \downarrow f \\ T & \xrightarrow[\text{smooth}]{p} & S \end{array} \quad \Rightarrow \quad q_\# g^* \longrightarrow f^* p_\# \text{ is an isomorphism.}$$

We deduce it from the corresponding property for $SH(\cdot)$ using the explicit description of the monoidal p-localization functor in [Bac18, Lemma 15]
$$SH(S) \longrightarrow SH(S)[p^{-1}]$$
$$\mathcal{E} \longmapsto \mathcal{E}[p^{-1}] := \operatorname{hocolim}_{n\geq 0} \mathcal{E}\wedge\mathbb{G}_m^{\wedge n}$$
$$\mathcal{E}\cong\mathcal{E}\wedge S^0 \xrightarrow{id\wedge p} \mathcal{E}\wedge\mathbb{G}_m \cong \mathcal{E}\wedge\mathbb{G}_m\wedge S^0 \xrightarrow{id\wedge p^2} \mathcal{E}\wedge\mathbb{G}_m^{\wedge 2} \dots$$

Since $q_\#, g^*, f^*, p_\#$ are all left adjoints, then they commute with homotopy colimit, hence
$$q_\# g^*(\mathcal{E}[p^{-1}]) \cong (q_\# g^*\mathcal{E})[p^{-1}] \xrightarrow{\cong} (f^* p_\#\mathcal{E})[p^{-1}] \cong f^* p_\#(\mathcal{E}[p^{-1}])$$
smooth base change for $SH(\cdot)$

⑤ Smooth projection formula:

$f: T \to S$ smooth $\implies \forall E \in SH(S)[p_S^{-1}]$ $\forall F \in SH(T)[p_T^{-1}]$

$f_\#(F \wedge f^*E) \cong f_\# F \wedge E$

Reasoning as in ④, since $f_\#$, f^*, $F \wedge \_$, $\_ \wedge E$ are all left adjoints, we can deduce it from the corresponding property for $SH(.)$.

⑥ Homotopy property: $\forall$ $p: \mathbb{A}^1_X \to X$

$p_\# \mathbb{S}^0_{\mathbb{A}^1_X}[p^{-1}] \to \mathbb{S}^0_X[p^{-1}]$ is an isomorphism

Immediately follows from the corresponding property for $SH(.)$ since it is the image via the p-localization functor of the isomorphism $p_\# \mathbb{S}^0_{\mathbb{A}^1_X} \xrightarrow{\simeq} \mathbb{S}^0_X$ in $SH(X)$.

⑦ Stability property: $\forall$ $q: \mathbb{P}^1_X \to X$

$\mathrm{cone}(q_\# \mathbb{S}^0_{\mathbb{P}^1_X}[p^{-1}] \to \mathbb{S}^0_X[p^{-1}])$ is invertible

Immediately follows from the corresponding property for $SH(.)$ since it is the image via the p-localization functor of $\mathrm{cone}(q_\# \mathbb{S}^0_{\mathbb{P}^1_X} \to \mathbb{S}^0_X)$, which is invertible, hence also its image is.

⑧ Localization property: $\forall$ $U \overset{j}{\hookrightarrow} X \overset{i}{\hookleftarrow} Z := X \setminus U$ $\forall E[p^{-1}] \in SH(X)[p^{-1}]$

$j_\# j^*(E[p^{-1}]) \to E[p^{-1}] \to i_* i^*(E[p^{-1}]) \to$

is a distinguished Δ

Notice that i_* commutes with filtered homotopy colimits because i^* preserves compact objects = $E \in SH(X)$ s.t. $\mathrm{Hom}_{SH(X)}(E, \_)$ preserves filtered homotopy colimits

([CD])

$\forall E \in SH(Z)$ compact object $\forall \{E_n\}$ filtered system

$\mathrm{Hom}_{SH(Z)}(E, i_* \mathrm{hocolim}_n E_n) = \mathrm{Hom}_{SH(X)}(i^*E, \mathrm{hocolim}_n E_n) = \mathrm{hocolim}_n \mathrm{Hom}_{SH(X)}(i^*E, E_n) =$

$= \mathrm{hocolim}_n \mathrm{Hom}_{SH(X)}(E, i_* E_n) = \mathrm{Hom}_{SH}(E, \mathrm{hocolim}_n i_* E_n)$

and this is sufficient to conclude that $i_* \mathrm{hocolim}_n E_n \cong \mathrm{hocolim}_n i_* E_n$ since $SH(Z)$ has a set of compact homotopy generators. ([CD])

Hence $\forall \mathcal{E} \in SH(U)$

$$i_*(\mathcal{E}[p^{-1}]) = i_*(\underset{n}{\mathrm{hocolim}}\, \mathcal{E} \wedge \mathbb{G}_m^n) \equiv \underset{n}{\mathrm{hocolim}}\, i_*(\mathcal{E} \wedge \mathbb{G}_m^n) = \underset{n}{\mathrm{hocolim}}\, i_*(\mathcal{E}) \wedge \mathbb{G}_m^n = i_*(\mathcal{E})[p^{-1}]$$

i.e. i_* commutes with p-localization functor.

Then localization property follows from the corresponding property for $SH(\cdot)$ since the triangle is the image of the distinguished triangle $j_\# j^* \mathcal{E} \to \mathcal{E} \to i_* i^* \mathcal{E} \to$ in $SH(X)$.

- $SH(S)[p^{-1}]$ is compactly generated, i.e. it has a set of compact homotopy generators

Since $SH(X)$ is compactly generated ([CD]), then also $SH(S)[p^{-1}]$ by [Bac 18, Lemma 15].

- continuity property: $\forall \{S_\alpha\}$ inverse system of schemes with affine morphisms s.t. the limit exists. Fix α_0 and let $f_\alpha : X_\alpha \to X_{\alpha_0}$ $\forall \alpha > \alpha_0$ $\forall \mathcal{E}[p^{-1}] \in SH(X_{\alpha_0})[p^{-1}]$ $\forall n \in \mathbb{Z}$
$p_\alpha : X \to X_\alpha$

$$\underset{\alpha > \alpha_0}{\mathrm{colim}}\, \mathrm{Hom}_{SH(S_\alpha)[p^{-1}]}(\mathbb{S}^0(i)[p^{-1}], f_\alpha^*(\mathcal{E}[p^{-1}])) \longrightarrow \mathrm{Hom}_{SH(S)[p^{-1}]}(\mathbb{S}^0(i)[p^{-1}], p_{\alpha_0}^*(\mathcal{E}[p^{-1}]))$$

is an isomorphism.

This immediately follows from the corresponding property for $SH(\cdot)$, using the fact that

$f^*(\mathcal{E}[p^{-1}]) = f^*\mathcal{E}[p^{-1}]$ $\longrightarrow$ this is the explicit description of f^* defined in 1!

and that the p-localization embeds fully faithfully in $SH(X)$. ■

*

Def: [Dug, §4.16]

- Given n variables $x_1, \dots, x_n$, a TENSOR WORD w is a pure tensor where each variable appears exactly once.

  e.g. $n=3$ $\quad w = (x_1 \otimes x_2) \otimes x_3$

- Given $(\mathcal{C}, \otimes, \mathbb{1})$ a symmetric monoidal model category, a TENSOR WORD w in n variables induces a functor

$$F_w : \mathcal{C}^n \to \mathcal{C}$$
$$(X_1, \dots, X_n) \mapsto \text{object obtained evaluating variables } x_i \text{ into } X_i$$

  e.g. $F_w : \mathcal{C}^3 \to \mathcal{C}$
$$(X_1, X_2, X_3) \mapsto (X_1 \otimes X_2) \otimes X_3$$

- For any $\sigma \in \Sigma_n$ permutation of n objects, we denote by σw the TENSOR WORD obtained from w by replacing x_i with $x_{\sigma(i)}$. It can be proved that there are natural isomorphisms

$$\eta_{\sigma w} : F_w \xrightarrow{\simeq} F_{\sigma w}$$

  given by a combination of the associativity (a) and commutativity (c) structural morphisms of $\mathcal{C}$.

  e.g. $\sigma = (123)$ $\quad (X_1 \otimes X_2) \otimes X_3 \xrightarrow[\simeq]{c} X_3 \otimes (X_1 \otimes X_2) \xrightarrow[\simeq]{a} (X_3 \otimes X_1) \otimes X_2$

- Given $X \in \mathcal{C}$ an object, we denote $F_w(X) := F_w(X, \dots, X)$.

  Notice that $F_w(X) = F_{\sigma w}(X)$ as objects, by $\eta_{\sigma w} : F_w(X) \to F_{\sigma w}(X)$ is not the identity!

  We say that X is n-SYMMETRIC if $\forall w$ TENSOR WORD of n variables

$$\eta^X_{\sigma w} : F_w(X) \to F_{\sigma w}(X) \text{ is homotopy equivalent to the identity}$$

  when $\sigma \in \Sigma_n$ is the n-cycle. $(1 \dots n)$

  We say X is SYMMETRIC if it is n-symmetric for some $n \geq 2$.

Rmk: (1) X, Y n-symmetric $\Rightarrow$ $X \otimes Y$ n-symmetric

(2) Let $\sigma, \tau \in \Sigma_n$ any two permutations.

If $\eta^X_{\sigma w} : F_w(X) \to F_{\sigma w}(X)$ and $\eta^X_{\tau w} : F_w(X) \to F_{\tau w}(X)$ are homotopy equivalent to the identity, then also $\eta^X_{\sigma\tau w} : F_w(X) \to F_{\sigma\tau w}(X)$ is.

Proof:

(1) Let w be a tensor word in n variables, σ the n-cyclic permutation.

Notice we can obtain $\eta^{X\otimes Y}_{\sigma w}$ as the composition:

$$F_w(X \otimes Y) \xrightarrow{\alpha} F_w(X) \otimes F_w(Y) \xrightarrow{\eta^X_{\sigma w} \otimes \eta^Y_{\sigma w}} F_{\sigma w}(X) \otimes F_{\sigma w}(Y) \xrightarrow{\alpha^{-1}} F_w(X \otimes Y)$$

where α is a combination of isomorphisms a and c that take all X's on the left and Y's on the right.

Since X and Y are n-symmetric, then $\eta^X_{\sigma w} \sim id$ & $\eta^Y_{\sigma w} \sim id$

hence $\eta^X_{\sigma w} \otimes \eta^Y_{\sigma w} \sim id$.

But then

$$\eta^{X\otimes Y}_{\sigma w} = \alpha^{-1} (\eta^X_{\sigma w} \otimes \eta^Y_{\sigma w}) \alpha \sim \alpha^{-1}\, id\, \alpha = id$$

that is, $X \otimes Y$ is n-symmetric.

(2) $\eta^X_{\sigma\tau w} = \eta^X_{\tau w} \circ \eta^X_{\sigma w} \sim id \circ id = id$ ∎

Proof (of the Prop):

We have to prove that $e_f \sim id$.

We do the following reductions:

- Notice that any $(2n-1)$ cycle is the product of 2 n-cycles. $(1 \ldots 2n-1) = (1 \ldots n)(1\; n+1 \ldots 2n-1)$
  So, by remark (2), we can assume that X is n-symmetric with n odd. because (2) tells that X n-symmetric $\Rightarrow$ X $(2n-1)$-symmetric.
- Since n is odd, hence $(1 \ldots n)$ is an even permutation, then I invertible is n-symmetric by [Dug, lemma 4.17].

By rmk (1), then $X \otimes I^{-1}$ is n-symmetric.