Gorenstein rings

Def Let A be a noetherian ring (not nec. commutative) of finite Krull dimension.
Say A is Gorenstein (strongly) if A has finite injective dimension as an A-module.
Equivalently: $\mathrm{Ext}^{n+1}_A(-,A)=0$ for some $n\geq 0$
($n = \mathrm{injdim}_A A$)

Recall (e.g. Matsumura "Commutative ring theory"
Thm 18.1) The following equivalent conditions for a commutative local ring $(A,\mathfrak{m},k)$

0) A is Gorenstein of dimension n
1) $\mathrm{injdim}_A A<\infty$
1') $\mathrm{injdim}_A A=n$
2) $\mathrm{Ext}^i_A(k,A)=0$ for $i\neq n$ & $\cong k$ for $i=n$
3) $\mathrm{Ext}^i_A(k,A)=0$ for some $i>n$
4) $\mathrm{Ext}^i_A(k,A)=0$ for $i<n$ and $\mathrm{Ext}^n_A(k,A)\cong k$

In addition a Gorenstein ring is Cohen-Macaulay
($\Leftrightarrow$ regular seq $x_1,\dots,x_n\in\mathfrak{m}$, $n=\dim A$), $\mathrm{depth}_A\,\mathfrak{m}=n$

Lemma 1 A a comm ring N, M A-modules, $x \in A$, $B = A/(x)$

suppose x is A-regular & M-regular and $xN = 0$ ②

Then $\operatorname{Ext}^{i+1}_A(N, M) \cong \operatorname{Ext}^i_B(N, M/xM)$

Pf $0 \to A \xrightarrow{\cdot x} A \to B \to 0$

Define the functor $T^n : B\text{-Mod} \to Ab$

$$T^n(N) = \operatorname{Ext}^{n+1}_A(N, M)$$

- $0 \to M \xrightarrow{\times x} M \to \bar{M} \to 0 \Rightarrow T^0(N) = \operatorname{Hom}_A(N, \bar{M})$

$\bar{M} = M/(x)M$

$$= \operatorname{Hom}_B(N, \bar{M})$$

- $0 \to A \xrightarrow{\times x} A \to B \to 0 \Rightarrow T^n(B) = \operatorname{Ext}^{n+1}_A(B, M) = 0$ for $n > 0$

$\Rightarrow T^n(P) = 0$ for P finite B-Mod

$\Rightarrow \quad T^n(N) = \operatorname{Ext}^n_B(N, \bar{M})$

Ex 1 (A, m, k) a regular local ring of dim n

Then $\operatorname{pd}_A k = n \Rightarrow \operatorname{Ext}^i_A(k, A) = 0$ $i > n \Rightarrow$ (Thm 1)

A is Gorenstein & $\operatorname{inj\,dim}_A A = n$

2. (A, m, k) local Gorenstein dim n (3)

$B = A/(x_1, \dots, x_r)$ $x_1, \dots, x_r$ a reg sequence in m

Then B is Gorenstein of dim $n-r$.

Pf By induction need only consider $r=1$ $B = A/(x)$

Then by lemma

$$\mathrm{Ext}^i_B(k, B) = \mathrm{Ext}^{i+1}_A(k, A) = \begin{cases} 0 & i \neq n-1 \\ k & i = n-1 \end{cases}$$

$\overset{\Rightarrow}{(\text{Thm } 8.1)}$ B is Gorenstein of dim $n-1$.

3. (A, m, k) local M f.gen A-mod then

$\mathrm{depth}_A M = r \Leftrightarrow \mathrm{Ext}^i_A(k, M) = 0$ $i < r$

and $\mathrm{Ext}^r_A(k, M) \neq 0$

(Cor to Thm 15.B Matsumura "Commutative algebra")

(Recall $\mathrm{depth}_A M =$ max length of M-reg seq in m)

Maximal Cohen-Macaulay modules

Def A Gorenstein M f.gen A-mod is MCM if

$\mathrm{Ext}^n_A(M, A) = 0$ for $\forall n > 0$.

Lemma 2 Let A be Gorenstein

i) A finitely generated projective A module is MCM

ii) If $0 \to M' \xrightarrow{f} M \to M'' \to 0$ is exact then

- $M, M'' \in$ MCM $\Rightarrow M' \in$ MCM
- $M', M'' \in$ MCM $\Rightarrow M \in$ MCM
- $M', M \in$ MCM. Then $M'' \in$ MCM $\Leftrightarrow$ $\mathrm{Hom}(f, A)$ is injective

iii) $M \in$ MCM $\Leftrightarrow$ $M^* := \underline{\mathrm{Hom}}_A(M, A)$ is in MCM

& $M \to M^{**}$ is an iso

iv) Let $N \in \mathrm{Mod}_A^{fg}$. Then N admits a resolution

$n = \dim A$

$$0 \to M \to P_n \to \cdots \to P_0 \to N \to 0$$

with the P_i projective f.gen & $M \in$ MCM

v) (A, m) local Gorenstein $n = \dim A$, N f.gen A-mod

$$\to F_k \xrightarrow{d_k} \cdots \xrightarrow{d_0} F_0 \xrightarrow{\varepsilon} N \to 0$$

exact with the F_i free finite rank and $F_{i+1} \xrightarrow{d_i} F_i$ with $d_i(F_{i+1}) \subset m F_i$ $\forall i$. Therefore $k \geq n$

$\mathrm{im}\, d_{k+1} =: M_{k+1} \subset F_k$ is MCM w.o. free summands

Pf (i) & (ii) are obvious

(iii) take projective res of M

$$F_{n+1} \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0 \qquad n = \dim A$$

& apply $\mathrm{Hom}(-, A)$ ...

$$0 \to M^* \to F_0^* \to \cdots \to F_n^* \to F_{n+1}^* \to \quad \text{is exact}$$

$$\Rightarrow 0 \to M^* \to F_0^* \to \cdots \to F_{n-1}^* \to K_n^* \to 0 \quad \text{for all } n$$

$$\Rightarrow \mathrm{Ext}_A^i(M^*, A) \cong \mathrm{Ext}_A^{i+n}(K_n^*, A) \quad \text{for } i > 0$$

$$= 0 \quad \text{for } i+n > \text{inj dim } A$$

$\Rightarrow M^*$ is Gorenstein

Dually again ...

$$\cdots \to F_1 \to F_0 \to M^{**} \to 0$$

$\Rightarrow M \to M^{**}$ is iso.

(iv) is similar $\quad 0 \to M_{k+1} \to F_k \to F_{k-1} \to \cdots F_0 \to N \to 0$

$\leadsto \mathrm{Ext}^i(M_{k+1}, A) \cong \mathrm{Ext}^{i+k}(N, A) = 0$ for $i+k >$ inj dim A, $i > 0$

so M_{k+1} is MCM if $k \geq$ inj dim $A = n$, $i > 0$

For (v) by (iv) $M_{(k+1)}$ is MCM if $k \geq n$. (6)

Let $(x_1, \dots, x_n)$ be a maximal A-reg sequence and let

$B = A/(x_1, \dots, x_n)$. Then B is Gorenstein of depth 0

$\Rightarrow m_B$ consists of 0-divisors, so $\operatorname{depth}_B m_B = 0 \Rightarrow \operatorname{Hom}(k, m_B)$ (Exc) $\neq 0$

$\Rightarrow \exists\ a \in m_B$ $a\, m_B = 0$

Moreover $\operatorname{pd}_A B = n$ (Koszul complex) so $\operatorname{Tor}_l^A(B, N) = 0$ for $l \geq n+1$

Also $\operatorname{Tor}_l^A(B, N) \simeq \operatorname{Tor}_{l-k}^A(B, M_k)$ for $l-k \geq 1 \Rightarrow \operatorname{Tor}_1^A(B, M_k) = 0$

for $k \geq n$. Then $0 \to M_{(k+1)} \to F_k \to M_k \to 0 \Rightarrow$

$$\begin{tikzcd} 0 \arrow[r] & B \otimes_A M_{k+1} \arrow[r] \arrow[dr] & B \otimes_A F_k \\ B \otimes_A F_{k+1} \arrow[u, "f"] \arrow[r] & & m(B \otimes F_k) = m_B \otimes_A F_k \arrow[u, hook] \end{tikzcd}$$

$\Rightarrow a(B \otimes_A M_{k+1}) = 0$, impossible if $M_{(k+1)}$ contains A as summand. $\square$

Lemma (A, m, k) local noetherian commutative $\dim A = n$

M is MCM $\Leftrightarrow \operatorname{depth} M = n$

pf ($\Leftarrow$) $\operatorname{depth} M = n \Leftrightarrow \operatorname{Ext}_A^i(k, M) = 0$ for $i < n$ & $\operatorname{Ext}^n(k, M) \neq 0$

$n = 0$ A Gorenstein $\Rightarrow \operatorname{inj}\dim A = 0$ ✓

$n > 0$: claim: $\exists x \in A$ that is M-regular s.t. A/x is Gorenstein and A-regular

By Ex 3 $\quad \mathrm{Ext}^i_A(k, M) = 0$ for $i < n$

A is CM $\Rightarrow \exists$ A-reg seq $x_1, \ldots, x_n \in \mathfrak{m} \Rightarrow \mathrm{Ext}^i(k, A) = 0$ for $i < n$

$\Rightarrow$ depth $A \oplus M \geq n \Rightarrow \exists\ x \in \mathfrak{m}$, $A \oplus M$ regular

Also x n.z.div on $M \Rightarrow \mathrm{Ext}^i_{A/(x)}(M/xM, A/x) = \mathrm{Ext}^i_A(M, A/(x))$

and

$\mathrm{Ext}^i_{A/(x)}(k, M/x) = \mathrm{Ext}^{i+1}_A(k, M)$ (Lemma 4) $\quad$ ($P_\cdot \to M$ res $\Rightarrow P_\cdot/xP_\cdot \to M/x$ res and $\mathrm{Hom}_A(P_n, A/x) = \mathrm{Hom}_{A/x}(P_n/x, A/x)$)

$\Rightarrow$ depth$_{A/x}$ $M/x = n-1 \Rightarrow M/x$ is MCM$_{A/x}$ by induction on dim A

Then $\quad \mathrm{Ext}^i_{A/x}(M/x, A/x) = 0 \quad$ for $i > 0$

$$\begin{array}{c} \mathrm{Ext}^i_{A/x}(M/x, A/x) \\ \| \\ \mathrm{Ext}^i_A(M, A/x) \end{array}$$

$$\begin{array}{c} \mathrm{Ext}^i_A(M, A) \\ \uparrow x \\ \mathrm{Ext}^i(M, A) \end{array}$$

$\Rightarrow x : \mathrm{Ext}^i_A(M, A) \to \mathrm{Ext}^i(M, A)$ surjective for $i > 0$

$\Rightarrow$ (Nakayama) $\mathrm{Ext}^i_A(M, A) = 0$ for $i > 0 \Rightarrow M$ is MCM$_A$

<u>Conversely</u> $M \in MCM_A$, A Gorenstein $\Rightarrow A$ CM $\Rightarrow \exists\, x \in m$ non-zerodivisor on A

$M = M^{**} = (M^{*})^{*} \Rightarrow M$ is a submodule of a free A-module

$\Rightarrow x$ is a non-zerodivisor on M and by the above

M/x is $MCM_{A/x}$ so has depth $n-1$ by induction

$\Rightarrow M$ has depth n.

<u>Note</u> M is CM by def if depth $M = \dim M$

<u>Ex</u> (A, m) regular $M \in MCM_A \Leftrightarrow M$ is a projective A-module

Pf Auslander Buchsbaum $\operatorname{pd}_A M + \operatorname{depth}_A M = \dim A$

Hypersurface singularities and MCM

<u>Prop</u> $(A, \mathfrak{m})$ regular $B = A/(x)$ $x \in \mathfrak{m}$
$M \in MCM_B$. Then $\exists$ free A-modules F_0, F_1
of the same finite rank
and injective A-linear maps $F_1 \xrightarrow{\varphi} F_0$, $F_0 \xrightarrow{\psi} F_1$ such that

1) $M \cong \operatorname{cok} \varphi$ as A-module

2) $M \cong \operatorname{coh} \bar{\varphi}$ $\quad \bar{\varphi} : \underset{\bar{F}_1}{\underbrace{F_1 / x F_1}} \to \underset{\bar{F}_0}{\underbrace{F_0 / x F_0}}$ as B-modules

3) $\varphi \circ \psi = x \cdot \mathrm{id}_{F_0}$ $\quad \psi \circ \varphi = x \cdot \mathrm{id}_{F_1}$

4) The complex

$$\cdots \to \bar{F}_1 \xrightarrow{\bar{\varphi}} \bar{F}_0 \xrightarrow{\bar{\psi}} \bar{F}_1 \xrightarrow{\bar{\varphi}} \bar{F}_0 \to M \to 0$$

is exact

5) $\bar{\psi}(\bar{F}_0) \subset \mathfrak{m} \bar{F}_1$ & $\bar{\varphi}(\bar{F}_1) \subset \mathfrak{m} \bar{F}_0 \Rightarrow$

M contains no free summands

Pf $M \in MCM_B \Rightarrow \operatorname{depth}_B M = \dim B$

$\Rightarrow \operatorname{depth}_A M = \dim A - 1$ since M is not A-projective

$\Rightarrow \operatorname{pd}_A M = 1$ (Auslander-Buchsbaum)

$\rightsquigarrow$ exact seq

$$\begin{tikzcd} 0 \arrow[r] & F_1 \arrow[r, "\varphi"] \arrow[d, "\cdot x"] & F_0 \arrow[r] \arrow[d, "\cdot x"] & M \arrow[r] \arrow[d, "\cdot x = 0"] & 0 \\ 0 \arrow[r] & F_1 \arrow[r, "\varphi"'] & F_0 \arrow[r] & M \end{tikzcd}$$

with $\operatorname{rk} F_0 = \operatorname{rk} F_1$

$\Rightarrow \cdot x : (F_1 \xrightarrow{\varphi} F_0) \rightsquigarrow (F_1 \to F_0)$ is 0 in $D^+(A)$

$\Rightarrow \exists$ homotopy ψ

$$\begin{tikzcd} F_1 \arrow[r, "\varphi"] \arrow[d, "\cdot x"'] & F_0 \arrow[d, "\cdot x"] \arrow[dl, "\psi"'] \\ F_1 \arrow[r, "\varphi"'] & F_0 \end{tikzcd}$$

i.e. $\varphi\psi = x \cdot \mathrm{id}_{F_0}$

$\psi\varphi = x \cdot \mathrm{id}_{F_1}$

if $\varphi(a) = x \cdot b$ for $a, b \in F_0$

then $\psi\varphi(a) = x \cdot \psi(b)$ but $x \cdot \mathrm{id}_{F_1}$ is injective

$\quad\;\; x \cdot a$ $\Rightarrow a = \psi(b)$

similarly $\psi(a') = x \cdot b'$ for $a', b' \in F_1$

$\Rightarrow a' = \varphi(b')$

given exactness of

$$\cdots \to \bar{F}_0 \xrightarrow{\bar{\psi}} \bar{F}_1 \xrightarrow{\bar{\varphi}} \bar{F}_0$$

as $\bar{F}_1 \xrightarrow{\bar{\varphi}} \bar{F}_0 \to M \to 0$ is exact by right exactness of $\otimes_A B$

for (5) we have

$$0 \to M \to \bar{F}_1 \to \bar{F}_0 \to \bar{F}_1 \to \bar{F}_0 \to \cdots \to \bar{F}_0 \to M \to 0$$

$\bar{F}_0$ so if $\bar{\psi}(\bar{F}_0) \subset m\bar{F}_1$ & $\bar{\varphi}(\bar{F}_1) \subset m F_0$ then M has no

free A-summand by Lemma 2 (i).

Def (A, m) local, $x \in m$

$MF(A, x)$ is the $\mathbb{Z}/2$ dg product category of complexes

$$\left(\begin{array}{c} F_1 \xrightarrow{\varphi} F_0 \\ \psi \\ F_0 \to F_1 \end{array} \right)$$

F_1, F_0 free A mod of the same rank with

$$\varphi\psi = x \cdot \mathrm{id}_{F_0}$$
$$\psi\varphi = x \cdot \mathrm{id}_{F_1}$$

Hom complex deg 0 $\left(F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0 \right) \to \left(F_1' \underset{\psi'}{\overset{\varphi'}{\rightleftarrows}} F_0' \right)$

$$\begin{tikzcd} F_1 \arrow[r, shift left, "\varphi"] \arrow[d, "f_1"'] & F_0 \arrow[l, shift left, "\psi"] \arrow[d, "f_0"] \\ F_1' \arrow[r, shift left, "\varphi'"] & F_0' \arrow[l, shift left, "\psi'"] \end{tikzcd}$$

$f = (f_0, f_1)$ $f_0 : F_0 \to F_0'$
$f_1 : F_1 \to F_1'$

deg 1 is $g = (g_0\ g_1)$ $g_0 : F_1 \to F_0'$
$g_1 : F_0 \to F_1'$

$$df = (\varphi' f_1 - f_0 \varphi,\ \psi' f_0 - f_1 \psi)$$

$$dg = (\varphi' g_1 + g_0 \psi,\ \psi' g_0 + g_1 \varphi)$$

$$\begin{tikzcd} F_0 \arrow[r, shift left, "\psi"] \arrow[d, "g_1"'] & F_1 \arrow[l, shift left, "\varphi"] \arrow[d, "g_0"] \\ F_1' \arrow[r, shift left, "\varphi'"] & F_0' \arrow[l, shift left, "\psi'"] \end{tikzcd}$$

Lemma let $F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0$ be in $MF(A, x)$. Then

$\operatorname{coker}\varphi$ & $\operatorname{coker}\psi$ are both in MCM_B

Pf we have $0 \to F_1 \to F_0 \to \operatorname{coker}\varphi \to 0$

$0 \to F_0 \to F_1 \to \operatorname{coker}\psi \to 0$

and $x \operatorname{coker}\varphi = 0 = x \cdot \operatorname{coker}\psi$

Then $\operatorname{pd}_A \operatorname{coker}\varphi = 1 \Rightarrow \operatorname{depth}_A \operatorname{coker}\varphi = \dim A - 1 = \dim \operatorname{coker}\varphi$

$\Rightarrow \operatorname{depth}_B \operatorname{coker}\varphi = \dim B$

& same for $\operatorname{coker}\psi$

Def. The trivial MF are $(\varphi = x \cdot id_A, \psi = 1_A)$ and $(\varphi = 1_A, \psi = x \cdot id_A)$

- given two MF (φ, ψ), (φ', ψ') we have the sum $(\varphi \oplus \varphi', \psi \oplus \psi')$
- isomorph of MF is the obvious notion
- A MF (φ, ψ) is reduced if $(\varphi, \psi) \not\cong (x, id) \oplus (\varphi_0, \psi_0)$ and $\not\cong (id, x) \oplus (\varphi_0, \psi_0)$

Lemma Suppose x is a n.z.div in A then (φ, ψ) is reduced $\Leftrightarrow \varphi(F_1) \subset m F_0$ and $\psi(F_0) \subset m F_1$

Pf Gaussian elimination

Eisenbud's periodicity theorem

For (A, m) local and M f.gen A-module

A minimal free resolution of M is a free resolution

$$\cdots \to F_2 \xrightarrow{d_1} F_1 \xrightarrow{d_0} F_0 \xrightarrow{\varepsilon} M \to 0$$

s.t.

$$\bar{d}_n : \frac{F_{n+1}}{mF_{n+1}} \xrightarrow{\sim} \frac{\ker(d_{n-1})}{m\ker(d_{n-1})}$$

&

$$\varepsilon : \frac{F_0}{mF_0} \xrightarrow{\sim} \frac{M}{mM}$$

This is the same as $\ker(d_{n-1}) \subset mF_n$

$\operatorname{rk} F_{n+1} = \min \#$ of generators of $\ker(d_{n-1})$

A minimal resolution is unique up to isomorphism

Thm (Eisenbud) (A, m) reg local $x \in m$, $B = A/(x)$ 15

$N \in \mathrm{Mod}_B$ let

$$\to \bar F_n \to \bar F_{n-1} \cdots \xrightarrow{d_1} \bar F_1 \xrightarrow{d_0} \bar F_0 \to N \to 0$$

$$\to \bar F_{n+1} \to \bar F_n \to \bar F_{n-1} \to \bar K_{n-2} \to$$

be a minimal free resolution. Then

the resolution is isomorphic to a periodic resolution

$$\bar F_1 \xrightarrow{\bar\varphi} \bar F_0 \xrightarrow{\bar\psi} \bar F_1 \xrightarrow{\bar\varphi} \bar F_0 \to$$

with

$$F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0 \in MF(A, x)$$

Pf let $\bar K_i = \ker(d_i) \subset \bar F_i$

B has depth $n-1 \Rightarrow \mathrm{Ext}^i_B(k, \bar F_m) = 0 \qquad i < n-1$

$$0 \to \bar K_n \to \bar F_n \to \bar K_{n-1} \to 0 \Rightarrow \mathrm{Ext}^i_B(k, \bar K_{n-1}) \xrightarrow{\sim} \mathrm{Ext}^{i+1}_B(k, \bar K_n)$$

for $i < n-1$

for all $0 \leq i \leq n-2$ $0 \overset{\sim}{=}$ $\mathrm{Ext}^0_B(k, \bar K_i) \xrightarrow{\sim} \mathrm{Ext}^1_B(k, \bar K_{i+1})$

$$\Rightarrow \mathrm{Ext}^{n-2-i}(k, \bar K_{n-2}) \Rightarrow \mathrm{depth}_B \bar K_{n-2} = n-1$$

$$\cdots \to \bar{F}_{n+1} \xrightarrow{d_n} \bar{F}_n \xrightarrow{d_{n-1}} \bar{F}_{n-1} \to \bar{K}_{n-1} \to 0$$

(minimal)

but $\bar{K}_{n-1}$ is MCM$_B$ so has a resolution

$$0 \to F_1' \xrightarrow{\varphi} F_0' \to K_{n-2} \to 0$$

Then have $\bar{F}_1' \xrightarrow{\bar{\varphi}} \bar{F}_0' \to \bar{K}_{n-2}$ minimal

similarly and have $F_0' \xrightarrow{\psi} F_1' \to \ker \varphi \to 0$

& $\bar{F}_0' \xrightarrow{\bar{\psi}} \bar{F}_1' \to \operatorname{coh}\psi \to 0$ is minimal

$\ker \bar{F}_0 \to \bar{K}_{n-1}$

so

$$\xrightarrow{\bar{\varphi}} \bar{F}_0' \xrightarrow{\bar{\psi}} \bar{F}_1' \xrightarrow{\bar{\varphi}} \bar{F}_0' \to \bar{K}_{n-2} \to 0$$

is minimal have so to

$$\cdots \to \bar{F}_{n+1} \to \bar{F}_n \to \bar{F}_{n-1} \to \bar{K}_{n-1} \to 0$$

<u>Cor</u> (A, m) regular $0 \neq x \in m$. There are bijections between

i) minimal 2 periodic resolutions over $A/(x)$

ii) reduced matrix factorizations in $MF(A, x)$

iii) MCM$_{A/(x)}$ without free summands

Stable categories

Let (A, m, k) be a Gorenstein local ring

We define several "stable" categories

Def. Let S be a noetherian ring. Define $\underline{\mathrm{Mod}}_S$ to have objects the finitely generated S modules with

$$\underline{\mathrm{Hom}}_S(M,N) = \mathrm{Hom}_S(M,N) \Big/ \left\{ f \;\middle|\; \exists \text{ a projective module } P \text{ and a factoring } \begin{array}{c} M \xrightarrow{f} N \\ {\scriptstyle i}\searrow \;\; \nearrow{\scriptstyle p} \\ P \end{array} \right\}$$

composition is induced from composition in Mod_S

For S Gorenstein define $\underline{\mathrm{MCM}}_S$ as the full subcategory of $\underline{\mathrm{Mod}}_S$ with objects the maximal Cohen Macaulay modules

- $D^{perf}(S) \subset D^b(S)$ (complexes of S-mod with f.g. H^*) is the full subcategory with objects iso to bounded complexes of projective S-modules
  $D^{perf}(S)$ is a thick subcategory of $D^b(S)$

Define the "singularity category" $\underline{D}^b(S)$ as the Verdier quotient $D^b(S) / D^{perf}(S)$

- we have the dg category of complexes of S-modules (unbounded) $C(S)$, and the homotopy category $K(S) = H^0(C(S))$

- let $\underline{APC(S)} \subset K(S)$ be the full subcategory with objects the acyclic complexes of finitely generated projective S modules. $\underline{APC}(S)$ is a thick subcategory of $K(S)$, in particular, triangulated