Matrix Factorizations

(A, m, k) regular local $x \in m$. We have matrix factorizations $F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0$; F_i free fin gen A-modules $\psi\varphi = x\,\mathrm{id}_{F_0}$, $\varphi\psi = x\,\mathrm{id}_{F_1}$

Call $F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0$ <u>reduced</u> if $\varphi F_1 \subset mF_0$, $\psi F_0 \subset mF_1$

<u>Prop</u> Let (A, m, k) be a regular local ring $B = A/(x)$ and $M \in MCM_B$. Then $\exists$ a matrix factorization

$$F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0 \; ; \; F_i \text{ fin gen free } A\text{-mod } \varphi\psi = x\,\mathrm{id}_{F_0},\ \psi\varphi = x\,\mathrm{id}_{F_1}$$

with $M = \operatorname{cok} \varphi$. Moreover, if $F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0$ is reduced then M contains no free summands and conversely, if M contains no free summand, we can find a reduced $F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0$ with $M = \operatorname{cok}\varphi$

Pf We showed all of this except the last assertion last time.

Suppose $M \in MCM_B$ has no free summands. We have a free A mod F_0 with surjection $F_0 \to M$ inducing iso $F_0/mF_0 \cong M/mM$ (Nakayama). Then we have the free resolution $0 \to F_1 \xrightarrow{\varphi} F_0 \to M \to 0$ with $\varphi(F_1) \subset mF_0$, which we then extend to $F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0 \in MF(A, x)$. If $\psi(F_0) \not\subset mF_1$ (Gauss elim) we have $F_0 \cong F_0' \oplus A$, $F_1 \cong F_1' \oplus A$ with $\psi = \psi' \oplus \mathrm{id}_A$. $\varphi\psi = x\,\mathrm{id}_{F_0}$ $\psi\varphi = x\,\mathrm{id}_{F_1}$ $\Rightarrow \varphi = \varphi' \oplus x\,\mathrm{id}_A \Rightarrow M = M' \oplus A/x$ #.

Minimal resolution

<u>Def</u> Let (R, m, k) be a local (Noetherian) ring, N a f.gen R-module and

$$\xrightarrow{d_1} F_1 \xrightarrow{d_0} F_0 \to N \to 0$$

a resolution by free f.gen R-modules. Call the resolution <u>minimal</u> if

i) $d_k(F_{k+1}) \subset mF_k$

ii) the induced map $F_k/mF_k \xrightarrow{d_{k-1}}$ $\underline{\mathrm{coker}\, d_k}$ modulo is an iso $\forall k \geq 0$

<u>Note</u> Two minimal resolutions of N are isomorphic & each N admits a minimal resolution.

<u>Ex</u> Let $F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0$ with $M = \mathrm{coker}\,\varphi$ as above, let $\bar{(\ )} := (\ )$ mod x.

Then $\dots \to \bar F_0 \xrightarrow{\bar\psi} \bar F_1 \xrightarrow{\bar\varphi} \bar F_0 \to M \to 0$ is a free periodic resolution of M, and is a minimal resolution if $F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0$ is reduced.

We showed this last time.

<u>Thm (Eisenbud)</u> (A, n, k) regular local dim $= n$, $x \in m$, $x \neq 0$,

$B := A/(x)$, $N \in \mathrm{Mod}_B$ $\xrightarrow{\psi} G_2 \xrightarrow{d_1} G_1 \xrightarrow{d_0} G_0 \to N \to 0$

a minimal resolution. Then $G_{k+2} \xrightarrow{d_{k+1}} G_{k+1} \xrightarrow{d_k} G_k \xrightarrow{d_{k-1}} G_{k-1}$ $\;M_k$

is 2-periodic for $k \geq n+1$

If B is Gorenstein of dim $n-1$. Let $M_k = \operatorname{coker} d_{k-1}$

By Prop ?, we know that M_k is MCM, with no free summands, for $k \geq n+1$. Then, there is a reduced matrix factorization $F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0$ with $M_k = \operatorname{coker}\varphi$, giving the minimal resolution $\cdots \xrightarrow{\bar\psi} \bar F_1 \xrightarrow{\bar\varphi} \bar F_0 \to M_k \to 0$ so then

$(\cdots \to G_{k+1} \to G_k) \cong (\cdots \xrightarrow{\bar\psi} \bar F_1 \xrightarrow{\bar\varphi} \bar F_0)$ hence 2-periodic

Some stable categories R noetherian ring of finite Krull dimension

1. $D^b(R) \simeq$ homotopy category of bounded above complexes $P_\bullet$, P_i f.gen projective R-modules such that $H^*(P_\bullet)$ is a f.gen R-module

i.e. The truncation

$\sigma_{\le k} P_\bullet \to \operatorname{coker} d_k$ is a resolution of $\operatorname{coker} d_k$ for all $k \ll 0$.

$D^b(R) \supset D^{perf}(R)$ = homotopy category of finite complexes of f.gen projective, a thick subcat of $D^b(R)$

The stabilized derived category (singularity category) is

$$\underline{D}^b(R) := D^b(R) / D^{perf}(R)$$ a triangulated category

2. $\operatorname{Mod} R$ = f.gen R modules

$\underline{\operatorname{Mod}} R$ has the same obj with

$$\underline{\operatorname{Hom}}_R(M,N) = \operatorname{Hom}_R(M,N) / \{ f : M \to N \text{ , same projective} \}$$

$M \searrow P \nearrow N$

3. $APC(R)$ = dg category of (unbounded) acyclic complexes of f.gen proj R-modules

$\underline{APC}(R)$ = homotopy category of $APC(R)$

$\underline{\mathrm{Mod}}_R$ has a well defined "loops functor" for $N \in \underline{\mathrm{Mod}}_R$ (5)

take $P \overset{\pi}{\twoheadrightarrow} N$ and define $\Omega_\pi N := \ker \pi$

given $P' \overset{\pi'}{\twoheadrightarrow} N$ have $P \oplus P' \overset{\pi+\pi'}{\twoheadrightarrow} N$

$$\begin{tikzcd}
& \ker(\pi+\pi') \arrow[d, shift left] & P \oplus P' \arrow[d, shift left] \\
\ker\pi \oplus P' \arrow[ur, "\cong"] & \ker\pi \arrow[u, shift left] \arrow[r] & P \arrow[u, shift left]
\end{tikzcd}$$

so all $\Omega_\pi N$ in $\underline{\mathrm{Mod}}_R$ are canonically isomorphic

get $\Omega : \underline{\mathrm{Mod}}_R \to \underline{\mathrm{Mod}}_R$ since $(\Omega P = 0)$

If R is Gorenstein we have $\mathrm{MCM}_R \supseteq \mathrm{Proj}_R$ so we have

$\underline{\mathrm{MCM}}_R \subset \underline{\mathrm{Mod}}_R$ full subcat with obj in MCM_R

and (Lemma 2 from before) $\Omega_\pi M \in \mathrm{MCM}_R$ if $M \in \mathrm{MCM}_R$ so we have

$$\Omega : \underline{\mathrm{MCM}}_R \to \underline{\mathrm{MCM}}_R$$

Some functors

On $APC(R)$ we have the ith syzygies

$$\Omega_i(X) = \mathrm{cok}\,(d^{-i} : X^{-i-1} \to X^{-i})$$

with $\Omega_i(X[j]) = \Omega_{i-j}(X)$

Ω_i is clearly functorial in X, and for $X \xrightarrow{f} Y$ homotopic to 0 in $Z^0 APC(R)$

$$\begin{tikzcd} & & X^{-i+1} & \\ X^{-i-1} \arrow[r, "d_X^{-i}"] \arrow[d] & X^{-i} \arrow[r] \arrow[d, "f^{-i}"] \arrow[dl, "h^{-i}"'] & \Omega_i(X) \arrow[u] \arrow[d, "\Omega_i(f)"] \arrow[dl, "h^{-i+1}"] & = \mathrm{cok}\, d^{-i} = \ker d^{-i+1} \\ Y^{-i-1} \arrow[r, "d_Y^{-i}"] & Y^{-i} \arrow[r] & \Omega_i(Y) & \end{tikzcd}$$

$$f^{-i} = h^{-i+1} d_X^{-i+1} + d_Y^{-i} h^{-i}$$

$\Rightarrow$ $\Omega_i(f)$ factors through X^{-i+1} or Y^{-i} $\Rightarrow \Omega_i(f) = 0$ in $\underline{\mathrm{Mod}}(R)$

So we have

$$\Omega_i : \underline{APC}(R) \to \underline{\mathrm{Mod}}\,R$$

and if R is Gorenstein then

$$\Omega_i : \underline{APC}(R) \to \underline{MCM}_R \; ; \; \Omega_i(X) \text{ admits}$$

a co-projective resolution

$$0 \to \Omega_i(X) \to X^{-i+1} \to X^{-i+2} \to \cdots$$

$\Rightarrow \Omega_i(X) \in MCM(R)$ (Characterizations of last time)

$X^n \to \Omega_n(X) \to 0$

<u>Moreover</u> $\Omega_i(X[-1]) = \Omega\,\Omega_i X$

$$\left(\begin{tikzcd} \Omega\Omega_i X \arrow[r] & X^{-i} \arrow[r, two heads] & \Omega_i X \\ X^{-i-2} \arrow[r] & X^{-i-1} \arrow[u, "d^{-i}"'] \arrow[ul] & \end{tikzcd} \right)$$

<u>Truncation functors</u> $X \in C(R)$ have

$$\begin{tikzcd}
\sigma_{\le k}X = (\arrow[r] & \cdots \arrow[r] & X^{k-1} \arrow[r] & X^{k} \arrow[r] & 0 & \cdots) \\
 & & X^{k-1} \arrow[u, equal] \arrow[r] & X^{k} \arrow[u, equal] \arrow[r] & X^{k+1} \arrow[u] \arrow[r] & 0)
\end{tikzcd}$$

$\uparrow$

$\sigma_{\le k+1}X = (\to \cdots \to X^{k-1} \to X^{k} \to X^{k+1} \to 0)$

$\uparrow$

$\vdots$

with $\sigma_{\le k}(X[i]) = (\sigma_{\le k+i}X)[i]$ for $l \ge k$ have $\pi_{l,k} : \sigma_{\le k} \to \sigma_{\le l}$

<u>Lemma</u> i) for $X \in APC(R)$ we have

$\sigma_{\le k}X \xrightarrow{\varepsilon_k} \Omega_{-k}X \to 0$ a proj resolution of $\Omega_{-k}X$

ii) for $k \le l$ we have the triangle exact seq

$$\sigma_{\le l}X \xrightarrow{\pi_{k,l}} \sigma_{\le k}X \to P, \quad P \in D^{perf}(R)$$

so $\operatorname{cone}(\pi_{k,l}) \cong P$ in $D^b(R)$

iii) If $f : X \to Y$ is a map in $APC(R)$ homotopic to 0

then $\sigma_{\le k}f : \sigma_{\le k}X \to \sigma_{\le k}Y$ is 0 in $\underline{D}^b(R)$

If (i) & (ii) are clear. For (iii), $f \underset{h}{\sim} 0$ so $f^i = d_Y^{i} h^i + h^{i+1} d_X^{i}$

$$\begin{tikzcd}
X^{k-2} \arrow[r] \arrow[d, "f^{k-2}"] & X^{k-1} \arrow[r] \arrow[dl, "h^{k-1}"'] \arrow[d, "f^{k-1}"] & X^k \arrow[r] \arrow[dl, "h^k"'] \arrow[d, "f^k"] & {} \\
Y^{k-2} \arrow[r] & Y^{k-1} \arrow[r] & Y^k \arrow[r] & {}
\end{tikzcd}$$

so $\sigma_{\le k} X \xrightarrow{\sigma_{\le k} f} \sigma_{\le k} Y \xrightarrow{\pi^Y_{k-1,k}} \sigma_{\le k-1} Y$ is 0 in $\underline{D}^b(R)$

but have $Y^k[-k] \to \sigma_{\le k} Y \to \sigma_{\le k-1} Y \xrightarrow{+1}$ $\Rightarrow$ $\pi^Y_{k-1,k}$ is iso in $\underline{D}^b(R)$

$\Rightarrow \sigma_{\le k} f = 0$ in $\underline{D}^b(R)$

Cor We have the system of iso of functors

$\underline{APC}(R) \to \underline{D}^b(R)$

$$\xrightarrow{\sim} \sigma_{\le k+1} \xrightarrow[\pi_{k,k+1}]{\sim} \sigma_{\le k} \xrightarrow[\pi_{k-1,k}]{\sim} \sigma_{\le k-1} \xrightarrow{\sim}$$

Def $\sigma_{\le} : \underline{APC}(R) \to \underline{D}^b(R)$ is defined as

$\sigma_{\le} = \varprojlim_{k} \sigma_{\le k}$, and $\sigma_{\le k}$ is an exact functor of triangulated

categories

Note if $X' \xrightarrow{f} X \to X'' \xrightarrow{+1}$ is a cone seq in $\underline{APC}(R)$, $X'' = \mathrm{cone}(f)$

then $\mathrm{cone}(\sigma_{\le k} f) \to \sigma_{\le k}(\mathrm{cone}(f)) \to X'^{k+1}[-k] \xrightarrow{+1}$ is a dist. Δ in $D^b(R)$

$\Rightarrow \mathrm{cone}(\sigma_{\le k} f) \xrightarrow{\sim} \sigma_{\le k}(\mathrm{cone}(f))$ in $\underline{D}^b(R)$.

<u>A construction</u>: Take $X \in APC(R)$. The fact that $\Omega_i X \in MCM_R$ and $\mathrm{Ext}^i_R(M,R)=0$ for $M \in MCM_R$, $i>0$ implies

i) X^* is also in $APC(R)$

ii) $\Omega_{i+1}(X^*) \cong \Omega_{-i}(X)^*$

$$\cdots \to X^{i-1} \to X^i \to X^{i+1} \to \quad \xrightarrow{(-)^*} \quad \to X^{*,-i-1} \to X^{*,-i} \to$$

$$X^i \to \Omega_i(X) \to X^{i+1} \qquad X^{*,-i-1} \to \Omega_i(X)^* \to X^{*,-i}$$

iii) we have $\underbrace{(\to X_{k-1} \to X_k)}_{\sigma_{\le k}X} \xrightarrow{d^{k+1}} \underbrace{(X_{k+1} \to X_{k+2} \to \cdots)}_{\cong\, (X_{k+1}^{**} \to X_{k+2}^{**} \to \cdots) \,=\, (\sigma_{\le -k-1}X^*)^*}$

$$X_k \to \Omega_k(X) \cong \Omega_k(X)^{**} \cong \Omega_{k+1}(X^*)^* \to X_{k+1}^{**}$$

and $(\sigma_{\le k}X, d^{k+1}, (\sigma_{\le -k-1}X^*)^*)$ is isomorphic to X. Thus, if we choose projective resolutions $P^\bullet \to \Omega_k X$, $Q^\bullet \to \Omega_{k+1}(X^*)$, we have $\to P^0 \xrightarrow{\delta^{k+1}} Q^{0*} \to$, $P^0 \to \Omega_k X \cong \Omega_{k+1}(X^*)^* \to Q^{0*}$,
and the homotopy equivalences $P[k] \simeq \sigma_{\le k}X$, $Q^\bullet[k+1] \simeq \sigma_{\le -k-1}X^*$ induce a homotopy equivalence
$(\to P^1 \to P^0 \xrightarrow{\delta^{k+1}} Q^{0*} \to Q^{1*} \to \cdots) \sim X$. □

<u>Def</u> $f,g : X \to Y$ morphisms in $Z^0 APC(R)$, $k \in \mathbb{Z}$. A k-homotopy $f \sim_k g$ is a homotopy $\{h^i : X^i \to Y^{i-1} \mid d_Y^{i} h^i + h^{i+1} d_X^{i+1} = f$

such that $h^{k+1} = 0$. Let $\underline{APC}_k(R) = Z^0 APC(R)/\sim^k$ (10)

<u>Prop 1</u> For R Gorenstein the functor $\Omega_k : Z^0 APC(R) \to MCM(R)$ descends to an equivalence $\underline{\Omega}_k : \underline{APC}_k(R) \to \underline{MCM}(R)$

Pf. For $M \in MCM(R)$ let $P^\cdot \to M \to 0$, $Q^\cdot \to M^* \to$ be proj resolution giving

$$(\to P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^{(k)}} Q^{0*} \xrightarrow{d^{k+2}} Q^{1*} \to \cdots) \in APC(R)$$

$$P^0 \to M = M^{**} \to Q^{0*}$$

so Ω_k is essentially surjective. Given $f : M \to N$ get $P(f) : P^\cdot_M \to P^\cdot_N$ lifting f

<u>full</u>: $Q(f^*) : Q^\cdot_N \to Q^\cdot_M$ giving $(P(f), Q(f^*)^*)$ in $APC(R)$ $M \to N$ lifting f, a map up to k-homotopy

$$\begin{array}{ccc} Q^\cdot_N & \to & Q^\cdot_M \\ \downarrow & & \downarrow \\ N^* & \xrightarrow{f^*} & M^* \end{array}$$

<u>faithful</u>: each $f : X \to Y$ is iso up to k-homotopy to $(P(f), Q(f^*)^*)$ and if $(P(f), Q(f^*)) \sim^k 0 \iff P(f) \sim_h 0$ and $Q(f^*) \sim_h 0$ $\iff \Omega_k f : \Omega_k X \to \Omega_k Y$ is 0

We have the inclusion functor $i_R : Mod_R \to D^b(R) \to \underline{D}^b(R)$ send N to the complex N supported in deg 0, or to a resolution of N by f.gen projective R modules

If $N \xrightarrow{f} M$ factors through P then $i_R(f) = 0$

$$N \to P \to M$$

so we have $i_R : \underline{\mathrm{Mod}}_R \to \underline{D}^b(R)$ and if R is Gorenstein

$$i_R : \underline{\mathrm{MCM}}_R \to \underline{D}^b(R)$$

Note that

$$i_R(\Omega M) \cong (i_R M)[-1] \quad \text{and similarly for morphisms}$$

by $0 \to N \to P \to M \to 0 \rightsquigarrow M[-1] \to N \to P \xrightarrow{+1} M$ in $D^b(R)$

$\rightsquigarrow M[-1] \cong N$ in $\underline{D}^b(R)$

<u>Thm</u> Let R be Gorenstein. Then

1) $\Omega_0 : \underline{\mathrm{APC}}(R) \to \underline{\mathrm{MCM}}(R)$ is an equivalence of categories with $\Omega_0 \circ [-1] \cong \Omega \circ \Omega_0$

2) $i_R : \underline{\mathrm{MCM}}_R \to \underline{D}^b(R)$ is an equivalence of category with $i_R \circ \Omega \cong [-1] \circ i_R$

Moreover, the structures of triangulated category in $\underline{\mathrm{MCM}}(R)$ induced by (1) & (2) are the same.

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Proof i) Ω_0 is surjective on objects and is full

By Prop 1 $\Omega_0 : \underline{APC}(R) \to MCM(R)$ is an equivalence

$\Rightarrow \Omega_0 : \underline{APC}_b(R) \to MCM(R)$ is surjective on objects and full

(ii) $\sigma_{\leq} : \underline{APC}(R) \to \underline{D}^b(R)$ is an equivalence

Note that this implies (1) + (2) since

$$\sigma_{\leq} \cong \sigma_{\leq 0} \cong i_R \circ \Omega_0$$

so $\sigma_{\leq}$ an equivalence $\Rightarrow \Omega_0$ is faithful $\Rightarrow$ (1)

and $i_R \circ \Omega_0$ and Ω_0 both equivalences $\Rightarrow$ (2)

(ii)(a) $\sigma_{\leq}$ is essentially surjective. For $A \in D^b(R)$

$\sigma_{\leq k} A \to \Omega_{-k} A$ is a projective resolution of $\Omega_{-k} A$ for $k \ll 0$

and $A \to \sigma_{\leq k} A$ is an iso in $\underline{D}^b(R)$. By Prop 1

$\Omega_{-k} : \underline{APC}(R) \to \underline{MCM}(R)$ is essentially surjective

$\Rightarrow \exists X \in APC(R)$ w iso $\sigma_{\leq k} X \cong \sigma_{\leq k} A$ in $\underline{D}^b(R)$

$\Rightarrow \sigma_{\leq}$ is ess surjective

(ii)(iii) $\sigma_{\leq}$ is fully faithful

<u>Lemma</u> Take $A \in \mathrm{APC}(R)$, $P \in D^{perf}(R)$, supported in $[a,b]$. Then

$\mathrm{Hom}_{D^b(R)}(\sigma_{\le \ell} A, P) = 0$ for $\ell > b$.

Pf Can assume $P = (P^a \to \dots \to P^b)$ so reduce to show

$\mathrm{Hom}_{D^b(R)}(\sigma_{\le \ell} A, P[-i]) = 0$ for P projective $i < \ell$

$$\begin{array}{ccccccc} \to A^{i-1} & \xrightarrow{d^i} & A^i & \xrightarrow{d^{i+1}} & A^{i+1} & \to \dots \to & A^\ell \\ & & & \searrow^{\pi} \; \Omega^i A[-i] & & & \\ \downarrow & & \downarrow f^i & \swarrow g & & & \\ 0 & \to & P^i & \to 0 & & & \end{array}$$

$A^i \twoheadrightarrow \Omega^{i+1} A$

Then f^i factors through $\Omega^i A[-i] = \mathrm{coker}\, d^i[-i]$

$\Omega^{i+1} A \in \mathrm{MCM}_R \Rightarrow \mathrm{Ext}^1_R(P^i, \Omega^{i+1} A) = 0 \Rightarrow g$ extends to $h: A^{i+1} \to P^i$

so $f \sim 0$ in $D^b(R)$ □

<u>full</u> $\sigma_{\le h} X \xrightarrow{g} \sigma_{\le h} Y$ in $\underline{D}^b(R)$ is $\sigma_{\le h} X \xrightarrow{g'} C \leftarrow \sigma_{\le h} Y$ in $D^b(R)$ with $P \in D^{perf}$

Since $\mathrm{Hom}_{D^b(R)}(\sigma_{\le \ell} X, P) = 0$ for $\ell \gg 0$, we have $C \to P_{\ne}$

$$\begin{array}{ccc} \sigma_{\le h} X & & \\ \uparrow & \searrow g' & \\ & & C \\ & \nearrow g'' & \uparrow \\ \sigma_{\le \ell} X & \xrightarrow{g_{h,\ell}} & \sigma_{\le h} Y \end{array}$$

so $\sigma_{\le \ell} X \xrightarrow{\sim} \sigma_{\le h} X \to \sigma_{\le h} Y$ in $\underline{D}^b(R)$ lifts to $\sigma_{\le \ell} X \to \sigma_{\le h} Y$ in $D^b(R)$ and descends to $\sigma_{\le h} X \xrightarrow{\tilde g} \sigma_{\le h} Y$ in $D^b(R)$, with $\tilde g = g$ in $\underline{D}^b(R)$

By Prop 7 $\exists f: X \to Y$ in $\underline{\mathrm{APC}}(R)$ with $\tilde g = \sigma_{\le h} f$ in $D^b(R)$

faithful a) Given $f: A \to B$ in $\underline{APC}(R)$ such that $\sigma_{\le k} f = 0$ in $D^b(R)$ then $\Omega_{-k} f : \Omega_{-k} A \to \Omega_{-k} B$ is 0

By Prop 1 $f = 0$ in $\underline{APC}_k(R) \Rightarrow f = 0$ in $\underline{APC}(R)$

b) Suppose $\sigma_{\le k} f = 0$ in $\underline{D}^b(R)$, so there is an exact Δ

$\sigma_{\le k} B \xrightarrow{\alpha} C \to P^\cdot \xrightarrow{+}$ with $P^\cdot \in D^{perf}_{P[-l]_{\le +1}}$ and $\alpha \circ \sigma_{\le k} f = 0$ in $D^b(R)$

We have $\sigma_{\le k+l} A \xrightarrow{\pi_{k,k+l}} \sigma_{\le k} A \xrightarrow{\sigma_{\le k} f} \sigma_{\le k} B \xrightarrow{\alpha} C$ and by the Lemma (for $l >> 0$)

$\alpha \sigma_{\le k} f \pi_{k,k+l} = 0 \Rightarrow \sigma_{\le k} f \pi_{k,k+l} = 0 \Rightarrow \sigma_{\le k} f \circ \pi_{k,k+l} \sim_h 0$, ie

we have

$$\begin{tikzcd}
\to A^{k-1} \arrow[r] \arrow[d] & A^k \arrow[r] \arrow[d] \arrow[dl, "h^k"'] & A^{k+1} \arrow[r] \arrow[d, "0"] \arrow[dl, "h^{k+1}"'] & \cdots \arrow[r] & A^{k+l} \arrow[r] \arrow[d, "0"] & 0 \\
\to B^{k-1} \arrow[r] & B^k \arrow[r] & 0 \arrow[r] & \cdots & 0 \arrow[r] & 0
\end{tikzcd}$$

with $f^i = d_B^{i-1} h^i + h^{i+1} d_A^i$ in $D^b(R)$. Thus $\sigma_{\le k} f \circ \pi_{k,k+l} = 0$ in $D^b(R)$

This gives the diagram

$$\begin{tikzcd}
\sigma_{\le k} A \arrow[r, "p_A^k"] \arrow[d, "\sigma_{\le k} f"'] & \Omega_{-k} A[-k] \arrow[r, hook] \arrow[d, "\Omega_{-k} f[-k]"] & A^{k+1}[-k] = P[-k] \arrow[dl, "p_B^k \circ h^{k+1}"] \\
\sigma_{\le k} B \arrow[r, "p_B^k"] & \Omega_{-k} B[-k] &
\end{tikzcd}$$

deg k

Let $P(h) = (0 \to \dots \to 0 \to P \xrightarrow{id} P \to 0 \to \dots \to 0 \to)$, so we have 15

with $P[-h] = \sigma_{\le h} P(h)$

$$\begin{tikzcd} \sigma_{\le h} A \arrow[d, "\sigma_{\le h} f"'] \arrow[r] & \sigma_{\le h} P \arrow[dl] \\ \sigma_{\le B} & \end{tikzcd} \quad \cong \quad \begin{tikzcd} \Omega_{>h} A[-h] \arrow[d, "\Omega_{>h} f[-h]"'] \arrow[r] & P[-h] \arrow[dl] \\ \Omega_{>h} B[-h] & \end{tikzcd}$$

in $D^b R$

By Prop 1 this lifts to

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[r] & P(h) \arrow[dl] \\ B & \end{tikzcd}$$

in $Z^0 APC(R)$

But $id_P : P(h)^{h+1} \to P(h)^h$ gives a homotopy $id_{P(h)} \sim 0$, so $f = 0$ in $\underline{APC}(R)$

Matrix Factorization

Take (A, m, k) regular local ring and $R = A/(x)$ we have the $\mathbb{Z}/2$-category of matrix factorizations

Obj: $\left(F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0 \right)$ F_i free fg A mod

$\varphi\psi = x \cdot id_{F_0}$, $\psi\varphi = x \cdot id_{F_1}$

$Hom^0\left((F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0) \to (F_1' \underset{\psi'}{\overset{\varphi'}{\rightleftarrows}} F_0') \right) = \{ (f_0 : F_0 \to F_0', f_1 : F_1 \to F_1') \}$

$Hom^1(\quad) = \{ (g_0 : F_0 \to F_1', g_1 : F_1 \to F_0') \}$

$d^0 : Hom^1 \to Hom^0$

$d^0(f_0, f_1) = (d_0(f_0, f_1), d_1(f_0, f_1)) = (\varphi' f_1 - f_0 \varphi, \psi' f_1 - f_0 \psi)$

$d'(g_0,g_1) = (d_0(g_0,g_1), d_1(g_0,g_1)) = (\varphi' g_0 + g_1\psi, \psi' g_1 + g_0\varphi)$

Translation

$(F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0)[1] = (F_0 \underset{-\varphi}{\overset{-\psi}{\rightleftarrows}} F_1)$

Conversely for $(F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0) \in MF(A,x)$ the sequence

$\bar{(-)} = - \otimes_A B$

$(*)$ $\quad \dots \xrightarrow{\bar\psi} \bar F_1 \xrightarrow{\bar\varphi} \bar F_0 \xrightarrow{\bar\psi} \bar F_1 \to \dots$

is in $APC(B)$

If $\varphi\psi = x\cdot id_{F_0}$, $\psi\varphi = x\cdot id_{F_1}$ $\Rightarrow$ $(*)$ is a complex of f. gen projective B-modules

for $a \in F_1$, $\varphi(a) = xb \Rightarrow \psi\varphi(a) = x\psi(b) = \varphi(a) = \psi(b)$

similarly $b \in F_0$ $\psi(b) = xa \Rightarrow \psi(b) = \varphi(a)$ $\Rightarrow$ $(*)$ is acyclic.

This gives us the functor a dg category

$MF(A,x) \to APC(B)$ $\qquad \bar{(-)} = -\otimes_A B$

$(F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0) \mapsto (\dots \to \bar F_1 \xrightarrow{\bar\varphi} \bar F_0 \xrightarrow{\bar\psi} \bar F_1 \to \dots)$

etc., inducing the functor

res: $\underline{MF}(A,x) \to \underline{APC}(B)$, compatible with shift)

Thm res is an eq[uivalence] of categories

Pf Each $M \in \mathrm{MCM}_B$ admits an $(F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0)$ in $\mathrm{MF}(A, x)$

with $M \cong \operatorname{coker} \varphi = \Omega_0(\operatorname{res}(F_1 \underset{\psi}{\overset{\varphi}{\rightleftarrows}} F_0))$

and similar for morphisms $M \to N$. This since Ω_0 is an eq[uivalence]

res is essentially surjective and full

faithful: If $f = (f_0, f_1) \in Z^0 \mathrm{MF}(A, x)$ has $\operatorname{res}(f) \underset{h}{\sim} 0$

$: (F_1 \rightleftarrows F_0) \to (F_1' \rightleftarrows F_0')$

Then lifting h to $F_1 \rightleftarrows F_0$ we find $f = f' \cdot x$

$f_1 \downarrow h_0 \swarrow \swarrow h_1 \downarrow f_1$ for $f' = (f_0', f_1')$

$F_1' \rightleftarrows F_0'$ a morphism in $Z^0\mathrm{MF}(A, x)$

But $F_1 \rightleftarrows F_0$ is homotopic to zero

$$\begin{tikzcd} F_1 \arrow[r, shift left] \arrow[d, "x"'] & F_0 \arrow[l, shift left] \arrow[d, "x"] \\ F_1 \arrow[r, shift left] & F_0 \arrow[l, shift left] \end{tikzcd} \qquad (x, x) = d\left(\psi, 0\right)$$

$$\begin{matrix} \| & \| \\ (\varphi\psi, & \psi\varphi) \end{matrix}$$

$\Rightarrow f' x \sim 0$