Applications of compact generation

The main result is to identify $MF^{\infty}(R,W)$ with a category of dg modules over the dg algebra $\mathrm{End}_{MF}(k^{stab})$

§ Homotopy theory of 2-periodic dg category

We have had a review of the model structure on dg-cat$_k$ and a description of the morphism complexes in Ho dg. With suitable adjustment this passes to the setting of 2-periodic dg category

Recall a dg category $\mathcal{A}$ is a category enriched in complexes of k-vector spaces $\mathcal{C}(k)$. A 2-periodic dg category is a category enriched in $\frac{\mathbb{Z}}{2}$-graded complexes. Equivalently a dg category over $k[u,u^{-1}]$ with $|u|=2$, ie

$\mathrm{Hom}(x,y)^{*}$ has an iso $xu : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(X,Y)[2]$

and morphism a compatible with xu.

We have the usual free ⊣ forget pair of adjoint functors

$$\begin{tikzcd} \mathrm{dgcat}_k \arrow[r, shift left, "\otimes k[u,u^{-1}]"] \arrow[r, phantom, "\perp"] & \mathrm{dgcat}_{k[u,u^{-1}]} \arrow[l, shift left, "\text{forget}"] \end{tikzcd}$$

Model structures

via this adjunction the cofibrantly generated model structure on dgcat_k induces one on $\mathrm{dgcat}_{k[u,u^{-1}]}$ ②

fib = f s.t. forget(f) is fibration in dgcat_k

weq = " weq in "

Moreover the generating (trivial) cofibrations in $\mathrm{dgcat}_{k[u,u^{-1}]}$ are the images under $-\otimes k[u,u^{-1}]$ of the generating (trivial) cofibrations in dgcat_k. $\mathrm{dgcat}_{k[u,u^{-1}]}$ has a closed monoidal structure $-\otimes-$, $\underline{\mathrm{Hom}}(-,-)$:

$T\otimes T'$ has objects $\mathrm{Obj}(T)\times\mathrm{Obj}(T')$,

and morphism complexes

$$\mathrm{Hom}_{T\otimes T'}((t,t'),(s,s')) = \mathrm{Hom}_T(t,s)\otimes_{k[u,u^{-1}]}\mathrm{Hom}_{T'}(t',s')$$

$\underline{\mathrm{Hom}}(T,T')$ = dg cat of $C(k[u,u^{-1}])$ enriched functors

with adjunction

$$\mathrm{Hom}(T\otimes T',T'')\cong\mathrm{Hom}(T,\underline{\mathrm{Hom}}(T',T''))$$

Explicitly: $\mathrm{Obj}(\underline{\mathrm{Hom}}(T,T')) = \mathrm{Hom}_{\mathrm{dgcat}_k^{2\pi}}(T,T')$ and for $f_1,f_2 : T\to T'$

$$\mathrm{Hom}_{\underline{\mathrm{Hom}}}(f_1,f_2)^n = \{ \vartheta_t \in \mathrm{Hom}_{T'}(f_1(t),f_2(t))]^n \mid (\vartheta_t)_t \text{ and } (d\vartheta_t)_t \text{ are (graded) natural transformations} \}$$

where $(\vartheta_t\in\mathrm{Hom}_{T'}(f_1(t),f_2(t))^n]_{t\in T}$ is a graded natural transformation if for all $g \in \operatorname{Hom}_T(t,s)^n$, the diagram

$$\begin{tikzcd}
f_1(t) \arrow[r, "\vartheta_t"] \arrow[d, "f_1(g)"'] & f_2(t) \arrow[d, "f_2(g)"] \\
f_1(s) \arrow[r, "\vartheta_s"] & f_2(s)
\end{tikzcd}$$

$(-1)^{mn}$ commutes: $f_2(g)\vartheta_t = (-1)^{mn}\vartheta_s f_1(g)$.

__Definition__ for $T \in \operatorname{dgcat}_k^{\mathbb{Z}/2}$, we set

$$T\text{-mod} := \underline{\operatorname{Hom}}(T, C(k[u,u^{-1}]))$$

__Examples__ (co) representable objects.

The representable T^{op}-module $\underline{h}_x$:

$x \in T \rightsquigarrow \underline{h}_x := T(-,x) \in T^{op}\text{-Mod}$, $y \mapsto T(y,x)$

The corepresentable T-modules $\underline{h}^x$

$x \in T^{op} \rightsquigarrow \underline{h}^x := T(x,-) \in T\text{-Mod}$, $y \mapsto T(x,y)$

This gives the Yoneda embeddings

$$\underline{h}_- : T \to T^{op}\text{-Mod}, \quad \underline{h}^- : T^{op} \to T\text{-mod}$$

Standard model structure on $C(k[u,u^{-1}])$ (as same as $C(k)$)
induces (projective) model structure on T-mod

fibration & weq defined objectwise over T^{op}
generating (trivial) cof = $\underline{h}^i$ i generating (trivial) cofibration in T^{op}

This gives us the associated homotopy category Ho(T-Mod) with localization Ho : T-Mod → Ho(T-Mod)

<u>Def</u> $D(T) := \mathrm{Ho}(T\text{-mod}) \in \mathrm{cat}$

$$\mathrm{Int}(T\text{-mod}) := T\text{-mod}_{fib,cof} \subset T\text{-Mod}$$

We have the functor $H^0 : \mathrm{dgcat}_k \to \mathrm{cat}$. The main theorem of model categories says we have a canonical equivalence $H^0(\mathrm{Int}(T\text{-Mod})) \simeq D(T)$. Note that all obj in T^{op}-mod are fibrant and all representables are cofibrant, so we have

$$\begin{tikzcd} T \arrow[r, "\underline{h}"] \arrow[dr] & \mathrm{Int}(T^{op}\text{-mod}) \arrow[d, phantom, "\cap"] \\ & T^{op}\text{-mod} \end{tikzcd}$$

Moreover $\underline{h}^x$ is compact for all $x \in T$

<u>Def</u> $\hat{T} = \mathrm{Int}(T^{op}\text{-Mod})$ so $H^0(\hat{T}) \simeq D(T^{op}) = \mathrm{Ho}(T^{op}\text{-Mod})$

$$\begin{tikzcd} & \hat{T} \arrow[d, phantom, "\cup"] \\ T \arrow[r, phantom, "\subset"] & \hat{T}_{pe} \end{tikzcd}$$

$\hat{T}_{pe}$ = perfect obj in $\hat{T}$: $M \in \mathrm{Int}(T^{op}\text{-Mod})$ is perfect if $[M]$ is compact in $H^0(\mathrm{Int}(T^{op}\text{-Mod})) = D(T^{op})$

↑
(pre)-triangulated hull of T

<u>Def</u> T is pretriang iff $T \to \hat{T}_{pe}$ is quasi-equiv

Note $H^0(\hat{T})$ is a triang cat with coproducts indexed from $C(k[u,u^{-1}])$
$\overset{\wr}{D}(T^{op})$

$H^0(\hat{T})^c = H^0(\hat{T}_{pe}) \supset H^0(T)$ is set of compact generators since $H^0(T)^\perp = \{0\}$
$\overset{\shortparallel}{D}(T^{op})^c$

Thm (Neeman) Let $\mathcal{T}$ be a triangulated category with a set S compact generators. Then the smallest triangulated subcategory of $\mathcal{T}$ containing S and closed under summands is $\mathcal{T}^c$, the full subcategory of compact objects in $\mathcal{T}$.

Consequence $H^0(\hat{T}_{pe})$ is the smallest triangulated subcategory of $\overset{H^0(\hat{T})}{\overset{\wr}{D}}(T^{op})$ containing $H^0(T)$ and closed under summands. If $H^0(T)$ is already a triangulated subcategory, then $H^0(\hat{T}_{pe})$ is the idempotent completion of $H^0(T)$.

Dg derived category Let W be a set of morphisms in a dg cat T. By Toën $\exists$ a functor

$$l : T \to L_W(T) \quad \text{in } Ho(dgcat_{k[u,u^{-1}]})$$

with the universal property: for $T' \in \mathrm{dgcat}_{k\mathbb{U},i}$ ⑥

$l^*: \mathrm{Map}(L_W(T), T') \to \mathrm{Map}(T,T')$ is injective with image the union of connected components $\mathrm{Map}(T,T')_f$, with $f \in \mathrm{Hom}(T,T')$

$H^0(f)(W) \subset \mathrm{Iso}\, H^0(T')$. $L_W(T) = \mathrm{dg}^{\mathbb{U}}_k$-homotopy pushout in

$$\begin{tikzcd} \coprod_{f\in W} I_k \arrow[r, "i"] \arrow[d] & T, \\ \coprod_{f\in W} k & \end{tikzcd} \qquad i(i_f) = f$$

<u>Def</u> Let $W = \{$w.eqv in T-Mod$\}$ The dg-derived cat of T is $L_W(T\text{-Mod})$

$$\mathrm{Int}(T\text{-Mod}) \xrightarrow{i} T\text{-Mod} \xrightarrow{l} L_W(T\text{-Mod})$$

<u>Prop*</u> The composition is a quasi-equivalence, i.e., an iso in $\mathrm{Ho}(\mathrm{dgcat}^{\mathbb{U}}_k)$. In particular

$$\begin{tikzcd} H^0(\mathrm{Int}(T\text{-Mod})) \arrow[r, "H^0(li)"] \arrow[d, phantom, "\wr"] & H^0(L_W(T\text{-Mod})) \\ D(T) \arrow[ur, "\mathrm{Ho}(l)"'] & \end{tikzcd}$$

are equivalences

pf (kindly supplied by B. Toën) We show more generally: Let C be a dg cat. with a full subcategory C_0. Let W be the set of "C_0-equivalences", i.e. maps $x \to y \in Z^0 C$ such that

$C(z,x) \to C(z,y)$ is a quasi iso for all z in C_0. Assume

That

(*) $\forall x \in C$, $\exists x_0 \in C_0$ and a map $x_0 \to x$ in W.

Then

(**) $C_0 \to C \to L_W C$ is a quasi equivalence.

To prove (**) the inclusion $j: C_0 \to C$ induces the Quillen adjunction of model categories

$$j_! : C_0\text{-Mod} \rightleftarrows C\text{-Mod} : j^*$$

(j^* = restriction functor
$j_!$ = left Kan extension of map induced by j^{op} on representables)

Now apply Bousfield localization of the model cat. C-Mod:
$C\text{-Mod} \to C\text{-Mod}[W^{-1}]$. Since $W|_{C_0}$ are all q iso we have $C_0\text{-Mod} \to C_0\text{-Mod}[W^{-1}]$ is a Quillen equivalence, so we extend the Quillen adjunction to

(1) $j_! : C_0\text{-Mod} \rightleftarrows C\text{-Mod}[W^{-1}] : j^*$

Claim (1) is a Quillen equivalence.

Pf $Lj_! = j_!$ on representables, since these are cofibrant

$j_!(h^{z}) = h^{j(z)}$, so $Lj_!$, $j_!$ are fully faithful on representables.

As left adjoints $Lj_!$, $j_!$ commute with homotopy colimits and

each $M \in C_0$-Mod is a homotopy colimit of representables. Thus $Lj_!$ is
fully faithful.

Next Rj^* is conservative: all $M \in C$-Mod are fibrant, so M is fibrant in $C\text{-Mod}[W^{-1}] \Leftrightarrow M$ sends $f \in W$ to a quasi-iso. If $M(z)$ is quasi-iso to 0 $\forall z \in C_0$ (i.e. $Rj^* M \simeq 0$) then by $(*)$ $M(z')$ is q-iso to 0 $\forall z' \in C$ so $M \simeq 0$. Since $Ho(C_0\text{-Mod})$ & $Ho(C\text{-Mod}[W^{-1}])$ are triangulated & Rj^* is exact, this shows Rj^* is conservative.

By the adjunction $Lj_! \dashv Rj^*$ this shows $Lj_!$ & Rj^* are inverse equivalences. We now apply two facts to finish the proof of $(**)$

Fact 1 A dg functor $C \xrightarrow{j} D$ is a quasi equivalence $\Leftrightarrow$ the induced Quillen adjunction $C\text{-Mod} \underset{j^*}{\overset{j_!}{\rightleftarrows}} D\text{-Mod}$ is a Quillen equivalence and $H^0(C) \to H^0(D)$ is essentially surjective

Fact 2 The map $C \xrightarrow{\ell} L_W C$ (C, W as above) induces a Quillen equivalence $\ell_! \; C\text{-Mod}[W^{-1}] \rightleftarrows L_W C\text{-Mod} : \ell^*$

Indeed the claim + Fact 2 $\Rightarrow$ $C_0 \to L_W C$ induces a Quillen equivalence $C_0\text{-Mod} \rightleftarrows L_W C\text{-Mod}$. By construction $H^0(C_0) \to H^0(L_W C)$ is essentially surjective & Fact 1 finishes the proof of $(**)$

Now apply this to $C_0 = \mathrm{Int}\, T\text{-Mod} \to T\text{-Mod} = C(\mathcal{S}_0)$
Then W is exactly the weqs in T-Mod and (∗) is satisfied since $\mathrm{Int}(T\text{-Mod}) = (T\text{-Mod})^{cof}$, and each $X \in T\text{-Mod}$ has a cofibrant replacement $X_0 \to X$

Proof of Fact 1 It is standard that a quasi equiv $C \xrightarrow{f} D$ Quillen equivalence $C\text{-Mod} \rightleftarrows D\text{-Mod}$ and induces an equiv $H^0(C) \to H^0(D)$ by definition. Conversely, the map $H^0(C) \to H^0(D)$ is an equivalence by definition of quasi equivalence

We have the commutative diag

$$\begin{tikzcd} C^{op} \arrow[r, "f^{op}"] \arrow[d, "h"] & D^{op} \arrow[d, "h"] \\ C\text{-Mod} \arrow[r, "f_!"] & D\text{-Mod} \end{tikzcd}$$

and $f_!$ the left adjoint for a Quillen equiv $\Rightarrow$
$Lf_! : \mathcal{H}_0 C\text{-Mod}(M, N) \to \mathcal{H}_0 D\text{-Mod}(f_! M, f_! N)$
is an iso for all cofibrant M, N (all N, $f_! N$ are fibrant). Apply this to $M = h^x$, $N = h^y$ we see that

$$f_{xy} : C(y,x)^{\vee} = C^{op}(x,y)^{\vee} = C\text{-Mod}(h^x, h^y)^{\vee} \to D\text{-Mod}(h^{f(x)}, h^{f(y)})^{\vee}$$
$$\| \quad D(f(y), f(x))^{\vee}$$

is a quasi-iso $\forall x, y \in C$ so f is a quasi-equiv. □

Proof of Fact 2 For dgcats A, B, we have
$(A \otimes B^{op}\text{-Mod})^{rqr} = \{ M \in A \otimes B^{op}\text{-Mod} \mid \forall e \in A,\ M(e,-) \in B^{op}\text{-Mod}$ is quasi representable$\}$.
Let $\mathbb{R}\underline{\mathrm{Hom}}(A,B) = \mathrm{Int}((A \otimes B^{op}\text{-Mod})^{rqr})$. We have
Thm (Toën) $\mathrm{Map}(A,B) \simeq N((A \otimes B^{op}\text{-Mod})^{rqr}_{qiso})$, where

N = nerve, $(\ )^{gr}_{qis}$ = category of qiso's between objects in $(A\otimes k^{op}\text{-Mod})^{gr}$

Thus the universal property of $\ell: C \to L_W C$ induces a fully faithful embedding $\ell^*: \underline{RHom}(L_W C, D) \to \underline{RHom}(C, D)$ (D a dg cat), with image the $f: C \to D^{op}\text{-Mod}$ such that for $g: c \to d$ in W, $f(g): f(c) \to f(d)$ in $D^{op}\text{-Mod}$ is a q-iso.

Take $D = Int(C(k))$ and use that

$$\underline{RHom}(X, Int(C(k))) \cong Int(X\text{-Mod})$$

for all dg cats X. Thus

$$\underline{RHom}(L_W(C), Int(C(k))) = Int(L_W(C)\text{-Mod}) \subset Int(C\text{-Mod})$$

as the full dg subcategory with objects those $M \in C\text{-Mod}$ such that

(***) $$H^0(M(f)) \in Iso\, H^0(C(k)) = qiso(C(k))$$

for all $f \in W$, since $M: C^{op} \to C(k)$ lifts to $L_W C^{op}$ $\Leftrightarrow$ (***) holds. But (***) is exactly the condition for $M \in C\text{-Mod}$ to be fibrant in $C\text{-Mod}[W^{-1}]$. Thus

$$Int(L_W C\text{-Mod}) \cong C\text{-Mod}[W^{-1}]^{bifib}$$

$$\Rightarrow \ell_!: C\text{-Mod}[W^{-1}] \rightleftarrows L_W C\text{-Mod}: \ell^*$$

is a Quillen equivalence $\square$

Here is dg version of Neeman's theorem ⑨

Recall T is pretriangulated if $T \xrightarrow{h} \hat{T}_{pe}$ is a q. equiv, (iso in Hodg $\Leftrightarrow H^0(T) \xrightarrow{\simeq} H^0(\hat{T}_{pe})$)

Thm 5.7 Suppose $T \in \text{dgcat}_k^{\mathbb{Z}/2}$ is pretriangulated with (small) coproducts, $S \subset T$ a full subcat of objects that are compact in $H^0(T)$. Suppose the smallest localizing subcat of $H^0(T)$ containing $H^0(S)$ is $H^0(T)$. Then

$$f : T \to L_W(S^{op}\text{-mod}) \quad x \mapsto \ell(h_x|_S)$$

is an isomorphism in $\text{Ho}(\text{dgcat}_k^{\mathbb{Z}/2})$ (i.e. quasi-equivalence)

Moreover, f induces an isomorphism $T_{pe} \underset{\text{(in Hodg)}}{\xrightarrow{\sim}} L_{W_S}(S^{op}\text{-mod})_{pe}$

Here $T_{pe} = \{x \in T \mid [x] \text{ is compact in } H^0(T)\}$

Pf follows Keller. The 2nd assertion is a direct consequence, since $f : T \xrightarrow{f} T'$, iso in Hodg, $H^0(f) : H^0(T) \to H^0(T')$ is an equivalence hence induces a equivalence on subcat of compact objects

Notation Let A be $\mathbb{Z}/2$-graded dg algebra/k $\rightsquigarrow$ $A\text{-Mod} \in \text{dgcat}_k^{\mathbb{Z}/2}$

Write $\tilde{A}$ for $\widehat{A\text{-mod}}$, so $H^0(\tilde{A}) = D(A^{op})$

Application to matrix factorization gas

(10)

Theorem 5.2 (R, w) with isolated singularities and let $A = \mathrm{End}_{MF}(k^{stab})$

Then we have iso's in $Ho(dgcat_k^{\mathbb{Z}/2})$

(1) $MF^{\infty}(R,w) \cong \widehat{MF(R,w)}$ $(= \mathrm{Ind}(MF(R,w)^{op}\text{-mod})$

(2) $MF^{\infty}(R,w) \cong \hat{A}$ $(= \mathrm{Ind}(A^{op}\text{-mod}))$

(3) $\widehat{MF(R,w)}_{pe} \cong \tilde{A}_{pe}$

pf By Thm 4.1, k^{stab} is compact generator for $H^0(MF^{\infty}(R,w))$

(2) Take $S = \{k^{stab}\}$ so $S^{op}\text{-Mod} = A^{op}\text{-Mod}$.

We have

$$\begin{tikzcd}
MF^{\infty}(R,w) \arrow[r, "\alpha"] & L_W(A^{op}\text{-Mod}) \\
& \hat{A} = \mathrm{Ind}A^{op}\text{-Mod} \hookrightarrow A^{op}\text{-Mod} \arrow[u, "l"] \arrow[ul, "l_i"]
\end{tikzcd}$$

$x \mapsto MF(k^{stab}, x) \in A^{op}\text{-Mod}$

α is a q.equiv by Theorem 5.1, l is a quasi-equivalence by Prop (*), so we have the iso $(l_i)^{-1}\alpha$ in $Hodg^{\mathbb{Z}/2}$

(1) Take $S = MF(R,w) \cup k^{stab}$, We have a similar diagram

$$\begin{tikzcd} MF^{\infty}(R,w) \arrow[r, "\alpha"] & L_w(MF(R,w)^{op}\text{-Mod}) \\ \widehat{MF(R,w)} = \mathrm{Int}\,MF(R,w)^{op}\text{-Mod} \arrow[ur, "li"] \arrow[r, hook, "i"] & MF(R,w)^{op}\text{-Mod} \arrow[u, "\iota"] \end{tikzcd}$$

with α, li quasi-equivalences

(3) follows from (1) & (2) and the fact that an iso $C \to D$ in $\mathrm{Hodg}_k^{\mathbb{Z}/2}$ between pretriang. objs induces an equivalence $H^0(C) \to H^0(D)$, hence an equiv on the full subcategories of compact objects □

<u>Cor 5.3</u> $H^0(\widehat{MF(R,w)}_{pe}) \subset H^0(MF^{\infty}(R,w))$ is the smallest triangulated subcategory containing k^{stab} and closed under summands

Pf k^{stab} is a compact generator of $H^0(MF^{\infty}(R,w))$, and

$$H^0(\widehat{MF(R,w)}_{pe}) = H^0(\widehat{MF(R,w)})^c \quad H^0(\widehat{MF}(R,w))$$

Then apply Neeman's theorem □.

<u>Cor 5.4</u> Take $X, Y \in MF^{\infty}(R,w)$ and $f \in Z^0 \mathrm{Hom}_{MF}(X,Y)$

Then f induces an iso in $H^0(MF^{\infty}(R,w)) \iff$

$\mathrm{id}_k \otimes f : k \otimes_R X \to k \otimes_R Y$ is a quasi-isomorphism (of $\mathbb{Z}/2$-graded complexes)

Pf By Theorem 5.2 f is an iso in $Ho(MF^{\infty}(R,w))$

$\iff f_* = MF(k^{stab}, f) : MF(k^{stab}, x) \to MF(k^{stab}, y)$ is a quasi iso. By Prop 4.9 $(n = \dim R)$

$$H_{n+i}(MF(k^{stab}, x)) = H_i(k \otimes_S \vec{x}) = \begin{cases} \ker(\bar{\psi}) / \mathrm{im}(\psi) & i > 0 \text{ odd} \\ \ker(\psi) / \mathrm{im}(\bar{\psi}) & i > 0 \text{ even} \end{cases}$$

& same for y. □

Explicit expression for $End(k^{stab})$

For $(R, m = (x_1 \cdots x_n))$ we recall $k^{stab} = \Lambda^{odd}(\oplus_1^n Re_i) \underset{s_0+s_1}{\overset{s_0+s_1}{\rightleftarrows}} \Lambda^{ev}(\oplus_1^n Re_i)$

The dg-algebra $End(k^{stab})$ has the following explicit presentation (assume char $k \neq 2$)

(*) $\Lambda^*(\oplus Re_i) = R\langle \theta_1 \cdots \theta_n \rangle$ with $\deg \theta_i = 1$

Let $\partial_i = \frac{\partial}{\partial \theta_i}$ acting on $R\langle \theta_1 \cdots \theta_n \rangle$ via super Leibniz rule

$\partial_i (ab) = \partial_i a \cdot b + (-1)^{|a|} a \, \partial_i b$, ie $\deg \partial_i = 1$

Let $R\langle \theta_1 \cdots \theta_n, \partial_1 \cdots \partial_n \rangle$ be the correspond (super) Weyl algebra of polynomial differential operators on $R\langle \theta_1 \cdots \theta_n \rangle$

so $[\partial_i, \theta_j]^{gr} = \begin{cases} 0 & i \neq j \\ \mathrm{id} & i = j \end{cases}$

Then via (*) we have $R\langle \theta_1 \cdots \theta_n, \partial_1 \cdots \partial_n \rangle \xrightarrow{\sim} End(k^{stab})$ (dg alg iso)

with differential $[\delta, -]$, $\delta = \sum x_i \partial_i + w_i \theta_i$

Thomas Sebastian Peam $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_m)$ (13.3)

Prop(~~**~~) let $R = k[x]_{(x)}$, $R' = k[y]_{(y)}$ $R\otimes R'_{loc} = k[x,y]_{(x,y)}$

Take $w_x \in (x)^2$, $w_y \in (y)^2$, so $w_x + w_y \in (x,y)^2$

suppose $V(w_x)$, $V(w_y)$, $V(w_x+w_y)$ have isolated singularities

Then $K^{stab}_{w_x+w_y} \simeq K^{stab}_{w_x} \otimes_k K^{stab}_{w_y} \in \mathrm{hf}(k[x,y]_{(x,y)}, w_x+w_y)$

and $A_{w_x+w_y} = (A_{w_x} \otimes_k A_{w_y}) \otimes (R\otimes_k R')_{(x,y)}$

Pf We have

$$K_\infty(R,x) \otimes_k K_\infty(R',y) \cong K_\infty(R\otimes R'_{loc}, (x,y))$$

$K^{stab}_{w_x} = K_\infty(R,x)^{odd} \overset{s_0^x+s_1^x}{\underset{s_0^x+s_1^x}{\rightleftarrows}} K_\infty(R,x)^{eve}$ $K^{stab}_{w_y} = K_\infty(R',y)^{odd} \overset{s_0^y+s_1^y}{\underset{s_0^y+s_1^y}{\rightleftarrows}} K_\infty(R',y)^{ev}$

where $w_x = \sum w_{x_i} x_i$ $w_y = \sum w_{y_i} y_i$

so $w_x+w_y = \sum w_{x_i} x_i + \sum w_{y_i} y_i$ This shows that

$$K^{stab}_{w_x+w_y} = K_\infty(R\otimes R'_{loc}, x,y)^{odd} \overset{s_0+s_1}{\underset{s_0+s_1}{\rightleftarrows}} K_\infty(R\otimes R'_{loc}, x,y)^{ev}$$

$$\begin{tikzcd} \| & & \\ K_\infty(R,x)^{odd} \otimes K_\infty(R',y)^{ev} \arrow[rr, shift left, "s_0+s_1"] & & K_\infty(R,x)^{ev} \otimes K_\infty(R',y)^{ev} \arrow[ll, shift left, "s_0+s_1"] \\ \oplus & & \oplus \\ K_\infty(R,x)^{e} \otimes K_\infty(R',y)^{odd} & & K_\infty(R,x)^{odd} \otimes K_\infty(R',y)^{odd} \end{tikzcd}$$

with $s_0 = \sum x_i \frac{\partial}{\partial e_i} + \sum y_j \frac{\partial}{\partial y_j}$ $s_1 = \left(\sum w_{ix} e_i + \sum w_{jy} f_j \right)_\wedge$ — ⑫

$$\Rightarrow k^{stab}_{w_x+w_y} = k^{stab}_{w_x} \oplus k^{stab}_{w_y}$$

$$A_{w_x+w_y} = \left(R \otimes_k R'_{loc} \langle \theta^x_1 \ldots \theta^x_n, \theta^y_1 \ldots \theta^y_n, \partial^x_1 \ldots \partial^x_n, \partial^y_1 \ldots \partial^y_m \rangle, \delta_{w_x+w_y} \right)$$

By the above computation $\delta_{w_x+w_y} = \delta_{w_x} + \delta_{w_y}$

so $A_{w_x+w_y} = \left(A_{w_x} \otimes_k A_{w_y} \right)_{(x,y)}$

Note in the complete setting, we have

$A_{w_x+w_y} = A_{w_x} \hat{\otimes}_k A_{w_y}$ $\wedge$ = (x,y)-adic completion

§ Knörrer periodicity

We use non-local extension of the above results, assume $V(w)$ has only isolated singularities (or pass always to the localisation of $V(w)_{\mathrm{sing}}$)

Theorem (Knörrer periodicity - local version)

Take (R, w) with $V(w)_{\mathrm{sing}} = \mathrm{Spec}(k)$. Then

$$MF^{\infty}(R, w) \text{ and } MF^{\infty}(R[[x,y]], w+xy)$$

are weakly equivalent

Pf Consider $MF^{\infty}(k[[x,y]], xy)$, $k(0)^{\mathrm{stab}}$ is a compact generator of $H^0 MF^{\infty}(k[[x,y]], xy)$.

let

$$X := \begin{tikzcd} k[[x,y]] \arrow[r, bend left, "x"] & k[[x,y]] \arrow[l, bend left, "y"] \end{tikzcd}$$

Then $k(0)^{\mathrm{stab}} \cong X \oplus X[1]$ in $MF(k[[x,y]], xy)$

$$0 \to k[[x,y]]e_1 \wedge e_2 \xrightarrow{s_0} \begin{matrix} k[[x,y]]e_1 \\ \oplus \\ k[[x,y]]e_2 \end{matrix} \xrightarrow{s_0} k[[x,y]] \to k(0)$$

$$s_1 = \wedge\left(\tfrac{1}{2} y e_1 + \tfrac{1}{2} x e_2\right) \neq (y e_1)\wedge \text{ or } (x e_2)\wedge$$

$$k^{\mathrm{stab}} \quad \begin{matrix} k[[x,y]]e_1 \\ \oplus \\ k[[x,y]]e_2 \end{matrix} \underset{\varphi = s_0 + s_1}{\overset{\psi = s_0 + s_1}{\rightleftarrows}} \begin{matrix} k[[x,y]] \\ \oplus \\ k[[x,y]]e_1 \wedge e_2 \end{matrix} \qquad \psi = \begin{pmatrix} \boxed{x} & y \\ -\frac{1}{2} x & \boxed{\frac{1}{2} y} \end{pmatrix}$$

$$\varphi = \begin{pmatrix} \boxed{\frac{1}{2} y} & +y \\ \frac{1}{2} x & \boxed{-x} \end{pmatrix}$$

$\sim$ [illegible]

$$\begin{tikzcd} \begin{matrix} k[[x,y]]e_1 \\ \oplus \\ k[[x,y]]e_2 \end{matrix} \arrow[r, shift left, "\begin{pmatrix} x & 0 \\ 0 & -y \end{pmatrix}"] \arrow[u] & \begin{matrix} k[[x,y]] \\ \oplus \\ k[[x,y]]e_1 \wedge e_2 \end{matrix} \arrow[l, shift left, "\begin{pmatrix} y & 0 \\ 0 & -x \end{pmatrix}"] \arrow[u, "\begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}"'] \end{tikzcd}$$

so X is also a ^{opt}generator of $H^0(MF^\infty(k[[x,y]], xy))$

let $B = \text{End}(X)$

$$B^0 = \begin{array}{c} \begin{array}{cc} 1 & 0 \end{array} \\ \begin{pmatrix} k[[x,y]] & 0 \\ 0 & k[[x,y]] \end{pmatrix} \end{array} \qquad B^1 = \begin{array}{c} \begin{array}{cc} 1 & 0 \end{array} \\ \begin{pmatrix} 0 & k[[x,y]] \\ k[[x,y]] & 0 \end{pmatrix} \end{array}$$

$d : B^0 \to B^1 \qquad B^1 \xrightarrow{d} B$

$$d\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} 0 & yg-gf \\ xf-gx & 0 \end{pmatrix} = 0 \Rightarrow f = g$$

$$d\begin{pmatrix} 0 & g \\ f & 0 \end{pmatrix} = \begin{pmatrix} fx+yg & 0 \\ 0 & xg+yf \end{pmatrix} = 0 \Rightarrow f = hy, g = xh$$

and $(y^2 - x^2)h = 0$ $\Rightarrow h = 0$

so
$$\begin{tikzcd} k \arrow[r, "1 \mapsto \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)"] \arrow[d, "0"'] & B^0 \arrow[d, "d"] \\ 0 \arrow[r] & B^1 \end{tikzcd}$$
is a iso of dg algebras

$\Rightarrow \tilde{B} \simeq k \simeq \overset{\mathbb{Z}/2}{C}(k)$

By Thom-Sebastiani

$$k^{stab}_{w+xy} \simeq k^{stab}_{w} \otimes k^{stab}_{xy} = k^{stab}_{w} \otimes_k (X \oplus X[1])$$

so $k^{stab}_{w} \otimes_k X$ is a compact generator of $H^0(MF^\infty(k[[x,y]], w+xy))$

Then $\operatorname{End}(k_w^{stab} \otimes X) = \operatorname{End}(k_w^{stab}) \otimes_k \operatorname{End}(X)$

$\simeq_{qis} \operatorname{End}(k_w^{stab}) \otimes_k k = \operatorname{End}(k_w^{stab})$

so $$MF^{\infty}(R[[x,y]], w+xy) \simeq \widehat{\operatorname{End}(k_{w+xy}^{stab})} \simeq \widehat{\operatorname{End}(k_w^{stab})} \simeq MF^{\infty}(R,w)$$

and $$\widehat{MF(R[[x,y]], w+xy)}_{pe} \simeq \widehat{MF(R,w)}_{pe}$$

Formal completion

We take (R,m) local with m-adic completion $(\hat{R}, \hat{m})$

Theorem 5.7 $\widehat{MF(R,w)}_{pe} \underset{\uparrow}{\simeq} MF(\hat{R}, w)$

in Hodg

and the idempotent completion of $H^0(MF(R,w))$ in $H^0(\widehat{MF(R,w)})$ is equivalent to $MF(\hat{R}, w)$

proof relies on

Lemma 5.6 The Yoneda embedding $MF(\hat{R},w) \to \widehat{MF(\hat{R},w)}_{pe}$ is an iso in Hodg, so $MF(\hat{R},w)$ is triangulated and $H^0(MF(\hat{R},w))$ is idempotent complete

Pf $H^0(MF(\hat{R},w))$ $\overset{\text{under Yoneda}}{\text{ }}$ is a set of cpt generators for

$H^0(\widehat{MF(\hat{R},w)})$ so by Neeman's Theorem

$H^0(\widehat{MF(\hat{R},w)}_{pe})$ is the smallest triangulated subcategory of $H^0(\widehat{MF(\hat{R},w)})$ containing $H^0(MF(\hat{R},w))$ and closed under summands

So : suffices to show $H^0(MF(\hat{R},w))$ is closed under summands.

We know that $\underline{MF}(\hat{R},w)$ is equivalent to $\underline{MCM}(\hat{R}/w)$

By a result of Swan, if $M \in Mod(\hat{R})$ is indecomposable then $End(M)$ is local : x,y nonunits $\Rightarrow x+y$ is a nonunit

Take $e \in End(M)$ s.t. the image $\bar{e} \in \underline{End}(M)$ is a nontrivial idempotent. Then $\bar{e}$ and $1-\bar{e}$ are nonunits, hence e & $1-e$ are nonunit in $End(M)$

and $e*(1-e)=1$, a contradiction. Thus $\underline{\mathrm{End}}(M)$ has no ⑰
non-trivial idempotents $\Rightarrow$ $\underline{MCM}(\hat{R}/w)$ is idempotent complete

proof of thm 5.7   let $A_{(R,W)} = \mathrm{End}(k^{stab}_{(R,W)})$, $A_{(\hat{R},W)} = \mathrm{End}(k^{stab}_{(\hat{R},W)})$
By 5.6 and Thm 5.2 we have

$$MF(\hat{R},W) \simeq \widehat{MF(\hat{R},W)}_{pe}$$
$$\simeq \left(\hat{A}_{(\hat{R},W)}\right)_{pe}$$

By Thm 5.2 we have

$$\widehat{MF(R,W)} \simeq \left(\hat{A}_{(R,W)}\right)_{pe}$$

However, $K_\infty(R,b) \otimes_R \hat{R} = K_\infty(\hat{R},b)$ ↗

We have the explicit description of $A_{(R,W)}$ as

$$R\langle \theta_1 \dots \theta_n, \frac{\partial}{\partial\theta_1} \dots \frac{\partial}{\partial\theta_n} \rangle \text{ with differential } [\delta,-]^{gr}$$

$$\delta = \sum x_i \frac{\partial}{\partial\theta_i} + w_i\theta_i \quad \text{where } W = \sum w_i x_i$$

and similar for $A_{(\hat{R},W)}$ We claim the natural

$A_{(R,w)} \hookrightarrow A^a_{(R,w)}$ is a quis of dg algebras

Filter $R(\theta_1 \dots, \frac{\partial}{\partial\theta_1} \dots)$ by $\overset{\text{total}}{\text{deg}}$ in $\frac{\partial}{\partial\theta_i}$

then $\overset{A_{(R,w)}}{gr^F_\cdot}\, A_{(R,w)}$ is $R(\theta_1 \dots \theta_n) \otimes_R R(\frac{\partial}{\partial\theta_1} \dots \frac{\partial}{\partial\theta_n})$

with 0 differential in $\frac{\partial}{\partial\theta_i}$ and the Koszul differen in the θ_i

the E_1 term is then $k\langle \frac{\partial}{\partial\theta_1}, \dots \frac{\partial}{\partial\theta_n}\rangle$ with differentials $[\delta, \cdot]$

the ~~map~~ same holds for $A_{(\hat{R},w)}$ & $A_{(R,w)} \to A_{(\hat{R},w)}$

is compatible with the filtration $\square$

Quadratic hypersurfaces

Take $w = \sum_1^n a_i x_i^2$ $\qquad \prod a_i \neq 0$

$= \sum x_i w_i \qquad w_i = a_i x_i$

Then $A = k(\theta_1 \dots \theta_n, \frac{\partial}{\partial\theta_1} \dots \frac{\partial}{\partial\theta_n})$ with

$d\theta_i = x_i \quad d(\frac{\partial}{\partial\theta_i}) = a_i x_i$ so $\tilde{\partial}_i = \frac{\partial}{\partial\theta_i} - a_i\theta_i$

are cycles and generate $H^0(A)$ as k-algebra

⑲

We have $\bar\partial_i^2 = -a_i$ $\bar\partial_i\bar\partial_j = -\bar\partial_j\bar\partial_i$ for $i \neq j$

so $H^*(A) = Cl(w)$

and $(Cl(w), 0) \hookrightarrow (A^{op}, d)$ is quis (A^{op} is formal)

Thm

$$\underline{\mathrm{MF}}(\hat{R}, w) \simeq H^0(\mathrm{Mod}^{\mathbb{Z}/2}_{Cl(w)})$$

$$\bar\partial_i \cdot \bar\partial_i = (\partial_i - a_i\theta_i)(\partial_i - a_i\theta_i)$$

$$= -a_i\theta_i\partial_i - a_i\partial_i\theta_i + a_i^2\theta_i^2 + \partial_i^2$$

Thom-Sebastiani

$$k^{\mathrm{stab}}_{w\otimes 1 + 1\otimes w'} \simeq k^{\mathrm{stab}}_w \otimes_k k^{\mathrm{stab}}_{w'}$$

$$\Rightarrow A_{w\otimes 1 + 1\otimes w'} \simeq A_w \otimes_k A_{w'}$$

Thm Sebastiani Aaron

~~Lemma~~ $MF^{\infty}(k[[\underline{x},\underline{y}]], w\otimes 1+1\otimes w') = (\widehat{A_w\otimes_k A_{w'}})$

$MF(k[[\underline{x},\underline{y}]], w\otimes 1+1\otimes w') \simeq (\widehat{A_w\otimes_k A_{w'}})_{pe}$

Pf We have

$$A_w\otimes_k A_{w'} = (k[[\underline{x}]][\theta_{x_1},\dots,\theta_{x_n},\frac{\partial}{\partial\theta_{x_1}},\dots,\frac{\partial}{\partial\theta_{x_n}}],[\delta_x,-])\otimes_k(k[[\underline{y}]][\theta_{y_1},\dots,\theta_{y_n},\frac{\partial}{\partial\theta_{y_1}},\dots,\frac{\partial}{\partial\theta_{y_n}}],[\delta_y,-])$$
$$= A_{w\otimes 1+1\otimes w'}$$

Then apply Thm 5.2 and Lemma 5.6

Hochschild cohomology

We recall some results of Toën

- $\otimes^L$: A and B dg categories, $A\otimes^L B := QA\otimes_k B$, where $QA\to A$ is cofibrant replacement in dgcat$_k$. $\otimes^L$ has right adjoint $R\underline{Hom}_{Hodg}(-,-)$:

$$Hom_{Hodg}(A\otimes^L B, C) \cong Hom_{Hodg}(A, R\underline{Hom}_{Hodg}(B,C))$$

If $QB\to B$ is cofibrant replacement in dgcat then $R\underline{Hom}_{Hodg}(B,C)$ is represented by the dg category internal Hom $\underline{Hom}(B,C)$.

Theorem 6.1 (Toën)

$$\mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{Hodg}}(A,B) \simeq \mathrm{Int}\left(\left(A\otimes B^{op}\text{-Mod}\right)^{rqr}_{0}\right)$$

Note Given dg functor $f: A \to B$ the corresponding $A\otimes B^{op}$-module is $M_f(a,b) = \mathrm{Hom}_B(b, f(a))$, $M_f(a,-) = h^{f(a)}$ is representable

- for $M, N \in A$-Mod $\mathbb{R}\mathrm{Hom}_A(M,N)$ is the $\underset{\overset{\wr}{D(k)}}{H^0(C(k))}$ enriched Hom's in $Ho(A\text{-Mod}) = H^0(\mathrm{Int}(A\text{-Mod}))$, Explicitly, for $M \in \mathrm{Int}(A\text{-Mod})$, $N \in A$-Mod $\mathbb{R}\underline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_{A\text{-Mod}}(M,N)$ *

  In general $\mathbb{R}\underline{\mathrm{Hom}}_A(M,N) = \mathrm{Hom}_{A\text{-Mod}}(QM, N)$

- C a dg category, C is a $C\otimes C^{op}$-module by left-right multiplication, making C an object in $C\otimes^{\mathbb{L}} C^{op}$-Mod

Def The Hochschild (cochain) complex of C is

$$HH(C) := \mathbb{R}\underline{\mathrm{Hom}}_{C\otimes^{\mathbb{L}}C^{op}}(C, C) \in D(k)$$

and the Hochschild cohomology of C is

$$HH^i(C) = H^i(HH(C)) \in k\text{-Mod}$$

This all transfers to the 2-periodic setting. We apply to the category MF^∞

Given (R,w), we have $k^{stab} \in MF(R,w)$, $A = \mathrm{End}_{MF}(k^{stab})$

and the genuine equiv $MF^\infty(R,w) \simeq \hat{A} = \mathrm{Int}(A^{op}\text{-Mod})$

in $\mathrm{Hodg}^{\mathbb{Z}/2}_{\mathbb{Z}/2}$

<u>Claim</u> $MF^\infty(R,-w) \simeq \widehat{A^{op}}$. Let $n = \dim R$

$$\begin{tikzcd} V = \bigoplus_i R e_i \quad \text{so} \quad k^{stab}_w = \Lambda^{odd} V \arrow[r, shift left, "s_0+s_1"] & \Lambda^{ev} V \arrow[l, shift left, "s_0+s_1"] \end{tikzcd} \qquad s_0(e_i) = x_i, \quad s_1(1) = \sum w_i \theta_i$$

$w = \sum x_i w_i$ n even

$$\begin{tikzcd} (k^{stab}_w)^\vee = \Lambda^{odd} V \arrow[r, shift right, "(s_0+s_1)^\vee"'] & \Lambda^{ev} V \arrow[l, shift right, "(s_0+s_1)^\vee"'] \end{tikzcd} = k^{stab}_{-w} \text{ in } MF(R,-w)$$

$-(s_0+s_1)$

$\Lambda^n V = R \Rightarrow (\Lambda^{odd} V)^\vee = \Lambda^{odd} V$ $\quad (s_0+s_1)^\vee = s_0+s_1$

$(\Lambda^{ev} V)^\vee = \Lambda^{ev} V$

n odd $(\Lambda^{odd} V)^\vee = \Lambda^{ev} V$, $(\Lambda^{ev} V)^\vee = \Lambda^{odd} V$

$$\begin{tikzcd} k^{stab}_w[1] = \Lambda^{ev} V \arrow[r, shift left, "-(s_0+s_1)"] & \Lambda^{odd} V \arrow[l, shift left] \end{tikzcd}$$

$$\begin{tikzcd} (k^{stab}_w[1])^\vee = \Lambda^{odd} V \arrow[r, shift right, "-(s_0+s_1)^\vee"'] & \Lambda^{ev} V \arrow[l, shift right, "-(s_0+s_1)^\vee"'] \end{tikzcd} = k^{stab}_{-w} \text{ in } MF(R,-w)$$

$-(s_0+s_1) \quad s_0+s_1 = -(s_0+s_1)^\vee$

$(k^{stab}_w)^\vee[1]$ $\qquad -(s_0+s_1)^\vee = -(s_0+s_1)$

Thm $\mathrm{End}_{MF(R,-w)}(k^{stab}_{-w}) = A^{op}$

so $MF^{\infty}(R,-w) = \widehat{A^{op}}$

and $MF^{\infty}(k[[x,y]], w_x - w_y) = \widehat{A \otimes A^{op}}$

(by Thom-Sebastiani)

<u>Def</u> $\Delta^{stab} \in MF(R \hat{\otimes} R, w\otimes 1 - 1\otimes w)$ is the stabilized obj corresponding to R as $R \hat{\otimes} R$ module in $k[[x,y]]/(\dots x_i - y_i \dots)$ as $k[[x,y]]/_{w_x - w_y}$-module

<u>Proposition</u> Under the assumptions

$$\underrightarrow{\mathrm{RHom}}_{\mathrm{Hodg}^{\mathbb{Z}/2}_{k}}(MF^{\infty}(R,w), MF^{\infty}(R,w))$$
$$= \mathrm{Ind}(MF^{\infty}(R,w) \otimes MF^{\infty}(R,w)^{op}\text{-Mod})$$
$$\cong \widehat{A \otimes A^{op}}$$
$$\cong MF^{\infty}(R \hat{\otimes} R, w\otimes 1 - 1\otimes w)$$

Δ^{stab} maps to M_{id} : $M_{id}(a,b) = \mathrm{Hom}_{MF^\infty}(a,b)$

ie, in $H^0(\mathrm{R}\underline{\mathrm{Hom}}(MF^\infty, MF^\infty))$ Δ^{stab} maps to id_{MF^∞}

$$\parallel$$

$$\mathrm{Hom}_{\mathrm{Ho}(dg^{\mathbb{Z}/2}_{k})}(MF^\infty, MF^\infty)$$

Pf By definition $\mathrm{coh}(\Delta^{stab}) \cong R$ in $\underline{\widetilde{\mathrm{MCM}}}(R\tilde{\otimes} R / w\otimes 1 - 1\otimes w)$

By Lemma 21.2, we have the iso in $D(A\otimes A^{op})$

$= \mathrm{Ind}(A\otimes A^{\infty}\text{-Mod})$

$$MF_{w_x - w_y}((k^{stab})^\vee \otimes k^{stab}, \Delta^{stab}) \cong \mathrm{Hom}^{\mathbb{Z}/2}_{R\tilde{\otimes}_k R}((k^{stab})^\vee \otimes k^{stab}, R)$$

$$\cong \mathrm{Hom}^{\mathbb{Z}/2}_{R}(k^{stab}, k^{stab})$$

$$= A$$

On the other hand, the equivalence

$$\mathrm{Ho}(MF^\infty(R\hat{\otimes} R, w\otimes 1 - 1\otimes w)) \xrightarrow{\sim} D(A\otimes A^{op})$$

is given by $\mathrm{Hom}((k^{stab})^\vee \otimes k^{stab}, -) = \mathrm{Hom}(k^{stab}_{w\otimes 1 - 1\otimes w}, -)$

the identity map on $MF^\infty(R,w)$, viewed as an object of $\mathrm{R}\underline{\mathrm{Hom}}(MF^\infty(R,w), MF^\infty(R,w))$ corresponds to

So the object $M_{id} \in D(A\otimes A^{op})$ corresponds to id_A,

ie

$$M_{id}(*,*) \simeq Hom_A(*,*)^* = A,$$

with $A\otimes A^{op}$ module structure $(a\otimes a')f = afa'$,

is the canonical $A\otimes A^{op}$ module structure on A.

From the above we have the isomorphism in $D(A\otimes A^{op})$

$$Hom((k^{stab})^\vee \otimes k^{stab}, -) \cong A$$

so $\Delta^{stab} \in MF^\infty(R\hat{\otimes} R, w\otimes 1 - 1\otimes w)$

maps to $id_{MF^\infty(R,w)}$ under

$$\begin{tikzcd} \underline{RHom}_{Hodg^{\mathbb{Z}/2}_k}(MF^\infty(R,w), MF^\infty(R,w)) \arrow[r, phantom, "\cong"] & \underline{RHom}_{Hodg^{\mathbb{Z}/2}_k}(\tilde{A},\tilde{A}) \arrow[d, phantom, "\wr"] \\ MF^\infty(R\hat{\otimes} R, w\otimes 1 - 1\otimes w)) \cong \widehat{A\otimes A^{op}} & = Int(A\otimes A^{op}\text{-Mod}) \end{tikzcd}$$

□

Write $w_x - w_y = \sum \tilde{w}_i \cdot (x_i - y_i)$, let $V = \bigoplus_i^n \hat{R}\hat{\otimes}\hat{R} e_i$

so $$\Delta^{stab} = \Lambda^{odd} V \underset{s_0+s_1}{\overset{s_0+s_1}{\rightleftarrows}} \Lambda^{ev} V \quad \Delta_0 \quad \text{with } \mathbb{Z}$$

$$s_0 = \sum_{i=1}^n \frac{\partial}{\partial x_i} \qquad s_1 = \left(\sum \tilde{w}_i e_i\right) \wedge -$$

Suppose $V(w)$ has an isolated singularity, Then

<u>Cor 6.4</u>
- $HH(MF^\infty(R,w)) \cong \mathrm{Kos}_R^{\mathbb{Z}/2}\left(\frac{\partial w}{\partial x_1}, \dots, \frac{\partial w}{\partial x_n}\right)$
- $HH^{odd}(MF^\infty(R,w)) = 0$
- $HH^{ev}(MF^\infty(R,w)) = R/\left(\frac{\partial w}{\partial x_1}, \dots, \frac{\partial w}{\partial x_n}\right)$, the Jacobian ring of w.

Pf $HH(MF^\infty(R,w)) = \mathrm{End}_{MF^\infty \otimes MF^{\infty\, op}}(MF^\infty)$

$\cong \mathrm{End}_{\tilde{A} \otimes \tilde{A}^{op}}(\tilde{A}) \cong MF(\Delta^{stab}, \Delta^{stab})_w \cong \mathrm{Hom}^{\mathbb{Z}/2}_{\hat{R}\hat{\otimes}R}(\Delta^{sta}, R)$ (Lemma 4.2)

using $\mathrm{Hom}_{R\otimes R^{op}}(\hat{R}\hat{\otimes}R, R) = R$, this is

$$\Lambda^{odd} \bigoplus_i^n Re_i \underset{\bar{s}_1}{\overset{\bar{s}_1}{\rightleftarrows}} \Lambda^{ev} \bigoplus_i^n Re_i, \quad \bar{s}_1 = s_1 \bmod (\dots x_i - y_i \dots)$$

i.e. the $\mathbb{Z}/2$ complex induced by

$Kos(\tilde{\omega}_1(x,x), \dots \tilde{\omega}_n(x,x))$

Claim $\tilde{\omega}_i(x,y) = \frac{\partial \omega}{\partial x_i}$

pf for $x = (x_1, \dots, x_n)$, let $y^{(i)} = (x_1, \dots x_{i-1}, y_i, x_{i+1}, \dots x_n)$

so $\omega(x) - \omega(y^{(i)}) = \tilde{\omega}_i(x, y^{(i)}) \cdot (x_i - y_i)$

$$\Rightarrow \frac{\omega(x) - \omega(y^{(i)})}{x_i - y_i} = \tilde{\omega}_i(x, y^{(i)}) = \tilde{\omega}_i(x,x) + (x_i - y_i) \cdot e(x, y^{(i)})$$

Also $$\frac{\partial \omega}{\partial x_i}(x) = \frac{\omega(x) - \omega(y^{(i)})}{x_i - y_i} + (x_i - y_i) \cdot f(x, y^{(i)})$$

so $$\frac{\partial \omega}{\partial x_i}(x) = \tilde{\omega}_i(x,x) + (x_i - y_i)(e + f)$$

then set $y_i = x_i \Rightarrow \frac{\partial \omega}{\partial x_i}(x) = \tilde{\omega}_i(x,x)$.

Thus $HH(MF^{\infty}(R, \omega)) = Kos_R^{\mathbb{Z}/2}\left(\frac{\partial \omega}{\partial x_1}, \dots \frac{\partial \omega}{\partial x_n}\right)$

and since $\frac{\partial \omega}{\partial x_1}, \dots, \frac{\partial \omega}{\partial x_n}$ is a regular sequence we have

$$H^i\left(Kos\left(\dots \frac{\partial \omega}{\partial x_i} \dots\right)\right) = \begin{cases} 0 & i \neq 0 \\ R/\left(\partial \omega/\partial x_1 \dots \partial \omega/\partial x_n\right) & i = 0 \end{cases}$$

□