# Talk 4: Classical invariant theory

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## § Motivation:

Let $k=\bar{k}$, $G$ a linear algebraic gp acting on an affine scheme $X$ of finite type over $k$. Let $\mathcal{O}(X) := \mathcal{O}_X(X)$ be the ring of regular functions on $X$.

$$\rightsquigarrow \quad G \circlearrowright \mathcal{O}(X) \quad (g\cdot f)(x) = f(g^{-1}\cdot x) \quad \forall g\in G,\ \forall x \in X.$$

Consider a map

$$\begin{aligned} \varphi : X &\longrightarrow \mathbb{A}^n \\ x &\longmapsto (f_1(x), \dots, f_n(x)) \end{aligned}$$

given by some $G$-inv. functions $f_1, \dots, f_n \in \mathcal{O}(X)^G$.

$f_i$ is $G$-invariant $\Rightarrow$ $f_i$ is constant on $G$-orbits

$\rightsquigarrow$ $\varphi$ sends each orbit to a single point.

Question: Can we choose $f_1, \dots, f_n$ in such a way that $\varphi$ separates the orbits?

Answer: This is often not the case!

e.g.: $\mathbb{G}_m \curvearrowright \mathbb{A}^2 = \operatorname{Spec} k[x,y]$
$(x,y) \longmapsto (tx, t^{-1}y)$ for $t \in \mathbb{G}_m$.

$\rightsquigarrow$ 4 types of orbits:

(a) $\{(0,0)\}$ (closed)
(b) Punctured $x$-axis
(c) Punctured $y$-axis.
(d) Hyperbolas $\{xy = \alpha\}$ for $\alpha \in k^\times$. (closed)

$\mathbb{G}_m \curvearrowright k[x,y]$, $k[x,y]^{\mathbb{G}_m} = k[xy] \simeq k[z]$

$\rightsquigarrow$ $\varphi : \mathbb{A}^2 \longrightarrow \mathbb{A}^1$
$(x,y) \longmapsto xy$.

$\varphi$ separates the orbits (d), yet cannot distinguish the 3 other orbits (a), (b), (c), they all map to $0$!

⚠ We overlooked that $\varphi$ is continuous!
Every fibre of $\varphi$ is a closed set.

So whenever we have non-closed orbits $\Rightarrow$ Answer to question 1 is automatically negative.

## § First notion of quotients :

Let $G$ be a linear algebraic gp acting on a scheme $X$ of finite type / $k = \bar{k}$.

Can always construct a quotient $X \to X/G$ in the category of topological spaces. Relax the idea of having an orbit space to get a quotient with better geometric properties.

$\rightsquigarrow$ Ask for a categorical qt in the category of schemes of finite type / $k$.

**Def:** A categorical quotient for the $G$-action on $X$ is a $G$-invariant morphism $\varphi : X \to Y$ of schemes which is universal, i.e

$$\begin{tikzcd} \varphi: X \arrow[r] \arrow[dr, "G\text{-inv}"'] & Y \arrow[d, dashed, "\exists ! h"] \arrow[dl, phantom, "\circlearrowright"] \\ & Z \end{tikzcd}$$

Further, if the preimage of each $k$-pt in $Y$ is a single orbit, then we say that $\varphi$ is an orbit space.

$\varphi$ continuous + constant on orbits $\Rightarrow$ $\varphi$ is constant on orbit closures.

$\Rightarrow$ Categorical quotient is an orbit space only if $G.x \subset X$ closed $\forall x \in X$.

**Rmk:** If $\varphi: X \to Y$ is $G$-invariant, $Y = \bigcup_i U_i$ s.t $\varphi^{-1}(U_i) \to U_i$ is a cat. quotient $\forall i$

$\Rightarrow$ $\varphi: X \to Y$ is a categorical quotient.

**Example:** $\mathbb{G}_m \circlearrowright \mathbb{A}^n$ $t.(x_1, \dots, x_n) = (tx_1, \dots, tx_n)$. Since the origin is in the closure of every single orbit, any $G$-inv. morphism $\mathbb{A}^n \to Z$ must be constant.

**Claim:** The cat. qt is the structure map $\varphi: \mathbb{A}^n \to \operatorname{Spec} k$.

$\varphi$ is $G$-inv ✓. Any other morphism $f: \mathbb{A}^n \to Z$ is a constant morphism to $z \in Z(k)$. $\rightsquigarrow$ $\exists!\ z: \operatorname{Spec} k \to Z$ s.t $f = z \circ \varphi$

## § Quick recollection on reductive gps: (à l'arrache)

**Def:** Let $G$ be a linear algebraic gp.

- $g \in G$ is unipotent if $\exists$ faithful linear rep $\rho: G \hookrightarrow GL_n$ s.t. $\rho(g)$ is unipotent. A unipotent subgp is a subgp of unipotent elts.

- $R_u(G) :=$ unipotent radical of $G$ = the maximal connected unipotent normal linear alg. subgp. of $G$.

- $G$ is reductive if $R_u(G) = \{1\}$ (e.g. $\mathbb{G}_m$, $GL_n$, $SL_n, \dots$)

• $G$ is linearly reductive if every finite dimens. linear rep

$\rho : G \to GL(V)$ is semisimple. (e.g. $(\mathbb{G}_m)^r$ alg. tori.)

$\iff$ $(-)^G$ is right exact.

For every finite dim $\rho : T = (\mathbb{G}_m)^r \longrightarrow GL(V)$ we have a weight space decomposition

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi$$

where $V_\chi = \{ v \in V : t.v = \chi(t) \cdot v \quad \forall t \in T \}$

Rmk:

$$\text{reductive} \Leftarrow \text{linearly reductive}$$

char $k = 0$
Weyl

§ Second notion of quotients:

Let $G$ be a linearly reductive gp acting on a scheme of finite type $X$.

Def: $\varphi : X \longrightarrow Y$ is a good quotient if
- $\varphi$ is $G$-invariant
- $\varphi$ is affine
- $\mathcal{O}_Y \xrightarrow{\sim} (f_* \mathcal{O}_X)^G$ is an isomorphism

amounts $\varphi$ is a good quotient $\iff$ locally $\operatorname{Spec} A \xrightarrow{\varphi|_U} \operatorname{Spec}(A^G)$

If moreover the preimage of each pt is a single orbit, $\varphi$ is called a geometric qt.

# What is so "good" about good quotients?

**Rem:** If $\varphi : X \to Y$ is $G$-inv, $Y = \bigcup_i U_i$

$\Rightarrow$ $\varphi$ is a good quotient $\Leftrightarrow$ $\varphi_{|\varphi^{-1}(U_i)}$ are good quotients.

**Fact:** $\varphi : X \to Y$ good quotient, $Z_1, Z_2 \subseteq X$ $G$-inv. closed

Then $f(Z_1) \cap f(Z_2) = f(Z_1 \cap Z_2)$.

**Prop:** Good quotients are preserved under arbitrary base change if $G$ is linearly reductive.

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\text{good quotient}"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\text{good quotient}"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

**Cor:** $\varphi : X \to Y$ is surjective.

**Pf:** $y \in Y$ $\quad$ $\{y\} \hookrightarrow Y$ base change $\Rightarrow$ $f^{-1}\{y\} \to \{y\}$ is a good quotient.

$\Rightarrow f^{-1}\{y\} \neq \emptyset$ $\blacksquare$

**Pf of Prop.:**

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "\varphi'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "p\ \text{good}"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

- affine: clear.
- To show: $\mathcal{O}_{Y'} \xrightarrow{\sim} (\varphi'_* \mathcal{O}_{X'})^G$

WLOG: $X = \operatorname{Spec} A \longrightarrow Y = \operatorname{Spec} A^G$ and $Y' = \operatorname{Spec} B$.

**Claim:** $B \xrightarrow{\sim} (B \otimes_{A^G} A)^G$

**Fact:** let $M$ be an $A^G$-module.
$$\Rightarrow \quad M \cong (M \otimes_{A^G} A)^G$$

**Pf:** $E \to F \to M \to 0$ free resolution.

If $N = \bigoplus_i A^G$ (free) $\Rightarrow N \otimes_{A^G} A = \bigoplus_i A^G \otimes_{A^G} A = \bigoplus_i A$

$$\Rightarrow (N \otimes_{A^G} A)^G = N \otimes_{A^G} A^G = N.$$

- $$\begin{tikzcd} E \arrow[r] \arrow[d, "\wr"] & F \arrow[r] \arrow[d, "\wr"] & M \arrow[r] \arrow[d] & 0 \arrow[d, "\wr"] \\ (E \otimes_{A^G} A)^G \arrow[r] & (F \otimes_{A^G} A)^G \arrow[r] & (M \otimes_{A^G} A)^G \arrow[r] & 0 \end{tikzcd}$$

$\rightarrow$ tensor product $+ (-)^G$ are exact.

$\Rightarrow$ 5-Lemma.

**Cor:** $\varphi : X \to Y$ good quotient.

- $\overline{G x_1} \cap \overline{G.x_2} \neq \emptyset \iff \varphi(x_1) = \varphi(x_2)$
- $\forall y \in Y$, $\varphi^{-1}(y)$ contains a unique closed orbit.

If moreover $G \curvearrowright X$ is closed, $\varphi$ is a geometric quotient.

**Pf:** $\varphi$ continuous + constant on orbit closures. $\rightsquigarrow$ "$\Rightarrow$"

"$\Leftarrow$": $\varphi(\overline{Gx_1} \cap \overline{G.x_2}) \overset{\text{Fact}}{=} \varphi(\overline{G.x_1}) \cap \varphi(\overline{G.x_2}) \overset{\text{assumption}}{\neq} \emptyset$.

$$\Rightarrow \overline{G.x_1} \cap \overline{G.x_2} \neq \emptyset$$

- Assume $\varphi^{-1}(y)$ contains 2 distinct orbits $W_1, W_2$

$$\varphi(W_1) = \{y\} = \varphi(W_2)$$

$$\Rightarrow \quad \varphi(W_1 \cap W_2) = \varphi(W_1) \cap \varphi(W_2)$$
$$\phi = \{y\}$$
$$\#$$

**Cor:** $Z \subseteq X$ closed + $G$-inv $\Rightarrow$ $\varphi(Z)$ is closed.

**Pf:** WLOG $X = \operatorname{Spec} A$, $Z = V(I) = \operatorname{Spec} A/I$.

$$\begin{tikzcd}
Z = \operatorname{Spec} A/I \arrow[r, hook] \arrow[d, "\text{surj}"', "\Leftarrow \text{good quotient}"] & X \arrow[d] \\
\operatorname{Spec}(A/I)^G = \operatorname{Spec} A^G/I^G \arrow[r, hook] & \operatorname{Spec} A^G
\end{tikzcd}$$

$\uparrow$ $(-)^G$ exact

$$\Rightarrow \varphi(Z) \text{ is closed.}$$

**Prop:** A good quotient is categorical quotient if $G$ is reductive.

**Pf:** $\varphi$ $G$-inv ✓. ETS universality.

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, "f"', "G\text{-inv}"] & Y \\
Z &
\end{tikzcd}$$

$Z = \bigcup_i U_i$ $\qquad W_i := X - f^{-1}(U_i)$ closed + $G$-inv.

$$\overset{\text{lemma}}{\Longrightarrow} \quad \varphi(W_i) \text{ is closed.}$$

$$V_i := Y - \varphi(W_i) \text{ open } \Rightarrow \varphi^{-1}(V_i) \subset f^{-1}(U_i)$$

$$\overset{U_i \text{ cover}}{\Longrightarrow} \quad \bigcap_i W_i = \emptyset$$

$$\Rightarrow \quad \bigcap_i \varphi(W_i) = \emptyset$$

$$\Rightarrow \quad V_i \text{ open cover of } Y$$

$$\mathcal{O}_Z(U_i) \to \mathcal{O}_X(f^{-1}(U_i))^G \to \mathcal{O}_X(\varphi^{-1}(V_i))^G$$
$$\cong \mathcal{O}_Y(V_i)$$

$$\rightsquigarrow \quad h_i : V_i \to U_i$$

$$\rightsquigarrow \quad f| = h_i \circ \varphi| : \varphi^{-1}(U_i) \to U_i$$

$\rightsquigarrow$ glue to $h: Y \to Z$ s.t. $f = h \circ \varphi$

<u>Fact</u>: indep. choice of affine open cover of $Z$. ∎

## § GIT Quotients in the Affine case:

Let $G$ be a reductive gp $\circlearrowright$ affine scheme (of finite type) $X$.

$$\rightsquigarrow \quad G \circlearrowright \mathcal{O}(X)$$

(Nagata) $\mathcal{O}(X)^G$ is a finitely generated $k$-algebra. (G-reductive)

$\rightsquigarrow$ Spec $\mathcal{O}(X)^G$ is an affine scheme of finite type.

**Def:** The affine GIT quotient is the morphism $\varphi: X \to X /\!/ G := \operatorname{Spec} \mathcal{O}(X)^G$ of affine schemes associated to the inclusion $\varphi^*: \mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$.

(tautologically a good quotient)

e.g. $G = \mathbb{G}_m$, $X = \mathbb{A}^2$,

$$t.(x,y) = (tx, t^{-1}y)$$

$$\rightsquigarrow \quad \mathcal{O}(X)^G = k[x,y]^G = k[xy]$$

$$\Rightarrow \quad Y = \mathbb{A}^1$$

$$\rightsquigarrow \quad \text{GIT qt} \quad \varphi : X \to Y, \ (x,y) \mapsto xy \quad (\text{not geometric!})$$

Rem: $X = \mathbb{A}^4$. $\rho : \mathbb{G}_a \to GL_4$.

$$s \longmapsto \begin{pmatrix} 1 & s & & 0 \\ & 1 & & \\ & & 1 & s \\ 0 & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & s & 0 & 0 \\ 0 & 1 & & \\ & & 1 & s \\ & & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + s x_2 \\ x_2 \\ x_3 + s x_4 \\ x_4 \end{pmatrix}$$

$$\mathbb{C}[x_1, x_2, x_3, x_4]^{\mathbb{G}_a} \cong \underbrace{\mathbb{C}[x_2, x_4, x_1x_4 - x_2x_3]}_{\text{f. gen}}$$

Yet $\varphi : X \to X /\!/ \mathbb{G}_a = \mathbb{A}^3$ is not surjective.

e.g. The punctured line $\{(0,0,\lambda) : \lambda \in k^*\} \not\subset \operatorname{Im} \varphi$.

Note: A geometric qt doesn't always exist.

If $|G| < \infty \Rightarrow$ "good qt = geom. qt".

$\rightsquigarrow$ define open subset $X^s \subseteq X$ for which $\exists$ a geometric quotient!

**Def:** $x \in X$ is <u>stable</u> if $G.x \underset{\text{closed}}{\subset} X$ and

$$\dim G_x = 0.$$

$$X^s := \underline{\text{stable locus}}.$$

**Prop:** $G$ reductive $\circlearrowleft$ affine scheme $X$ (of finite type $/k$)

det. $\varphi : X \to Y := X /\!\!/ G$ (GIT qt)

$\Longrightarrow$ $X^s \subset X$ is <u>open</u> + $G$-inv.

and $Y^s := \varphi(X^s)$ is open in $Y$ and $X^s = \varphi^{-1}(Y^s)$

+ $\underline{\varphi : X^s \longrightarrow Y^s}$ is a geometric <u>qt</u>.

e.g. $G = \mathbb{G}_m$, $X = \mathbb{A}^2$

The closed orbits are $\{xy = \alpha\}$ for $\alpha \in \mathbb{A}^1 \setminus \{0\}$

and the origin.

But $\dim G_{(0,0)} \neq 0$

$$\Longrightarrow \quad X^s = \{(x,y) \in \mathbb{A}^2 : xy \neq 0\} = X_{xy}$$

open!

(Insist on $\dim G_x = 0$ so that $X^s$ is open)