

Talk 4: Classical invariant theory

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§ Notation:

Let $k = \bar{k}$, G a linear algebraic grp acting on an affine scheme X of finite type over k . Let $\mathcal{O}(X) := \mathcal{O}_X(X)$ be the ring of regular functions on X .

$$\leadsto G \curvearrowright \mathcal{O}(X) \quad (g \cdot f)(x) = f(g^{-1} \cdot x) \quad \forall g \in G, \forall x \in X.$$

Consider a map

$$\begin{aligned} \Psi: X &\longrightarrow \mathbb{A}^n \\ x &\longmapsto (f_1(x), \dots, f_n(x)) \end{aligned}$$

given by some G -inv. functions $f_1, \dots, f_n \in \mathcal{O}(X)^G$.

f_i is G -invariant $\Rightarrow f_i$ is constant on G -orbits

$\leadsto \Psi$ sends each orbit to a single point.

Question: Can we choose f_1, \dots, f_n in such a way that Ψ separates the orbits?

Answer: This is often not the case!

e.g.: $G_m \curvearrowright A^2 = \text{Spec } k[x, y]$
 $(x, y) \mapsto (tx, t^{-1}y)$ for $t \in G_m$.

\leadsto 4 types of orbits:

(a) $\{(0,0)\}$ (closed)

(b) Punctured x -axis

(c) Punctured y -axis.

(d) Hyperbolas $\{xy = a\}$ for $a \in k^\times$. (closed)

$$G_m \curvearrowright k[x, y], \quad k[x, y]^{G_m} = k[xy] = k[z]$$

$\leadsto \psi : A^2 \rightarrow A^1$
 $(x, y) \mapsto xy$.

ψ separates the orbits (d), yet cannot distinguish the 3 other orbits (a), (b), (c), they all map to 0!

\triangle We overlooked that ψ is continuous!
 Every fibre of ψ is a closed set.

So whenever we have non-closed orbits \Rightarrow Answer to question 1 is automatically negative.

§ First notion of quotients:

Let G be a linear algebraic gp acting on a scheme X of finite type / $k = \bar{k}$.

Can always construct a quotient $X \rightarrow X/G$ in the category of topological spaces. Relax the idea of having an orbit space to get a quotient with better geometric properties.

\leadsto Ask for a categorical qt in the category of schemes of finite type / k .

Def: A categorical quotient for the G -action on X

is a G -invariant morphism $\varphi: X \rightarrow Y$ of schemes

which is universal, i.e

$$\begin{array}{ccc} \varphi: X & \longrightarrow & Y \\ & \searrow & \uparrow \exists! h \\ & G\text{-inv} & Z \end{array}$$

Further, if the preimage of each k -pt in Y is a single orbit, then we say that φ is an orbit space.

φ continuous + constant on orbits $\Rightarrow \varphi$ is constant on orbit closures.

\Rightarrow Categorical quotient is an orbit space only if $G \cdot x \subset X$ closed $\forall x \in X$.

Prmk: If $\varphi: X \rightarrow Y$ is G -invariant, $Y = \bigcup_i U_i$ s.t. $\varphi^{-1}(U_i) \rightarrow U_i$ is a cat. quotient $\forall i$
 $\Rightarrow \varphi: X \rightarrow Y$ is a categorical quotient.

Example: $G_m \curvearrowright \mathbb{A}^n$ t. $(x_1, \dots, x_n) = (tx_1, \dots, tx_n)$. Since the origin is in the closure of every single orbit, any G -inv. morphism $\mathbb{A}^n \rightarrow \mathbb{Z}$ must be constant.

Claim: The cat. qt is the structure map $\varphi: \mathbb{A}^n \rightarrow \text{Spec } k$.

φ is G -inv. \checkmark . Any other morphism $f: \mathbb{A}^n \rightarrow \mathbb{Z}$ is a constant morphism to $z \in \mathbb{Z}(k)$. $\leadsto \exists ! z: \text{Spec } k \rightarrow \mathbb{Z}$ s.t. $f = z \circ \varphi$

Quick recollection on reductive gps: (à l'arrache)

Def. Let G be a linear algebraic gp.

• $g \in G$ is unipotent if \exists faithful linear rep $\rho: G \hookrightarrow GL_n$ s.t. $\rho(g)$ is unipotent. A unipotent subgp is a subgp of unipotent elts.

• $R_u(G) :=$ unipotent radical of G = the maximal

connected unipotent normal linear alg. subgp. of G .

• G is reductive if $R_u(G) = \{1\}$ (e.g. O_n, GL_n, SL_n, \dots)

• G is linearly reductive if every finite dimens. linear rep

$\rho: G \rightarrow GL(V)$ is semisimple. (e.g. $(G_m)^r$ alg. tori.)

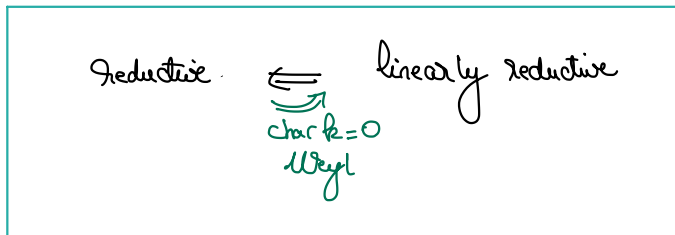
$\Leftrightarrow (-)^G$ is right exact.

For every finite dim $\rho: T = (G_m)^r \rightarrow GL(V)$ we have a weight space decomposition

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi$$

where $V_\chi = \{v \in V : t \cdot v = \chi(t) \cdot v \ \forall t \in T\}$

Prmk:



§ Second notion of quotients:

Let G be a linearly reductive gp acting on a scheme of finite type X .

Def: $\varphi: X \rightarrow Y$ is a good quotient if

- φ is G -invariant
- φ is affine
- $\mathcal{O}_Y \xrightarrow{\sim} (f_* \mathcal{O}_X)^G$ is an isomorphism

φ is a good quotient \Leftrightarrow locally $\text{Spec } A \xrightarrow{f} \text{Spec } (A^G)$

If moreover the preimage of each pt is a single orbit, φ is called a geometric qt.

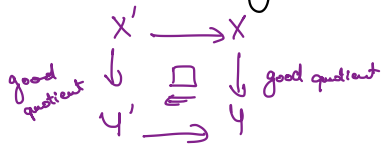
What is so "good" about good quotients?

Def: If $\varphi: X \rightarrow Y$ is G -inv, $Y = \bigcup U_i$
 $\Rightarrow \varphi$ is a good quotient $\Leftrightarrow \varphi|_{\varphi^{-1}(U_i)}$ are good quotients.

Fact: $\varphi: X \rightarrow Y$ good quotient, $Z_1, Z_2 \subseteq X$ G -inv. closed

Then $f(Z_1) \cap f(Z_2) = f(Z_1 \cap Z_2)$.

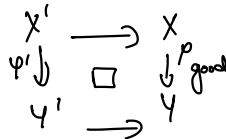
Prop: Good quotients are preserved under arbitrary base change if G is linearly reductive.



Cor: $\varphi: X \rightarrow Y$ is surjective.

Pr: $y \in Y$ $\{y\} \hookrightarrow Y$ base change $\Rightarrow f^{-1}\{y\} \rightarrow \{y\}$ is a good quotient.
 $\Rightarrow f^{-1}\{y\} \neq \emptyset$

Pr of Prop.:



• affine: clear.

• to show: $\mathcal{O}_{Y'} \xrightarrow{\sim} (\varphi'_* \mathcal{O}_{X'})^G$

WLOG: $X = \text{Spec } A \longrightarrow Y = \text{Spec } A^G$ and $Y' = \text{Spec } B$.

Claim: $B \xrightarrow{\sim} (B \otimes_{A^G} A)^G$

Fact: let M be an A^G -module.

$$\Rightarrow M \cong (M \otimes_{A^G} A)^G$$

P: $E \rightarrow F \rightarrow M \rightarrow 0$ free resolution.

$$\text{If } N = \bigoplus_i A^G \text{ (free)} \Rightarrow N \otimes_{A^G} A = \bigoplus_i A^G \otimes_{A^G} A = \bigoplus_i A$$

$$\Rightarrow (N \otimes_{A^G} A)^G = N \otimes_{A^G} A^G = N.$$

$$\begin{array}{ccccccc} \bullet & E & \rightarrow & F & \rightarrow & M & \rightarrow 0 \\ & \downarrow \wr & & \downarrow \wr & & \downarrow & \downarrow \wr \\ & (E \otimes_{A^G} A)^G & \rightarrow & (F \otimes_{A^G} A)^G & \rightarrow & (M \otimes_{A^G} A)^G & \rightarrow 0 \end{array}$$

\rightarrow tensor product + $(-)^G$ are exact.

\Rightarrow 5-lemma.

Cor: $\Psi: X \rightarrow Y$ good quotient.

- $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset \Leftrightarrow \Psi(x_1) = \Psi(x_2)$
- $\forall y \in Y, \Psi^{-1}(y)$ contains a unique closed orbit.

If moreover $G \curvearrowright X$ is closed, Ψ is a geometric quotient.

P: Ψ continuous + constant on orbit closures. $\leadsto \Rightarrow$

$$\Leftarrow: \Psi(\overline{G \cdot x_1} \cap \overline{G \cdot x_2}) \stackrel{\text{Fact}}{=} \Psi(\overline{G \cdot x_1}) \cap \Psi(\overline{G \cdot x_2}) \stackrel{\text{assumption}}{\neq} \emptyset.$$

$$\Rightarrow \overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$$

• Assume $\varphi^{-1}(y)$ contains 2 distinct orbits ω_1, ω_2
 $\varphi(\omega_1) = \{y\} = \varphi(\omega_2)$

$$\Rightarrow \varphi(\omega_1 \cap \omega_2) = \varphi(\omega_1) \cap \varphi(\omega_2)$$

$$\emptyset = \{y\} \quad \#$$

Cor: $Z \subseteq X$ closed + G -inv $\Rightarrow \varphi(Z)$ is closed.

Prf: WLOG $X = \text{Spec } A$, $Z = V(I) = \text{Spec } A/I$.

$$\begin{array}{ccc} Z = \text{Spec } A/I & \longleftrightarrow & X \\ \swarrow \text{good quotient} & & \downarrow \\ \text{Spec } (A/I)^G & \longleftrightarrow & \text{Spec } A^G \\ \uparrow (-)^G \text{ exact} & & \end{array}$$

$\Rightarrow \varphi(Z)$ is closed.

Prop: A good quotient is categorical quotient if G is reductive.

Prf: φ G -inv \checkmark . ETS universality.

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ f \downarrow G\text{-inv} & & \\ Z & & \end{array}$$

$$Z = \bigcup_i U_i \quad \omega_i := X - f^{-1}(U_i) \text{ closed. + } G\text{-inv.}$$

$\xrightarrow{\text{lemma}} \varphi(\omega_i) \text{ is closed.}$

$$V_i := Y - \Psi(W_i) \text{ open} \Rightarrow \Psi^{-1}(V_i) \subset f^{-1}(U_i)$$

U_i cover
 \Rightarrow

$$\bigcap_i W_i = \emptyset$$

\Rightarrow

$$\bigcap_i \Psi(W_i) = \emptyset$$

\Rightarrow

V_i open cover of Y

$$\mathcal{O}_Z(U_i) \rightarrow \mathcal{O}_X(f^{-1}(U_i))^G \rightarrow \mathcal{O}_X(\Psi^{-1}(U_i))^G \cong \mathcal{O}_Y(V_i)$$

$$\leadsto h_i : V_i \rightarrow U_i$$

$$\leadsto f_i = h_i \circ \Psi_i : \Psi^{-1}(U_i) \rightarrow U_i$$

\leadsto glue to $h : Y \rightarrow Z$ s.t. $f = h \circ \Psi$

Fact: indep. choice of affine open cover of Z . ▣

GIT Quotients in the Affine case

Let G be a reductive gp \curvearrowright affine scheme (of finite type) X .

$$\leadsto G \curvearrowright \mathcal{O}(X)$$

(Nagata) $\mathcal{O}(X)^G$ is a finitely generated k -algebra. (G -reductive)
 $\leadsto \text{Spec } \mathcal{O}(X)^G$ is an affine scheme of finite type.

Def: The affine GIT quotient is the morphism
 $\Psi : X \rightarrow X // G := \text{Spec } \mathcal{O}(X)^G$ of affine schemes associated to the inclusion $\Psi^* : \mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$.

(tautologically a good quotient)

e.g. $G = G_m, X = A^2,$

$$t \cdot (x, y) = (tx, t^{-1}y)$$

$$\leadsto \mathcal{O}(X)^G = k[x, y]^G = k[x, y]$$

$$\Rightarrow Y = A^1$$

$$\leadsto \text{GIT qt } \varphi: X \rightarrow Y, (x, y) \mapsto xy \quad (\text{not geometric!})$$

Rev: $X = A^4, \ell: G_a \rightarrow G_4.$

$$s \mapsto \begin{pmatrix} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + sx_2 \\ x_2 \\ x_3 + sx_4 \\ x_4 \end{pmatrix}$$

$$\mathbb{C}[x_1, x_2, x_3, x_4]^{G_a} \cong \mathbb{C}[x_2, x_4, x_1x_4 - x_2x_3]$$

f. gen

yet $\varphi: X \rightarrow X // G_a = A^3$ is not surjective.

e.g. The punctured line $\mathbb{A}^1 \setminus \{0, 1\} \subset \text{Im } \varphi.$

Note: A geometric qt doesn't always exist.

If $|G| < \infty \Rightarrow$ "good qt = geom. qt".

\leadsto define open subset $X^s \subseteq X$ for which \exists a geometric quotient!

Def. $x \in X$ is stable if $G \cdot x \subset X$ and closed
 $\dim G_x = 0$.

$X^s :=$ stable locus.

Prop. G reductive \odot affine scheme X (of finite type/k)

def. $\varphi: X \rightarrow Y := X // G$ (GIT qt)

$\Rightarrow X^s \subset X$ is open + G -inv.

and $Y^s := \varphi(X^s)$ is open in Y and $X^s = \varphi^{-1}(Y^s)$

+ $\varphi: X^s \rightarrow Y^s$ is a geometric qt.

e.g. $G = G_m, X = \mathbb{A}^2$

The closed orbits are $\{xy = \alpha\}$ for $\alpha \in \mathbb{A}^1 \setminus \{0\}$
 and the origin.

But $\dim G_{(0,0)} \neq 0$

$\Rightarrow X^s = \{(x,y) \in \mathbb{A}^2 : xy \neq 0\} = X_{xy}$
open!

(Insist on $\dim G_x = 0$ so that X^s is open)