§ The Hilbert-Mumford criterion.

• **Recall:** G red. group acting on $X = \mathrm{Proj}\left(\bigoplus_n H^0(X, \mathcal{L}^{\otimes n})\right)$, $\mathcal{L}$ ample linearisation of this action.

We constructed the GIT quotient
$$X/\!\!/_{\mathcal{L}} G = \mathrm{Proj}\left(\bigoplus_n H^0(X, \mathcal{L}^{\otimes n})^G\right)$$

If we want to check if some $x \in X$ is semistable, we would need to find some $f \in \bigoplus H^0(X, \mathcal{L}^{\otimes n})^G$ s.t. $f(x) \neq 0$.

• **Problem:** $\bigoplus_n H^0(X, \mathcal{L}^{\otimes n})^G$ is difficult to compute!

Our objective is to provide a criterion to check semistability in a much simpler way.

§ 1. The Hilbert-Mumford criterion.

• Let X be a proj. variety with an action by some red. gp. G, and $\mathcal{L}$ an ample linearisation.

• **Recall:** Last time, we saw that, for any $k \geq 2$,
$$X^{ss}(\mathcal{L}) = X^{ss}(\mathcal{L}^{\otimes k}), \quad X^{s}(\mathcal{L}) = X^{s}(\mathcal{L}^{\otimes k}).$$
Thus, choosing k s.t. $\mathcal{L}^{\otimes k}$ is very ample, we can consider the corresponding embedding $X \hookrightarrow \mathbb{P}^n$, and use the affine cone and other machinery to prove (semi) stability of points in X for $\mathcal{L}$.

· <u>Prop. 1.</u> Let $\tilde{x} \in \tilde{X}$ be a point lying over x. Then:

(i) x is semistable iff $0 \notin \overline{G \cdot \tilde{x}}$.

(ii) x is stable iff $\dim \underbrace{G_{\tilde{x}}}_{\text{stabilizer of } \tilde{x}} = 0$ and $G \cdot \tilde{x}$ is closed in $\tilde{X}$.

<u>Proof</u>

(i) <u>$\Rightarrow$</u> If x is semistable, $\exists f \in R(X)^G$ homogeneous and invariant pol. s.t. $f(x) \neq 0$. Since we can see f as a G-inv. function on $\tilde{X}$, then also $f(\tilde{x}) \neq 0$. Because f is continuous and invariant, it follows that $f(\overline{G \cdot \tilde{x}}) \neq 0$. Of course, because f is homogeneous, this means that 0 and $\overline{G \cdot \tilde{x}}$ are closed subvarieties separated by a function, and thus are disjoint.

<u>$\Leftarrow$</u> If $\overline{G \cdot \tilde{x}}$ and 0 are disjoint, then there exists some G-inv. polynomial $f \in A(\tilde{X})^G$ s.t.

$$f(\overline{G \cdot \tilde{x}}) = 1, \quad f(0) = 0.$$

Of course, f is a sum of G-inv. hom. polynomials f_r, and since $f(\overline{G \cdot \tilde{x}}) \neq 0$, $f_r(\overline{G \cdot \tilde{x}}) \neq 0$ for some r. But then, $f_r(x) = f_r(\tilde{x}) \neq 0$, showing that x is semistable (since f_r is non-const., G-inv., hom.).

(ii) <u>$\Rightarrow$</u> If x is stable, then $\dim G_x = 0$ and $\exists$ G-inv. homog. pol. $f \in R(X)^G$ s.t. $x \in X_f$ and $G \cdot x$ is closed in X_f. Since $G_{\tilde{x}} \subseteq G_x$, the stabiliser of $\tilde{x}$ is also 0-dim. Seeing f as a function on $\tilde{X}$, we consider the closed subvariety

$$Z := \{ z \in \tilde{X} \mid f(z) = f(\tilde{x}) \}.$$

Thus, it is ETS that $G \cdot \tilde{x}$ is closed in Z. The projection map $\tilde{X} \setminus \{0\} \to X$ restricts to a surj. finite morphism $\pi : Z \to X_f$. The preimage of $G \cdot x$ under π is closed and G-inv. Because π is finite, $\pi^{-1}(G \cdot x)$ is a finite number of G-orbits, which all lie over $G \cdot x$ and thus have dimension equal to $\dim G$. Therefore, they must be closed, or otherwise they would contain lower dimensional orbits in their closure, and so $G\cdot\tilde{x} \subseteq \pi^{-1}(G\cdot x)$ is closed in Z.

<u>$\Leftarrow$]</u> We assume $\dim G_{\tilde{x}} = 0$ and $G\cdot\tilde{x}$ is closed in $\tilde{X}$. Then, $0 \notin \overline{G\cdot\tilde{x}} = G\cdot\tilde{x} \Rightarrow x$ is semistable by (i). Thus, $\exists f$ hom. pol. non-const. and G-inv. s.t. $f(x) \neq 0$. As earlier, define

$$Z := \{ z \in \tilde{X} \mid f(z) = f(\tilde{x}) \}$$

and the finite surj. morphism $\pi : Z \to X_f$. Since $\pi(G\cdot\tilde{x}) = G\cdot x$, x must have fin. dim. stabiliser and $G\cdot x$ must be closed in X_f. Since f was arbitrary, $G\cdot x$ is closed in $\bigcup_f X_f = X^{ss}$, so it is stable. $\boxtimes$

· <u>Remark:</u> as previously mentioned, this result holds for any X linearised by an ample line bundle, since the result holds for $X^{(s)s}(\mathcal{L}^{\otimes k})$ with $\mathcal{L}^{\otimes k}$ very ample, and $X^{(s)s}(\mathcal{L}) = X^{(s)s}(\mathcal{L}^{\otimes k})$.

· <u>Example 2.</u> Over $\mathbb{C}$, $\mathbb{G}_m \curvearrowright \mathbb{P}^1$ by $t\cdot[x:y] = [tx : t^{-1}y]$.
The affine cone is $\mathbb{A}^2$, and has an action $t\cdot(x,y) = (tx, t^{-1}y)$.
As we have already seen, the orbits are:

- ·) conics $\{xy = \alpha\}$ for $\alpha \in \mathbb{C}^\times$,
- ·) punctured x-axis,
- ·) " y-axis,
- ·) the origin.

Looking at the closures, we have that:

- ·) the orbits $\{xy = \alpha\}$ are closed of dim. 1 and don't contain 0,
- ·) the closures of the other three orbits contain 0.

Therefore, by Prop. 1,

$$(\mathbb{P}^1)^{ss} = \{ [1:\alpha] \mid \alpha \in \mathbb{C}^\times \},$$
$$(\mathbb{P}^1)^{s} = \{ [1:\alpha] \mid \alpha \in \mathbb{C}^\times \}.$$

· **Definition 3.** A 1-parameter subgroup (1-PS) of G is a nontrivial gp. homomorphism

$$\lambda : \mathbb{G}_m \longrightarrow G.$$

· For any 1-PS λ and any $x \in \overline{X}$, we get a morph.

$$\begin{aligned} \lambda(-)\cdot x : \mathbb{G}_m &\longrightarrow \overline{X} \\ t &\longmapsto \lambda(t)\cdot x \end{aligned}$$

But $\overline{X}$ is a complete variety (since it is projective), and thus this map extends to a map $\mathbb{P}^1 \to \overline{X}$ (seeing $\mathbb{G}_m \hookrightarrow \mathbb{P}^1$ via $t \mapsto [1:t]$). If we write 0 for $[1:0]$ and ∞ for $[0:1]$, we write

$$\lim_{t\to 0} \lambda(t)\cdot x \quad \text{and} \quad \lim_{t\to\infty} \lambda(t)\cdot x$$

for the images of 0 and ∞ (respectively) under the map $\mathbb{P}^1 \to \overline{X}$. In particular, let $x_0 := \lim_{t\to 0} \lambda(t)\cdot x$. Clearly, x_0 is fixed by the action of $\lambda(\mathbb{G}_m)$, meaning that $\mathbb{G}_m$ acts on the fibre $\mathcal{L}_{x_0}$ by a character $t \mapsto t^r$ of $\mathbb{G}_m$, for some $r \in \mathbb{Z}$.

· <u>Definition 4.</u> r is the <u>weight of the λ-action</u> on $\mathcal{L}_{x_0}$, and we set

$$\mu^{\mathcal{L}}(x,\lambda) := r.$$

· <u>Example 5.</u> Consider the case when $\mathcal{L}$ is very ample and we have an embedding $\varphi: \bar{X} \hookrightarrow \mathbb{P}^n$.

The action $\lambda(-)$ induces an action on $\mathbb{P}^n$, and thus on $\mathbb{A}^{n+1}$.

Since G acts linearly (because $\mathcal{L}$ was a linearization), we can actually diagonalise this action: $\exists$ a basis $\{e_0, \dots, e_n\}$ of $\mathbb{A}^{n+1}$ s.t. $\lambda(t)\cdot e_i = t^{r_i} e_i$ for some $r_i \in \mathbb{Z}$.

If $\tilde{x} \in \tilde{X}$ lies over $x \in \bar{X}$, then $\tilde{x} = \sum_{i=0}^{n} a_i e_i$ for some $a_i \in \mathbb{C}$, and

$$\lambda(t)\cdot \tilde{x} = \sum_{i=0}^{n} t^{r_i}\cdot a_i \cdot e_i .$$

We define $\mu(x,\lambda) := -\min\{r_i : a_i \neq 0\}$. We claim that $\mu^{\mathcal{L}}(x,\lambda) = \mu(x,\lambda)$.

in the base induced by $\{e_0, \dots, e_n\}$

To see this, let $x_0 := \lim_{t\to 0} \lambda(t)\cdot x$ and assume $x_0 = [b_0 : \dots : b_n]$ via φ. Then,

$$b_i = \begin{cases} a_i & \text{if } r_i = -\mu(x,\lambda), \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, we must have that $\lambda(t)\cdot x_0 = x_0$, and since $\lambda(t)$ acts via

$$\lambda(t)\cdot[b_0 : \dots : b_n] = [t^{r_0} b_0 : \dots : t^{r_n} b_n],$$

in order for this to be equal to $[b_0 : \dots : b_n]$ for all t, we must have only the b_i's with the smallest r_i, and the rest must be zero. From the definition of x_0, it's clear that the non-zero b_i's are equal to a_i.

Thus, if $\tilde{x}_0 = (b_0, \dots, b_n)$ is a point over x_0, we have that

$$\lambda(t) \cdot \tilde{x}_0 = t^{-\mu(x,\lambda)} \tilde{x}_0 .$$

Now, the tautological line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ over $\mathbb{P}^n$ has as fibers over the point x_0 the line $\{\tilde{x}_0 \in \mathbb{A}^{n+1} \mid \tilde{x}_0 \text{ maps to } x_0\}$. Thusly, $\lambda(\mathbb{G}_m)$ acts on such a fiber by a character $t \longmapsto t^{-\mu(x,\lambda)}$.
Since $\mathcal{O}_{\mathbb{P}^n}(1)$ is the dual line bundle, then the action of $\lambda(\mathbb{G}_m)$ over the fiber over x_0 by $t \longmapsto t^{\mu(x,\lambda)}$. Therefore, because $\mathcal{L} \cong \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$,

$$\mu^{\mathcal{L}}(x,\lambda) = \mu(x,\lambda).$$

• Proposition 6.

(1) $\mu^{\mathcal{L}}(x,\lambda)$ is the unique $\mu \in \mathbb{Z}$ s.t. $\lim_{t\to 0} (t^{\mu} \lambda(t) \cdot x)$ exists and is non-zero.

(2) $\mu^{\mathcal{L}}(x, \lambda^n) = n \cdot \mu(x,\lambda)$ for $n \in \mathbb{Z}_{>0}$.

(3) $\mu^{\mathcal{L}}(g\cdot x, g\lambda g^{-1}) = \mu^{\mathcal{L}}(x,\lambda)$ $\forall g \in G$.

(4) $\mu^{\mathcal{L}}(x,\lambda) = \mu^{\mathcal{L}}(x_0,\lambda)$, for $x_0 := \lim_{t\to 0} \lambda(t)\cdot x$.

(5) $\mu^{\mathcal{L}^{\otimes n}}(x,\lambda) = n \cdot \mu^{\mathcal{L}}(x,\lambda)$ for all $n \in \mathbb{Z}_{>0}$.

Proof | Omitted. $\boxtimes$

• Remark 7. Again, if $X \subseteq \mathbb{P}^n$ we have an easier criterion for the sign of $\mu(x,\lambda)$.

•) $\mu(x,\lambda) < 0$ if and only if

$$\tilde{x} = \sum_{\substack{i=0 \\ r_i > 0}}^{n} a_i e_i$$ → same notation as Example 5

if and only if the limit $\lim_{t\to 0}(\lambda(t)\cdot\tilde{x})$ exists in $\tilde{X}$ and is zero.

•) $\mu(x,\lambda)=0$ if and only if $r_{i_0}=0$ for some i_0 and

$$\tilde{x} = a_{i_0}e_{i_0} + \sum_{\substack{i=0\\ i\neq i_0\\ r_i>0}}^{n} a_i e_i$$

with $a_{i_0}\neq 0$. This holds if and only if $\lim_{t\to 0}(\lambda(t)\cdot\tilde{x})$ exists and is equal to $a_{i_0}e_{i_0}\neq 0$.

•) $\mu(x,\lambda)>0$ if and only if $\lim_{t\to 0}(\lambda(t)\cdot\tilde{x})$ doesn't exist.

If we study λ^{-1} instead of λ, we get the converse results but with

$$\lim_{t\to 0}(\lambda^{-1}(t)\cdot\tilde{x}) = \lim_{t\to\infty}(\lambda(t)\cdot\tilde{x}).$$

• Theorem 8 (Hilbert–Mumford criterion). Let G be a red. gp. acting on a proj. var. X with a linearisation $\mathcal{L}$. Then,

$$x\in X^{ss}(\mathcal{L}) \iff \mu^{\mathcal{L}}(x,\lambda)\geq 0 \quad \forall\lambda \text{ 1-PS of } G,$$
$$x\in X^{s}(\mathcal{L}) \iff \mu^{\mathcal{L}}(x,\lambda)> 0 \quad \forall\lambda \text{ 1-PS of } G.$$

Proof| First, we observe that, for all $k\geq 1$,

$$X^{(s)s}(\mathcal{L}) = X^{(s)s}(\mathcal{L}^{\otimes k}),$$

and by Prop. 6 (5), $\mu^{\mathcal{L}^{\otimes k}}(x,\lambda) = k\cdot\mu^{\mathcal{L}}(x,\lambda)$. Since multiplying by k does not change the sign, this means that both sides of the equivalence hold for the linearisation $\mathcal{L}$ if and only if they hold for $\mathcal{L}^{\otimes k}$. We thus assume that $\mathcal{L}$ is very ample.

$\underline{\Rightarrow)}$ If x is (semi)stable for G, then it is (semi)stable for all $H \leq G$ (since G-invariant $\Rightarrow$ H-invariant).

Thus, if $x \in X^{(s)s}(\mathcal{L})$, then x is (semi)stable for the action of $\lambda(\mathbb{G}_m)$ $\forall \lambda$ 1-PS of G.

If x is semistable, then by Prop. 1, $0 \notin \overline{\lambda(\mathbb{G}_m)\cdot\tilde{x}}$ for any $\tilde{x} \in \tilde{X}$ over x. But then, this means that the limit $\lim_{t\to 0}(\lambda(t)\cdot\tilde{x})$ cannot exist and be equal to zero. By Remark 7,

$$\mu(x,\lambda) \geq 0.$$

On the other hand, if x is stable, then again by Prop. 1 we have that $\mu(x,\lambda)\geq 0$. We claim that the limit $\lim_{t\to 0}(\lambda(t)\cdot\tilde{x})$ cannot exist for any $\tilde{x}\in\tilde{X}$ over x.

If $\tilde{x}_0 := \lim_{t\to 0}(\lambda(t)\cdot\tilde{x})$ existed, then by Prop. 1, $\tilde{x}_0 \in \lambda(\mathbb{G}_m)\cdot\tilde{x}$, since the orbit is closed. Since $\tilde{x}_0$ is invariant under the action of $\lambda(\mathbb{G}_m)$, it follows that $\tilde{x}$ also is. But by Prop. 1, the stabilizer of $\tilde{x}$ (in $\lambda(\mathbb{G}_m)$) has dimension zero, so $\dim \lambda(\mathbb{G}_m) = 0$.

Since λ is non-trivial (by def. of 1-PS), we get a contradiction.

Therefore, $\tilde{x}_0$ doesn't exist and

$$\mu(x,\lambda) > 0.$$

$\underline{\Leftarrow)}$ Omitted. Here, we use the fact that reductive groups have "abundant 1-PS" (one uses the Cartan–Iwahori decomp.) $\boxtimes$

· $\underline{\text{Corollary 9.}}$ $Y \subseteq X$ G-invariant subvariety. Then,

$$Y^{ss}(\mathcal{L}|_Y) = X^{ss}(\mathcal{L}) \cap Y,$$

$$Y^{s}(\mathcal{L}|_Y) = X^{s}(\mathcal{L}) \cap Y.$$

Proof| If $y \in Y$, then our definition of $\mu^{\mathcal{L}|_Y}(y,\lambda)$ depends only on the properties of the G-action on the fibers of $\mathcal{L}|_Y$, which agree with the fibers of $\mathcal{L}$ on Y. Thus,

$$\mu^{\mathcal{L}|_Y}(y,\lambda) = \mu^{\mathcal{L}}(y,\lambda). \quad \boxtimes$$

• Corollary 10. $X^{ss}(\mathcal{L})$ and $X/\!\!/_{\mathcal{L}} G$ only depend on $[\mathcal{L}] \in NS^G(X)$.
(for fixed X, G)

$\mathrm{Pic}^G(X) := \{\text{line bundles with a } G\text{-lin.}\}/_{\cong}$

$NS^G(X) := \{\text{line bundles with a } G\text{-lin.}\}/_{\text{alg. equiv.}}$

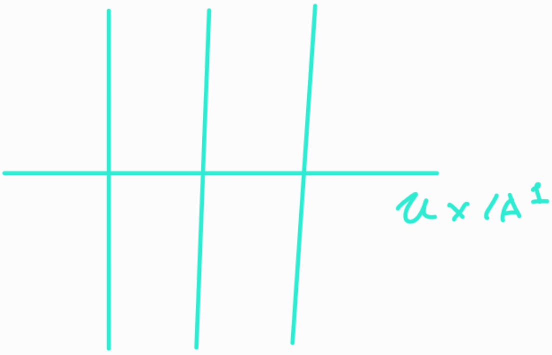
We say $\mathcal{L}_1$ and $\mathcal{L}_2$ G-lins. of X are alg. equiv. iff $\exists$ a conn. k-var. Y, pts. $y_1, y_2 \in Y(k)$, and a line bundle with a G-linearization $\mathcal{L} \to X \times Y$, with $G \curvearrowright X \times Y$ via $g\cdot(x,y) = (gx, y)$; s.t. $\mathcal{L}_i \cong \mathcal{L}|_{X_{y_i}}$.

• Rem: X normal and proper $\Rightarrow NS^G_{\mathbb{Q}} = NS_{\mathbb{Q}} \oplus (\hat{G} \otimes \mathbb{Q})$, so $NS^G_{\mathbb{Q}}$ is fin. gen. abelian

Proof| Firstly, clearly $X/\!\!/_{\mathcal{L}} G$ depends only on $X^{ss}(\mathcal{L})$. Indeed, this GIT quotient is in particular a categorical quotient of $X^{ss}(\mathcal{L})$, and thus it is unique.

Now, we show that $X^{ss}(\mathcal{L})$ depends only on $[\mathcal{L}]$. We let $x \in X$, and by Thm. 8, WLOG $G = \mathbb{G}_m$ and $x \in X^{\mathbb{G}_m}$.
Let Y be a conn. k-var., and $\mathcal{L}$ a l.b. on $X \times Y$ with a $\mathbb{G}_m$-lin.
Then, $\mathcal{L}|_{\{x\} \times Y}$ is a l.b. on Y, with $G \curvearrowright Y$ trivial.
Write $\mathcal{L}_i := \mathcal{L}|_{X \times \{y_i\}}$ for some $y_1, y_2 \in Y(k)$. Then, we want to study the action of $\mathbb{G}_m$ on $\mathcal{L}_i|_{\{x\}} = \mathcal{L}|_{\{x\} \times \{y_i\}}$. But since the

action of $\mathbb{G}_m$ on $\mathcal{Y}$ was trivial, then the weight of the action of $\mathbb{G}_m$ on $\mathcal{L}|_{\{s\}\times\mathcal{Y}}$ cannot "jump".



One looks at $\mathcal{U}\subseteq\mathcal{Y}$ s.t. $\mathcal{L}|_{\{s\}\times\mathcal{U}}$ is trivial, and sees that $\frac{t\cdot\sigma}{\sigma} = f(u)\in\mathcal{O}_{\mathcal{U}}(\mathcal{U})^\times$ is a funct. $f:\mathcal{U}(k)\to k$, cont., with values in $\{t^l \mid l\in\mathbb{Z}\}$.

action of $\mathbb{G}_m$ is the same in each vertical fiber

Thus, $\mu^{\mathcal{L}_1}(x,\lambda) = \mu^{\mathcal{L}_2}(x,\lambda)$. ⊠

§2. The weight polytope.

• $G = T = (\mathbb{G}_m)^r$ torus acting on X proj. var. with very ample linearisation $\mathcal{L}$. The action of T induces an action on $V := \mathbb{A}^{n+1}$, and thus a weight decomp.

$$V = \bigoplus_{\chi\in M} V_\chi,$$

where $M := \mathrm{Hom}(T,\mathbb{G}_m)\cong\mathbb{Z}^r$ is the group of characters of T, and $V_\chi := \{v\in V \mid t\cdot v = \chi(t)\cdot v \quad \forall t\in T\}$.

• Definition 11. If $x\in X$, and $\tilde{x}\in\tilde{X}\subseteq V$ lies over x, then we may write $\tilde{x} = \sum_{\chi\in M} x_\chi$. The *T-weight set of x* is

$$wt_T(x) := \{\chi\in M \mid x_\chi\neq 0\}.$$

The $\underline{\text{T-weight polytope of } x}$ is the convex hull of $wt_T(x)$ in $\mathbb{R}^r \cong M \otimes_{\mathbb{Z}} \mathbb{R}$, and we denote it by $conv(wt_T(x))$.

· <u>Remark 12.</u> There is a clear pairing:

$$\langle -,- \rangle : Hom(\mathbb{G}_m, T) \times M \longrightarrow \mathbb{Z}$$
$$(\lambda, \chi) \longmapsto \underbrace{\chi \circ \lambda}_{\text{map } \mathbb{G}_m \to \mathbb{G}_m}.$$

Moreover, due to our definition of μ (as in Ex. 5), we have that for any 1-PS λ, and any $x \in X$:

$$\mu(x,\lambda) = -\min\{\langle \lambda, \chi \rangle \mid \chi \in wt_T(x)\} =$$
$$= -\min\{\langle \lambda, \chi \rangle \mid \chi \in conv(wt_T(x))\}$$

the minimum is reached on the boundary of the convex hull

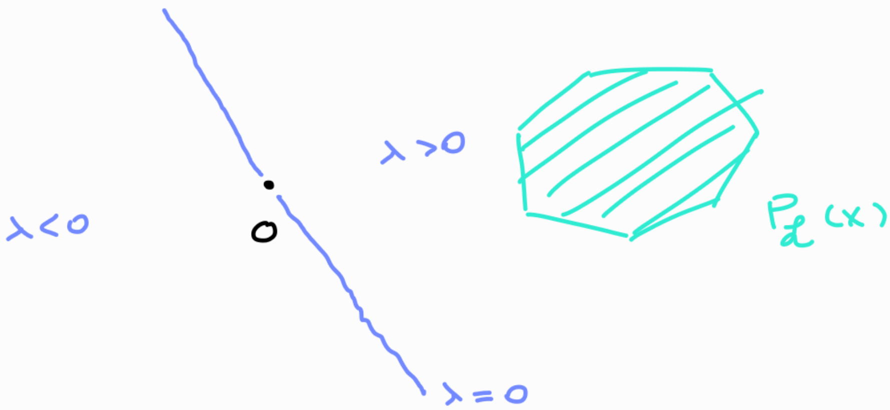
· <u>Prop. 13.</u> (Hilbert–Mumford crit. for tori). Let $x \in X$. Then,

(i) $x \in X^{ss}(\mathcal{L}) \iff \mu(x,\lambda) \geq 0 \;\; \forall \lambda$ 1-PS $\iff 0 \in conv(wt_T(x))$.

(ii) $x \in X^{s}(\mathcal{L}) \iff \mu(x,\lambda) > 0 \;\; \forall \lambda$ 1-PS $\iff 0 \in \underbrace{Int}_{\text{interior}}(conv(wt_T(x)))$.

<u>Proof</u>

(i) Consider the following picture:



We claim that we can always partition $\mathbb{R}^r$ by a hyperplane given by $\{\langle \lambda, - \rangle = 0\}$. Then, one "side" of the hyperplane will be $\{\langle \lambda, - \rangle > 0\}$ and the other will be $\{\langle \lambda, - \rangle < 0\}$.

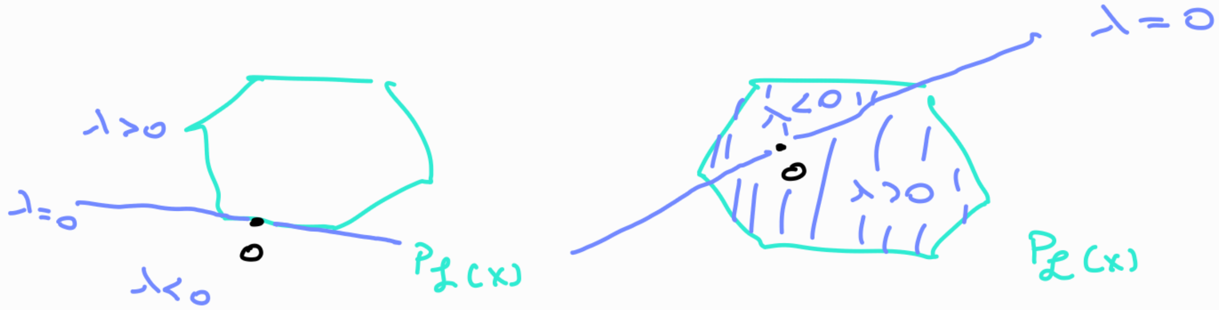
If $0 \in \mathrm{conv}(\mathrm{wt}_T(x))$, then clearly $\mu(x,\lambda) \geq 0$ for all possible λ, since

$$\min\{\langle \lambda, x\rangle \mid x \in \mathrm{conv}(\mathrm{wt}_T(x))\} \leq \langle \lambda, 0\rangle = 0.$$

Conversely, if $0 \notin \mathrm{conv}(\mathrm{wt}_T(x))$, then we can choose a λ s.t. $\{\langle \lambda, -\rangle = 0\}$ does not intersect the convex hull, and moreover, s.t. $\mathrm{conv}(\mathrm{wt}_T(x)) \subseteq \{\langle \lambda, - \rangle > 0\}$. But then, for such a λ,

$$\mu(x,\lambda) = -\min\{\underbrace{\langle \lambda, x\rangle}_{>0} : x \in \mathrm{conv}(\mathrm{wt}_T(x))\} < 0.$$

(ii) This is clear from the previous reasoning, by observing that if $0 \in \mathrm{Int}(\mathrm{conv}(\mathrm{wt}_T(x)))$, then $\mathrm{conv}(\mathrm{wt}_T(x)) \cap \{\langle \lambda, - \rangle < 0\} \neq \emptyset$, and viceversa.



$\boxtimes$

- <u>Example 14.</u> Consider the action of $G = \mathbb{G}_m^2$ on $\mathbb{P}^3$ corresponding to the rep.

$$\mathbb{G}_m^2 \longrightarrow GL_4(k)$$

$$(s,t) \longmapsto \begin{pmatrix} st & & & \\ & s^{-1}t & & \\ & & s^{-1}t^{-1} & \\ & & & st^{-1} \end{pmatrix}$$

$M := \mathrm{Hom}(\mathbb{G}_m^2, \mathbb{G}_m)$ is gen. by $x_1 : (s,t) \longmapsto s$, and

$x_2 : (s,t) \mapsto t$. Then, $V := \mathbb{A}^4$ decomposes as

$$V = \bigoplus_{n,m \in \mathbb{Z}} V_{x_1^n x_2^m} .$$

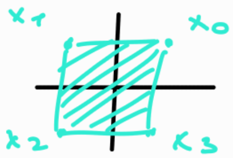
Since clearly
$$V = \underbrace{V_{x_1 x_2}}_{(x,0,0,0)} \oplus \underbrace{V_{x_1^{-1} x_2}}_{(0,x,0,0)} \oplus \underbrace{V_{x_1^{-1} x_2^{-1}}}_{(0,0,x,0)} \oplus \underbrace{V_{x_1 x_2^{-1}}}_{(0,0,0,x)} ,$$

this is our weight decomposition.

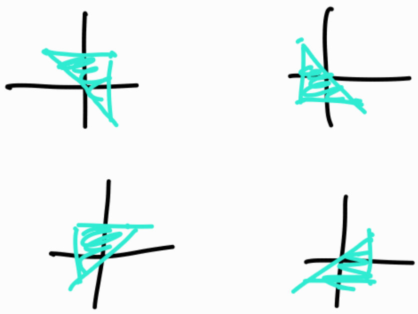
Thus, for any $x = [x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3$, we can write $\tilde{x} = (x_0, x_1, x_2, x_3)$ as

$$\tilde{x} = \underbrace{x_{x_1 x_2}}_{x_0} + \underbrace{x_{x_1^{-1} x_2}}_{x_1} + \underbrace{x_{x_1^{-1} x_2^{-1}}}_{x_2} + \underbrace{x_{x_1 x_2^{-1}}}_{x_3} .$$

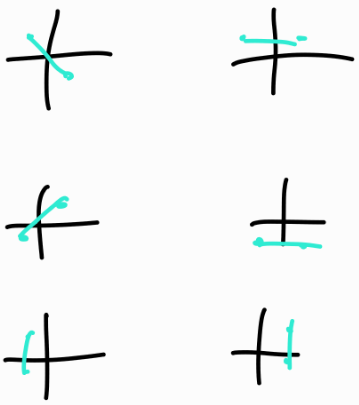
It follows that the convex hull of $\mathrm{wt}_T(x)$ must be one of the following:



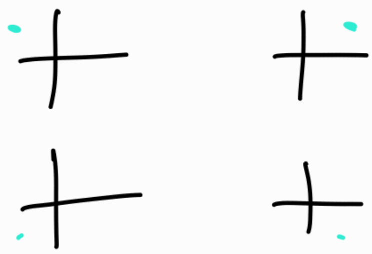
all coord. non-zero



one coord. is zero



two coord. are zero



three coord. are zero

By Prop. 13, we must have that

$$(\mathbb{P}^3)^s = \{ [x_0 : x_1 : x_2 : x_3] \mid x_i \neq 0 \ \forall i \}$$

$$(\mathbb{P}^3)^{ss} = \{ [x_0 : x_1 : x_2 : x_3] \mid x_0 x_2 \neq 0 \quad \text{or} \quad x_1 x_3 \neq 0 \}.$$

The unstable points are those when $x_0 x_2 = 0$ and $x_1 x_3 = 0$.

Now, consider the structure ring $k[T_0, T_1, T_2, T_3]$, with an action of $\mathbb{G}_m^2$ by $(s,t) \cdot f(T_0, T_1, T_2, T_3) = f(stT_0, s^{-1}tT_1, s^{-1}t^{-1}T_2, st^{-1}T_3)$.
Then,

$$k[T_0, T_1, T_2, T_3]^G = k[T_0T_2, T_1T_3] \cong k[X, Y].$$

Therefore, the GIT quotient is

$$\mathbb{P}^3 \dashrightarrow \mathbb{P}^3 /\!/ G \cong \mathrm{Proj}(k[X, Y]) \cong \mathbb{P}^2,$$

which means that we get a GIT quot.

$$\begin{aligned} (\mathbb{P}^3)^{ss} &\longrightarrow \mathbb{P}^2 \\ [x_0 : x_1 : x_2 : x_3] &\longmapsto [x_0 x_2 : x_1 x_3]. \end{aligned}$$

· <u>Remark 15.</u> The Prop. 13 can be extended to the general case when G is a reductive gp., but when now one only needs to check if $0 \in \mathrm{conv}(wt_T(x))$ for the maximal torus T of G and $\forall g \in G$.

• <u>Rmk. 16.</u> What happens when X is semiprojective?

In this case, the limits $\lim_{t\to 0} \lambda(t)\cdot x$ may not always exist.

Then, we change our definition of $\mu^{\mathcal{L}}$ to be

$$\mu^{\mathcal{L}}(x,\lambda) := \begin{cases} \text{the usual def. if } \lim_{t\to 0} \lambda(t)\cdot x \text{ exists,} \\ +\infty \text{ if the limit doesn't exist.} \end{cases}$$

Using this, all the prev. results are correct.

• <u>Example 17.</u> Consider G red. $\curvearrowright X = \operatorname{Spec} R$, with the trivial linearisation $\mathcal{L}$. But then,

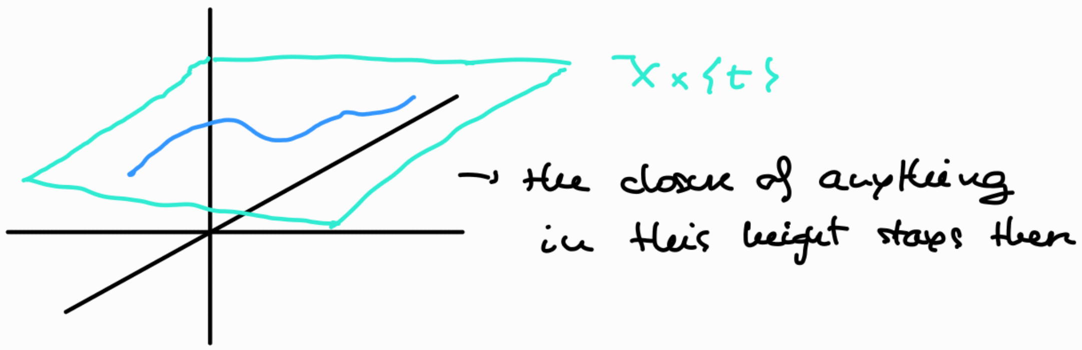
$$X = \operatorname{Proj}\left(\bigoplus_{n=0}^{\infty} R\right), \text{ and thus the affine cone is}$$

$$\tilde{X} = \operatorname{Spec}\left(\bigoplus_{n=0}^{\infty} R\right) = \operatorname{Spec}(R[T]) \cong X \times \mathbb{A}^1$$

If $x \in R$, $x \neq 0$, then any $\tilde{x} \in \tilde{X}$ over x is $\tilde{x} = (x,t)$ for some $t \in k^\times$. Since G acts trivially on $X \times \mathbb{A}^1$, this means that

$$G\cdot\tilde{x} = (G\cdot x) \times \{t\} \subseteq X \times \{t\}.$$

Therefore, 0 <u>cannot</u> be in $\overline{G\cdot\tilde{x}}$, meaning that all x are semistable.



§3. Toric varieties.

• Recall:

-) M lattice, $M := X^*(T)$ [torus $\mathbb{G}_m^r$], P polyhedron with
$$P := \{ m \in M_{\mathbb{R}} \mid \forall F \in P(1),\ \langle m, u_F \rangle \geq -a_F \},$$
with $P(1)$ the facets of P. In particular,
$$F = \{ m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle = -a_F \}.$$

.) Corresponding toric variety X_P, which we assume has no torus factor.

-) There is a s.e.s.
$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{P(1)} \longrightarrow Cl(X_P) \longrightarrow 0,$$
and by taking duals we obtain
$$1 \longleftarrow T \longleftarrow \mathbb{G}_m^{P(1)} \longleftarrow G \longleftarrow 1$$
def. as the kernel of the left map

Therefore, by choosing $D = \left[\sum c_F \cdot D_F\right] \in Cl(X_P)$, we obtain a char. $\theta_D : G \to \mathbb{G}_m$.

.) $G \curvearrowright \mathbb{A}^{P(1)}$ with a linearisation $\mathcal{L}_{\theta_D}$ twisted by θ_D.

Last time, we saw that
$$\mathbb{A}^{P(1)} /\!\!/_{\mathcal{L}_{\theta_D}} G := \mathrm{Proj}\Big(\bigoplus_{k=0}^{\infty} \underbrace{H^0(\mathbb{A}^{P(1)}, \mathcal{L}_{\theta_D}^{\otimes k})^G}_{\text{Global sections of } \mathcal{O}_{\mathbb{A}^{P(1)}} \text{ which are in the } \theta_D \text{ eigensp. of } G} \Big) \cong X_{P_D},$$
where
$$P_D := \{ m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -c_F \}.$$

- **Today**: we want to study the semistable locus in $\mathbb{A}^{P(1)}$.

- **Definition 18.** For a D as before, the *virtual facets of P_D* are defined as

$$\mathcal{F}_F := P_D \cap \{ m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle = -c_F \}$$

for all $F \in P(1)$.

- **Lemma 19.**

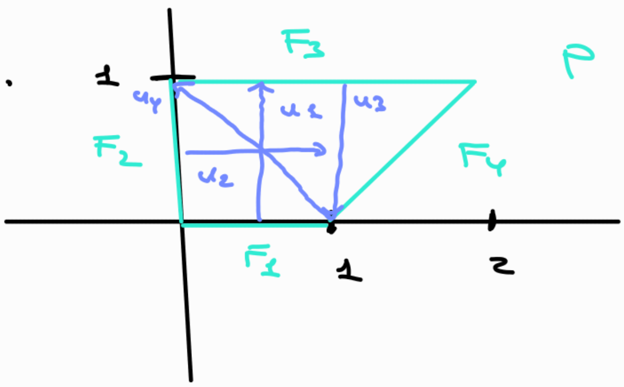
$$(\mathbb{A}^{P(1)})^{us}_{\Theta_D} = \bigcup_{\substack{I \subseteq P(1) \\ \bigcap_{F \in I} \mathcal{F}_F = \emptyset}} \{ x \in \mathbb{A}^{P(1)} \mid x_F = 0 \iff F \in I \}$$

- **Example 20.** $\mathbb{G}_m^{P(1)}$ can be identified with $\{ x \in \mathbb{A}^{P(1)} \mid x_F \neq 0 \ \forall F \}$.
Therefore, $\mathbb{G}_m^{P(1)}$ is semistable if and only if

$$\bigcap_{F \in I} \mathcal{F}_F \neq \emptyset$$

when $I = \emptyset$. But in this case, the empty intersection is P_D, which is non-empty iff $X_{P_D} \neq \emptyset$.

- **Example 21.**



← polytope for the blowup of $\mathbb{P}^2$ at a point.

$$\underbrace{\begin{matrix} u_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ u_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ u_4 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{matrix}}_{\text{minimal normal vector to } F_i \text{ in } P} , \quad \underbrace{\begin{matrix} a_1 = 0 \\ a_2 = 0 \\ a_3 = 1 \\ a_4 = 1 \end{matrix}}_{\text{determines how big the polytope is}} \quad \begin{matrix} \rightsquigarrow y \geq 0 \\ \rightsquigarrow x \geq 0 \\ \rightsquigarrow -y \geq 1 \\ \rightsquigarrow -x+y \geq 1 \end{matrix}$$

We consider the s.e.s. from before:

$$1 \longrightarrow \overbrace{\mathbb{Z}^2}^{M} \longrightarrow \overbrace{\mathbb{Z}^4}^{\mathbb{Z}^{P(1)}} \longrightarrow Cl(X_P) \longrightarrow 0 \qquad (*)$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

this sends m to $(\langle m, u_i \rangle)_i$

Then, we have the following relations in $Cl(X_P)$:

$$[D_2 - D_4] = 0 \Rightarrow [D_2] = [D_4].$$
$$[D_1 - D_3 + D_4] = 0 \Rightarrow [D_1 + D_2] = [D_3].$$

It is clear from this that $Cl(X_P) \cong \mathbb{Z}^2$, gen. by $[D_1], [D_2]$.

Now, taking the dual of $(*)$, we have that

$$1 \longleftarrow T \longleftarrow \mathbb{G}_m^{P(1)} \longleftarrow \mathbb{G}_m^2 \longleftarrow 1$$

$$\begin{pmatrix} s \\ t \\ st \\ t \end{pmatrix} \longleftarrow\!\shortmid (s,t)$$

corresponds to writing

$D_i = sD_1 + tD_2$

We then know how $G := \mathbb{G}_m^2$ acts on $\mathbb{A}^{P(1)}$. Now, to determine the linearisation, we choose a divisor.

In particular, we look at divisors of the form $D = \lambda D_2 + D_3 + D_4$.

•) $A := D_3 + D_4 \sim_{\text{lin}} D_1 + 2D_2 \longrightarrow$ ample (prev. talk).

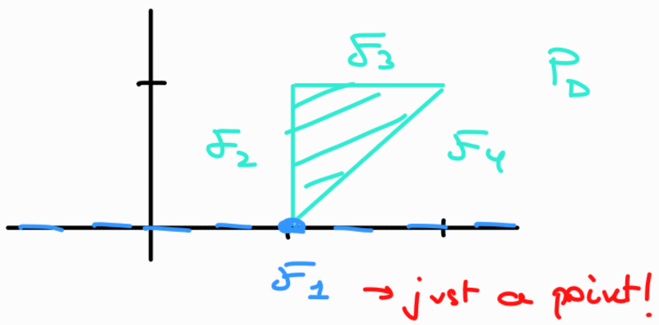
also from a prev. talk

Therefore, $\mathbb{A}^{P(1)} /\!/_{\Theta_A} G = X_{P_A} = X_P$.

(in fact, we have something even stronger: since the coeff. of A are the a_i's, $P_A = P$).

The unstable locus is $(\mathbb{A}^{P(1)})^{us}_{\Theta_A} = \left\{ x \in \mathbb{A}^{P(1)} \;\middle|\; \begin{array}{l} x_1 = x_3 = 0, \text{ or} \\ x_2 = x_4 = 0 \end{array} \right\}$

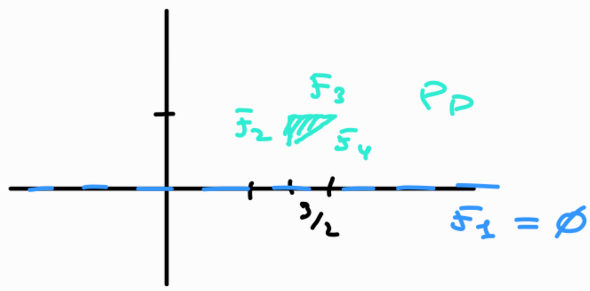
•) $D := D_2 + D_3 + D_4$.



In this case, $\mathbb{A}^{P(1)} /\!/_{\Theta_D} G = X_{P_D} = \mathbb{P}^2$, and the unstable locus is

$$(\mathbb{A}^{P(1)})^{us}_{\Theta_D} = \{ x \in \mathbb{A}^{P(1)} \mid x_1 = x_3 = 0 \}.$$

•) $D := \frac{3}{2} D_2 + D_3 + D_4$.



In this case, $\mathbb{A}^{P(1)} /\!\!/_{\theta_D} G = X_{P_D} \cong \mathbb{P}^2$ again, but the unstable locus is bigger:

$$\left(\mathbb{A}^{P(1)}\right)^{us}_{\theta_D} = \{ x \in \mathbb{A}^{P(1)} \mid x_1 = 0 \}.$$