Stability sets and GIT-classes

<u>Recall</u>: Let X be a projective variety over an algebraically closed field k.
If $\mathcal{L}$ is a very ample line bundle on X, then $X \cong \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} H^0(X, \mathcal{L}^{\otimes n})\right)$ and we have an embedding of X into a projective space $\varphi: X \hookrightarrow \mathbb{P}(H^0(X,\mathcal{L}))$ with the property that $\mathcal{L} = \varphi^*(\mathcal{O}(1))$.
If $\mathcal{L}$ is ample, there is a similar embedding into $\mathbb{P}(H^0(X,\mathcal{L}^{\otimes n}))$ for a suitable $n>0$.
Suppose that we now have a reductive algebraic group G acting on X. For simplicity, assume that $\mathcal{L}$ is very ample. The choice of a G-linearization for $\mathcal{L}$ corresponds to the choice of an action of G on $H^0(X,\mathcal{L})$ in such a way that the embedding $\varphi: X \hookrightarrow \mathbb{P}(H^0(X,\mathcal{L}))$ is G-equivariant. We therefore obtain a linear action of G on the affine cone of X, i.e. on $\tilde{X} = \{\tilde{x} \in H^0(X,\mathcal{L}) \setminus \{0\} \mid \pi(\tilde{x}) \in X\} \cup \{0\}$ (where $\pi: H^0(X,\mathcal{L}) \setminus \{0\} \twoheadrightarrow \mathbb{P}(H^0(X,\mathcal{L}))$ is the canonical projection).
Given a G-linearized ample line bundle $\mathcal{L}$, we have defined the GIT-quotient of X modulo G via $\mathcal{L}$ as $X /\!\!/_{\mathcal{L}} G = \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} H^0(X,\mathcal{L}^{\otimes n})^G\right)$, which comes with a rational map $X \dashrightarrow X /\!\!/_{\mathcal{L}} G$.
The locus where this map is defined consists of the semistable points of X with respect to $\mathcal{L}$ and is denoted by $X^{ss}(\mathcal{L})$.

Given a 1-parameter subgroup of G, say $\lambda: \mathbb{G}_m \to G$, for every $x \in X$ there exists $x_0 = \lim_{t\to 0} \lambda(t).x$ in X, since X is proper. The point x_0 is λ-invariant, so $\mathbb{G}_m$ acts linearly on the fiber $\mathcal{L}|_{x_0}$, according to the G-linearization that has been chosen for $\mathcal{L}$. The weight of this action is denoted by $\mu^{\mathcal{L}}(x,\lambda)$. Varying $\mathcal{L}$ gives a map $\mu^{\cdot}(x,\lambda): \mathrm{Pic}^G(X) \to \mathbb{Z}$.

<u>Proposition</u>: Let $x \in X$, $\tilde{x} \in \tilde{X}$ such that $\pi(\tilde{x}) = x$.

1) $\mu^{\mathcal{L}}(x,\lambda) < 0$ if and only if $\lim_{t\to 0} \lambda(t).\tilde{x}$ exists and is zero.
2) $\mu^{\mathcal{L}}(x,\lambda) = 0$ if and only if $\lim_{t\to 0} \lambda(t).\tilde{x}$ exists and is non-zero.
3) $\mu^{\mathcal{L}}(x,\lambda) > 0$ if and only if $\lim_{t\to 0} \lambda(t).\tilde{x}$ does not exist.

<u>Theorem</u> (Hilbert-Mumford numerical criterion)

For every $x \in X$ we have:

1) x is semistable for $\mathcal{L}$ if and only if $\mu^{\mathcal{L}}(x,\lambda) \geq 0$ for all 1-parameter subgroups λ.
2) x is stable for $\mathcal{L}$ if and only if $\mu^{\mathcal{L}}(x,\lambda) > 0$ for all 1-parameter subgroups λ.

Def: $\mathrm{Pic}(X) = \{\text{line bundles on } X\}/\cong$, $\mathrm{Pic}^G(X) = \{G\text{-linearized line bundles on } X\}/\cong$

$NS(X) = \{\text{line bundles on } X\}/(\text{algebraic equivalence})$

$NS^G(X) = \{G\text{-linearized line bundles on } X\}/(G\text{-algebraic equivalence})$

We will be interested in $NS^{(G)}(X)_{\mathbb{Q}} = NS^{(G)}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, $NS^{(G)}(X)_{\mathbb{R}} = NS^{(G)}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

There is an exact sequence $0 \to \mathfrak{X}^*(G)_{\mathbb{R}} \to NS^G(X)_{\mathbb{R}} \to NS(X)_{\mathbb{R}} \to 0$, where $\mathfrak{X}^*(G) = \mathrm{Hom}(G, \mathbb{G}_m)$.

The function $\mu^\cdot(x,\lambda) : \mathrm{Pic}^G(X) \to \mathbb{Z}$ for a 1-parameter subgroup λ of G descends to a well-defined map $\mu^\cdot(x,\lambda) : NS^G(X) \to \mathbb{Z}$, which we extend by linearity to $\mu^\cdot(x,\lambda) : NS^G(X)_{\mathbb{R}} \to \mathbb{R}$.

For $\ell \in NS^G(X)_{\mathbb{R}}$ we set $M^\ell(x) = \inf_\lambda \frac{\mu^\ell(x,\lambda)}{\|\lambda\|}$, which defines a concave and positively homogeneous function $M^\cdot(x) : NS^G(X)_{\mathbb{R}} \to \mathbb{R}$. (here, $\|\cdot\|$ is a fixed norm invariant under the action of the Weyl group)

We set $X^{ss}(\ell) = \{x \in X \mid M^\ell(x) \geq 0\}$ and $X^s(\ell) = \{x \in X \mid M^\ell(x) > 0\}$ for $\ell \in NS^G(X)_{\mathbb{R}}$.

Def: $NS^G(X)^+_{\mathbb{R}}$: cone in $NS^G(X)_{\mathbb{R}}$ generated by the classes of ample G-linearized line bundles.

$C^G(X) = \{\ell \in NS^G(X)^+_{\mathbb{R}} \mid X^{ss}(\ell) \neq \emptyset\}$ (G-ample cone).

We will also need the Hesselink stratification for proving finiteness results.

Given $x \in X$, $\mathcal{L}$ an ample G-linearized line bundle, we say that a 1-parameter subgroup $\lambda : \mathbb{G}_m \to G$ is "adapted to $\mathcal{L}$ and x" if it is primitive and $\frac{\mu^{\mathcal{L}}(x,\lambda)}{\|\lambda\|} = M^{\mathcal{L}}(x)$.

$\Lambda^{\mathcal{L}}(x) = \{\lambda : \mathbb{G}_m \to G \mid \lambda \text{ is a 1-parameter subgroup adapted to } \mathcal{L} \text{ and } x\}$.

Proposition: $\Lambda^{\mathcal{L}}(x)$ is non-empty and any two elements of $\Lambda^{\mathcal{L}}(x)$ are conjugate by an element of a certain subgroup of G (defined in the previous talk)

For $d<0$, $[\lambda]$ a G-conjugacy class of a 1-parameter subgroup of G, we define

$S^{\mathcal{L}}_{d,[\lambda]} = \{x \in X \mid M^{\mathcal{L}}(x) = d \text{ and } \Lambda^{\mathcal{L}}(x) \cap [\lambda] \neq \emptyset\}$.

This is a stratification by locally closed subsets of $X^{us}(\mathcal{L})$. So $X = X^{ss}(\mathcal{L}) \cup \bigcup_{d,[\lambda]} S^{\mathcal{L}}_{d,[\lambda]}$

Theorem: The set $\{S^{\mathcal{L}}_{d,[\lambda]}\}_{\mathcal{L},d,[\lambda]}$ is finite.

In particular, only finitely many open subsets of X can be realized as $X^{ss}(\mathcal{L})$ for some G-linearized ample line bundle $\mathcal{L}$.

§ The stability set of a point.

Let X be a projective variety over an algebraically closed field k.

<u>Def:</u> For every $x \in X$ we define the "stability set of x" to be

$$\Omega(x) = \{ \ell \in NS^G(X)^+_{\mathbb{R}} \mid x \in X^{ss}(\ell) \} \; \left(= \{ \ell \in NS^G(X)^+_{\mathbb{R}} \mid M^{\ell}(x) \geq 0 \} = (M^{\cdot}(x))^{-1}([0,+\infty)) \right).$$

<u>Goal:</u> study the geometry of $\Omega(x)$ for a fixed $x \in X$

This means: fix $x \in X$ and study its semistability with respect to a ($\mathbb{R}$-linear combination of ample G-linearized) line bundle ℓ which varies in $NS^G(X)^+_{\mathbb{R}}$.

We start with a technical lemma.

<u>Lemma 1:</u> Let ℓ be a rational point in the G-ample cone (i.e. $\ell \in C^G(X) \cap NS^G(X)_{\mathbb{Q}}$).

Let $x \in X^{ss}(\ell)$ and $\lambda \in \mathfrak{X}_*(G) = \mathrm{Hom}(\mathbb{G}_m, G)$. Set $x_0 = \lim\limits_{t \to 0} \lambda(t).x$.

If x_0 is unstable for ℓ, then $\mu^{\ell}(x,\lambda) > 0$.

<u>Proof:</u> Since ℓ is rational, we can write $n \cdot \ell = \sum_i n_i \mathcal{L}_i$ in $NS^G(X)_{\mathbb{Q}}$ where $n, n_i \in \mathbb{Z}$, $n \neq 0$ and $\mathcal{L}_i$ is (the class of) a G-linearized ample line bundle on X. The element $\sum n_i \mathcal{L}_i$ in $NS^G(X)_{\mathbb{Q}}$ corresponds to the class of the G-linearized ample line bundle $\mathcal{L} = \bigotimes_i \mathcal{L}_i^{\otimes n_i}$, so it actually belongs to $NS^G(X)$. Since $X^{ss}(\ell) = X^{ss}(n\cdot\ell)$, we may assume that $n=1$ for our purposes. Thus, we have a G-equivariant embedding $\varphi: X \hookrightarrow \mathbb{P}(V)$ for some G-module V (more precisely, $V = H^0(X, \mathcal{L})$ and the action of G on V depends on the G-linearization of $\mathcal{L}$) such that $\varphi^*\mathcal{O}(1)$ is algebraically equivalent to ℓ.

Assume that $\mu^{\ell}(x,\lambda) \leq 0$. Since x is ℓ-semistable, $\mu^{\ell}(x,\lambda) \geq 0$, thus $\mu^{\ell}(x,\lambda) = 0$. Let $\tilde{x} \in \tilde{X}$ be a point in the affine cone of X in V under φ, chosen so that $\tilde{x} \neq 0$ and $\pi(\tilde{x}) = x$. Since $\mu^{\ell}(x,\lambda) = 0$, $\tilde{x}_0 = \lim\limits_{t \to 0} \lambda(t).\tilde{x}$ exists and it is non-zero. In particular, $\tilde{x}_0$ lies in $\overline{G\tilde{x}}$ (the closure of the G-orbit of $\tilde{x}$ in V). Since x is semistable, we have $0 \notin \overline{G\tilde{x}}$. It follows that $0 \notin \overline{G\tilde{x}_0}$, so x_0 is semistable for ℓ. This contradicts the assumption on x_0. $\square$

<u>Proposition 2:</u> Let $x \in X$. Then:

1) $\Omega(x)$ is a convex cone and is closed in $NS^G(X)^+_{\mathbb{R}}$;

2) the span of $\Omega(x)$ in $NS^G(X)_{\mathbb{R}}$ is a rational vector subspace of $NS^G(X)_{\mathbb{R}}$.

In particular, $\Omega(x)$ is the closure of its rational points.

Proof: 1) The function $M^{\cdot}(x): NS^G(X)^+_{\mathbb{R}} \to \mathbb{R}$ is convex and positively homogeneous, in particular also continuous. Since $\Omega(x) = (M^{\cdot}(x))^{-1}([0,+\infty))$, concavity of $M^{\cdot}(x)$ implies the convexity of $\Omega(x)$, the positive homogeneity of $M^{\cdot}(x)$ shows that $\Omega(x)$ is a cone and continuity thereof ensures that $\Omega(x)$ is closed in $NS^G(X)^+_{\mathbb{R}}$.

2) Let $F \subseteq NS^G(X)_{\mathbb{R}}$ be the minimal rational vector subspace of $NS^G(X)^+_{\mathbb{R}}$ such that $\Omega(x) \subseteq F$. Suppose that $\Omega(x)$ does not span F. Since $\Omega(x)$ is convex, it follows that the interior of $\Omega(x)$, as a subset of F, is empty. This implies that $M^{\ell}(x) = 0$ for every $\ell \in \Omega(x)$. Indeed, if $\ell \in \Omega(x)$ and $M^{\ell}(x) > 0$, then $(M^{\cdot}(x))^{-1}((0,\infty))$ is a non-empty open subset of F contained in $\Omega(x)$, which is not possible.

Let $\ell \in \Omega(x)$. We may find a sequence $\{\ell_n\}_n$ of points in $F \setminus \Omega(x)$ which tends to ℓ, again because $\Omega(x)$ has empty interior. Since F is rational, we may assume that the ℓ_n's are rational.

We know that the strata $\{S^{\mathcal{L}}_{d,\langle\tau\rangle}\}_{\mathcal{L},d,\langle\tau\rangle}$ of the Hesselink stratification of X are only finitely many. Up to passing to a subsequence of $\{\ell_n\}_n$, we may assume that all ℓ_n's induce the same stratification. Notice that for every ℓ_n we have $M^{\ell_n}(x) < 0$ (otherwise we would have $\ell_n \in \Omega(x)$), so there is a non-open stratum S in the chosen stratification which contains x. Let $\lambda_0 \in \Lambda^{\ell_0}(x)$, so that $M^{\ell_0}(x,\lambda_0) = \frac{\mu^{\ell_0}(x,\lambda_0)}{\|\lambda_0\|}$, and set $y = \lim_{t\to 0} \lambda(t).x$. We have $\lambda_0 \in \Lambda^{\ell_0}(y)$ and $M^{\ell_0}(x) = M^{\ell_0}(y)$ (reference in the paper), hence $y \in S$. Since S is in the stratification of all the ℓ_n's, it follows that $M^{\ell_n}(x) = M^{\ell_n}(y)$. By continuity of $M^{\cdot}(x)$, here we have $M^{\ell}(y) = M^{\ell}(x) = 0$.

Since λ_0 fixes y, the weights of both λ_0 and $-\lambda_0$ on $\mathcal{L}|_y$ appear in $M^{\ell}(y)$, and $\mu^{\ell}(y,-\lambda_0) = -\mu^{\ell}(y,\lambda_0)$. Thus $\mu^{\ell}(x,\lambda_0) = \mu^{\ell}(y,\lambda_0) = 0$ since $M^{\ell}(y) = 0$. But $\mu^{\ell_0}(x,\lambda_0) < 0$, so $\mu^{\cdot}(x,\lambda_0)$ is not identically zero on F. Thus, $\{\mu^{\cdot}(x,\lambda_0) = 0\}$ is a hyperplane of F containing ℓ.

Summing up, we have proved that every point $\ell \in \Omega(x)$ is contained in a hyperplane of F of the form $\{\mu^{\cdot}(x,\lambda) = 0\}$ for some 1-parameter subgroup λ of G.

Now, each function $\mu^{\cdot}(x,\lambda)$ is a linear function on $NS^G(X)_{\mathbb{Q}}$ with values in $\mathbb{Q}$, and there are only countably many linear maps $NS^G(X)_{\mathbb{Q}} \to \mathbb{Q}$. As a result, the set of the functions $\mu^{\cdot}(x,\lambda)$ for λ running through all 1-parameter subgroups of G is countable.

By what we have shown before, $\Omega(x)$ lies in the union of countably many hyperplanes of F. However, $\Omega(x)$ is convex: given $\ell_1, \ell_2 \in \Omega(x)$, the segment from ℓ_1 to ℓ_2 is contained in $\Omega(x)$. A linear subspace of F either contains this segment or intersects this segment in at most one point. Since said segment is contained in a countable union of hyperplanes of F, one of these hyperplanes must contain it completely. This applies to all segments of $\Omega(x)$, so we see that $\Omega(x)$ must be contained in a rational hyperplane of F. However, this contradicts the minimality of F. $\square$

Corollary 3: There are only finitely many stability sets.
Proof: If ℓ_1 and ℓ_2 are GIT-equivalent (i.e. $X^{ss}(\ell_1) = X^{ss}(\ell_2)$), it is clear that $\Omega(x)$ is a union of GIT classes. On the other hand, there are only finitely many open subsets of X which can be realized as $X^{ss}(\mathcal{L})$ for some $\mathcal{L} \in NS^G(X)$. It follows that there are only finitely many sets of the form $\Omega(x) \cap NS^G(X)_{\mathbb{Q}}$. Proposition 2 shows that $\Omega(x) = \overline{\Omega(x) \cap NS^G(X)_{\mathbb{Q}}}$, hence the statement. $\square$

Corollary 4: The G-ample cone $C^G(X)$ is closed in $NS^G(X)^+_{\mathbb{R}}$.
Proof: We have $C^G(X) = \bigcup_{x\in X} \Omega(x)$. By Proposition 2 each $\Omega(x)$ is closed and by Corollary 3 the number of the $\Omega(x)$ for $x \in X$ is finite. $\square$

We now turn to the study of the geometry of stability sets.
Lemma 5: Let $x \in X$, $z \in \overline{G\cdot x} \setminus G\cdot x$. Assume that there is a rational point $\ell_0 \in C^G(X)$ such that $G\cdot z$ is closed in $X^{ss}(\ell_0)$. Then, there exists $\lambda \in \mathcal{X}_*(G) = \mathrm{Hom}(\mathbb{G}_m, G)$ such that:

1) $\lim_{t\to 0} \lambda(t)\cdot x \in G z$;
2) $\Omega(x) \subseteq \{\ell \in NS^G(X)_{\mathbb{R}} \mid \mu^\ell(x,\lambda) \leq 0\}$;
3) $\Omega(z) = \{\ell \in NS^G(X)_{\mathbb{R}} \mid \mu^\ell(x,\lambda) = 0\} \cap \Omega(x)$.

Proof: As we have seen in Lemma 1, we may assume that ℓ_0 is a very ample line bundle.
1) This was (the difficult) part of the proof of Hilbert-Mumford criterion.
2) Clear from the definition of $\Omega(x)$.
3) Let us call $z' = \lim_{t\to 0} \lambda(t).x \in G\cdot z$. In particular, the image of λ fixes z'. If $\ell \in \Omega(z')$, then $z' \in X^{ss}(\ell)$; both λ and $-\lambda$ fix z', because $\lim_{t\to 0} -\lambda(t).z = \lim_{t\to\infty} \lambda(t).z = z'$. The actions of λ and $-\lambda$ on the same fiber are inverse to each other, so we have $\mu^\ell(z,-\lambda) = -\mu^\ell(z,\lambda)$. But since $z' \in X^{ss}(\ell)$, we have both $\mu^\ell(z',\lambda) \geq 0$ and $\mu^\ell(z',-\lambda) \geq 0$, which implies $\mu^\ell(z',\lambda) = 0$. We have shown that if $\ell \in \Omega(z')$, then $\mu^\ell(z',\lambda) = 0$, so

$\Omega(z') \subseteq \{ \ell \in NS^G(X)_{\mathbb{R}} \mid \mu^\ell(z',\lambda)=0 \}$. Now, since $M^\ell(-)$ is G-invariant and $z' \in G \cdot z$, we see that $\Omega(z) = \Omega(z')$. Also, $\mu^\ell(x,\lambda) = \mu^\ell(z',\lambda)$ because $\lim_{t\to 0} \lambda(t).x = z'$ and $\mu^\ell(x,\lambda)$ is precisely the weight of λ on the fiber of ℓ over such limit (when ℓ is a line bundle... otherwise we need to extend by linearity). These observations lead us to: $\Omega(z) \subseteq \{ \ell \in NS^G(X)_{\mathbb{R}} \mid \mu^\ell(x,\lambda)=0 \}$.

Let us now take $\ell \in \Omega(z) \cap NS^G(X)_{\mathbb{Q}}$, so a rational point in the stability set of ℓ. $X^{ss}(\ell)$ is open and $z \in \overline{G \cdot x}$, so $X^{ss}(\ell) \cap Gx \neq \emptyset$, and by G-invariance of $X^{ss}(\ell)$, we have $\ell \in \Omega(x)$. By Proposition 2, $\Omega(z)$ is the closure of its rational points and $\Omega(x)$ is closed, so we have $\Omega(z) \subseteq \Omega(x)$. By what we have shown before, we see that $\Omega(z) \subseteq \{ \ell \in NS^G(X)_{\mathbb{R}} \mid \mu^\ell(x,\lambda)=0 \} \cap \Omega(x)$. Let us prove the converse inclusion. Let us fix a rational point $\ell \in \Omega(x)$ such that $\mu^\ell(x,\lambda)=0$. By Lemma 1, z cannot be unstable for ℓ, so it is semistable. This means that $\ell \in \Omega(z)$. We have therefore shown that the rational points of $\{\ell \mid \mu^\ell(x,\lambda)=0\} \cap \Omega(x)$ are contained in $\Omega(z)$. However, these points are dense $\{\ell \mid \mu^\ell(x,\lambda)=0\} \cap \Omega(x)$ (for they are in $\Omega(x)$, and this set is the intersection thereof with a rational hyperplane). Since $\Omega(z)$ is closed, the claim follows. $\square$

<u>Def</u>: A polyhedral cone in $NS^G(X)^+_{\mathbb{R}}$ is a subset of $NS^G(X)^+_{\mathbb{R}}$ defined by a finite number of linear inequalities. A polyhedral cone is said "rational" if these inequalities are defined over the rationals.

Let C be a polyhedral cone, f is a linear form on $NS^G(X)^+_{\mathbb{R}}$ such that $f(c) \geq 0$ for all $c \in C$. Then $\{ c \in C \mid f(c) = 0 \}$ is a "face" of C.

(Notice that also C is a face of C itself).

<u>Proposition 6</u>: Let $x \in X$.

1) The stability set of x is a convex rational polyhedral cone in $NS^G(X)^+_{\mathbb{R}}$.
2) The faces of $\Omega(x)$ are exactly the sets $\Omega(y)$ with $y \in \overline{G \cdot x}$.
3) There exists $y \in \overline{G \cdot x}$ such that $\Omega(y) = \Omega(x)$ and y satisfies the following property: $\ell \in NS^G(X)_{\mathbb{R}}$ belongs to the relative interior of $\Omega(y)$ if and only if y is polystable for ℓ (i.e. $G \cdot y$ is closed in $X^{ss}(\ell)$)

Proof: 1) Let us start by fixing a rational point $\ell_0 \in \Omega(x)$ which lies in the relative interior of $\Omega(x)$. By definition of $\Omega(x)$, $X^{ss}(\ell_0) \neq \emptyset$, so we obtain a nonempty GIT quotient $X /\!/_{\ell_0} G$ (as usual, since ℓ_0 is a rational point of $NS^G(X)$, for our purposes we may assume that ℓ_0 is actually the class of a very ample line bundle over X, as we have explained in the proof of Lemma 1).

We know that the quotient map $\varphi : X^{ss}(\ell_0) \to X /\!/_{\ell_0} G$ is a good quotient. Given $x \in X$, the set $\varphi^{-1}(\varphi(x))$ consists of G-orbits, only one of which is closed in $X^{ss}(\ell_0)$ (because φ separates G-invariant closed subsets of X). Let us say that this orbit is the one of some $y \in X$. Thus, $y \in \overline{Gx}$ and y is polystable for ℓ_0.

If $y \in Gx$, then $\Omega(y) = \Omega(x)$, since the function $M^\ell(-)$ are G-invariant for every $\ell \in NS^G(X)$. If $y \notin Gx$, then $y \in \overline{Gx} \setminus Gx$ and by lemma 5 $\Omega(y)$ is the intersection of $\Omega(x)$ with $\{\ell \in NS^G(X) \mid \mu^\ell(x,\lambda) = 0\}$ for a fixed one parameter subgroup λ.

If $\ell \in \Omega(x)$, then $\mu^\ell(x,\lambda) \geq 0$, so $\mu^{\cdot}(x,\lambda)$ is non-negative on $\Omega(x)$. This means that $\Omega(y)$ is a face of the cone $\Omega(x)$.

On the other hand, ℓ_0 had been chosen in the relative interior of $\Omega(x)$, and $\ell_0 \in \Omega(y)$, so we must have $\Omega(y) = \Omega(x)$. This argument shows that we may always replace x by some point $y \in \overline{Gx}$ which is polystable with respect to some $\ell \in NS^G(X)_{\mathbb{Q}}$ in the relative interior of $\Omega(x)$.

By Lemma 5, the sets $\Omega(z)$ for $z \in \overline{Gy}$ are all faces of the cone $\Omega(y)$.

Claim: The relative boundary of $\Omega(y)$ is the union of its faces of the form $\Omega(z)$ for some $z \in \overline{G \cdot y}$.

Indeed, let $\ell \in \Omega(y)$ be a point in the relative boundary of $\Omega(y)$. As we have seen in Proposition 2, we may find a sequence of rational points $\{\ell_n\}_n$ in the vector space spanned by $\Omega(y)$ which converge to ℓ, but such that $\ell_n \notin \Omega(y)$; once again, we may assume that all the ℓ_n's induce the same stratification. Chosse $\lambda_\ell \in \Lambda^{\ell_1}(y)$ and set $z_\ell = \lim_{t \to 0} \lambda(t).y$. Then $z_\ell \in \overline{G \cdot y}$ and, exactly as in the proof of Proposition 2, we have that $\ell \in \Omega(z_\ell)$ and that $\Omega(z_\ell)$ is contained in the hyperplane $\{\mu^{\cdot}(x,\lambda_\ell) = 0\}$, whereas $\Omega(y)$ is not.

This means that every point ℓ in the relative boundary of $\Omega(y)$ is contained in a proper face of $\Omega(y)$ of the form $\Omega(z_\ell)$ for some $z_\ell \in \overline{G \cdot y}$.

By Corollary 1, we know that the sets of the form $\Omega(z)$ for $z \in X$ are only finitely many. Moreover, if $z \in \overline{G\cdot y} \setminus G\cdot y$, then by lemma 5 $\Omega(z)$ is a proper face of $\Omega(y)$. The above claim thus shows that $\Omega(y)$ consists of only finitely many faces, so $\Omega(y)$ is a rational polyhedral cone in $NS^G(X)^+_{\mathbb{R}}$.

2) We wish to prove that every face of $\Omega(y)$ is of the form $\Omega(z)$ for some $z \in \overline{G\cdot y}$. We have seen that any face of $\Omega(y)$ is covered by sets of the form $\Omega(z)$ for some $z \in \overline{G\cdot y}$. If a face has codimension 1 in $\Omega(y)$, then it must be of the form $\Omega(z)$ for some $z \in \overline{G\cdot y}$. Then one may argue by induction.

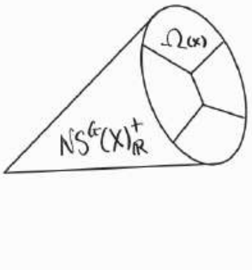
3) Suppose that $\ell \in NS^G(X)_{\mathbb{R}}$ belongs to the relative boundary of $\Omega(y)$. Then in point (1) we have found $z_\ell \in \overline{Gy} \setminus Gy$ such that $\ell \in \Omega(z_\ell)$, i.e. $z_\ell \in (\overline{Gy} \setminus Gy) \cap X^{ss}(\ell)$. This tells us that Gy cannot be closed in $X^{ss}(\ell)$, i.e. y is not polystable for ℓ. Conversely, let $\ell' \in \Omega(y)$ be such that $G\cdot y$ is not closed in $X^{ss}(\ell)$. This means that we can find $z' \in (\overline{Gy} \setminus Gy) \cap X^{ss}(\ell')$. On the other hand, $G\cdot y$ is closed in $X^{ss}(\ell_0)$, so z' cannot belong to $X^{ss}(\ell_0)$, which implies that $\Omega(z')$ is a proper face of $\Omega(y)$. But $\ell' \in \Omega(z')$, so ℓ is not in the interior of $\Omega(y)$. $\square$

<u>Remark:</u> For every $x \in X$, $\Omega(x)$ is a polyhedral cone inside $NS^G(X)^+_{\mathbb{R}}$, but not necessarily inside $NS^G(X)_{\mathbb{R}}$. Recall that we have an exact sequence

$$0 \to \mathfrak{X}^*(G)_{\mathbb{R}} \to NS^G(X)_{\mathbb{R}} \to NS(X)_{\mathbb{R}} \to 0.$$

Notice that $NS^G(X)^+_{\mathbb{R}}$ is precisely the preimage of the ample cone $\mathrm{Amp}(X) \subseteq NS(X)_{\mathbb{R}}$ in $NS^G(X)_{\mathbb{R}}$. Thus, $NS^G(X)^+_{\mathbb{R}} = \mathrm{Amp}(X) \times \mathfrak{X}^*(G)_{\mathbb{R}}$.

If E is a general elliptic curve, then $X = E \times E$ has an ample cone which is circular, so it cannot be a polyhedral cone overall. Only the internal faces of the $\Omega(x)$'s will be polyhedral cones.



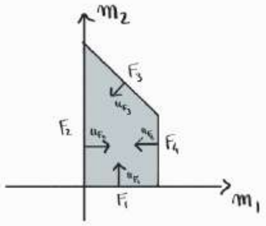
§ An example with toric varieties.

Let us fix a lattice M of rank n, $M_{\mathbb{R}} = M \otimes \mathbb{R}$, and let P be a polyhedron in $M_{\mathbb{R}}$. In the previous talks, we have associated to P a toric variety X_P to P, which comes with an action of a torus T such that $\mathfrak{X}^*(T) = M$.

Some facts known from the previous talks:

·) P is determined by a collection of vectors $u_F \in M^\vee = \mathfrak{X}_*(T)$ and real numbers $a_F \in \mathbb{R}$, one for each codimension-1 face F of P. Then
$P = \{ m \in M \mid \langle m, u_F \rangle \geq -a_F$ for all $F \in P(1) \}$, where $P(1) = \{$codim.-1 faces of $P\}$.

Example:



$u_{F_1} = (0,1)$ $\quad u_{F_2} = (1,0)$ $\quad u_{F_3} = (-1,-1)$ $\quad u_{F_4} = (-1,0)$

$a_{F_1} = 0$ $\quad a_{F_2} = 0$ $\quad a_{F_3} = 2$ $\quad a_{F_4} = 1$

·) The T-invariant prime divisors of X_P are in bijection with the codim.-1 faces of P. For $F \in P(1)$, we denote by D_F the corresponding divisor.

·) There is an exact sequence $\quad 0 \longrightarrow M \longrightarrow \mathbb{Z}^{P(1)} \longrightarrow \mathrm{Pic}(X) \longrightarrow 0$

$$m \longmapsto \sum \langle m, u_F \rangle D_F$$

Example: (continuing the example above)

A basis for M is given by $e_1 = (1,0)$ and $e_2 = (0,1)$. The map $M \to \mathbb{Z}^{P(1)}$ sends e_1 to $D_{F_2} - D_{F_3} - D_{F_4}$ and e_2 to $D_{F_1} - D_{F_3}$. Thus, $\mathrm{Pic}(X)$ is freely generated by $[D_{F_1}]$ and $[D_{F_2}]$, and we have $[D_{F_3}] = [D_{F_1}]$ and $[D_{F_4}] = [D_{F_2}] - [D_{F_1}]$.

Let $G \subseteq T$ be a subtorus and set for short $\widehat{G} = \mathfrak{X}^*(G)$. Define $T' = T/G$ and $M' = \mathfrak{X}^*(T')$. The exact sequence $1 \to G \to T \to T' \to 0$ yields a short exact sequence $0 \to M' \to M \to \widehat{G} \to 0$.

Since X_P is toric, then $NS(X_P) = \mathrm{Pic}(X_P)$. Thus, one can check that the two exact sequences $0 \to M \to \mathbb{Z}^{P(1)} \to \mathrm{Pic}(X_P) \to 0$ and $0 \to M = \mathfrak{X}^*(T) \to NS^T(X_P) \to NS(X_P) \to 0$ agree, so $NS^T(X_P) \cong \mathbb{Z}^{P(1)}$.

We want to understand $NS^G(X_P)$. We know that it fits into an exact sequence $0 \to \widehat{G} \to NS^G(X_P) \to NS(X_P) \to 0$. By comparing this with the above exact sequences, we obtain the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & M' \arrow[r, equal] \arrow[d] & M' \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & \mathbb{Z}^{P(1)} = NS^T(X_P) \arrow[r] \arrow[d] & NS(X_P) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \widehat{G} \arrow[r] \arrow[d] & NS^G(X_P) \arrow[r] \arrow[d] & NS(X_P) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

As a result, $NS^G(X_P) \cong \mathbb{Z}^{P(1)}/M'$.

This means that the choice of a G-linearized line bundle on X_P is equivalent to the choice of a T-invariant divisor up to elements of M'.

Given a T-invariant divisor D, let $\mathcal{O}_X(D)$ denote the corresponding G-linearization via $\mathbb{Z}^{P(1)}/M' \cong NS^G(X_P)$. We may define a new polytope in $M_{\mathbb{R}}$ as follows. If $D = \sum_{F\in P(1)} c_F D_F$, we set $P_D = \{ m \in M_{\mathbb{R}} \mid \langle m, u_F\rangle \geq -c_F$ for every $F \in P(1)\}$. We may then intersect P_D with $(M')_{\mathbb{R}}$ to obtain a polytope $P_D \cap (M')_{\mathbb{R}}$ in $(M')_{\mathbb{R}} = \mathcal{X}^*(T')_{\mathbb{R}}$. To this polytope we can associate the toric variety $X_{P_D \cap M_{\mathbb{R}}}$, which comes with an action of $T' = T/G$.

<u>Proposition:</u> $X_P /\!\!/_{\mathcal{O}_X(D)} G = X_{P_D \cap M'}$.

This proposition tells us how to recover the GIT qudient $X /\!\!/_{\mathcal{O}_X(D)} G$ for every point $[D]$ in $NS^G(X) \cong \mathbb{Z}^{P(1)}/M'$. By looking at which points have non-empty qudient, one can recover $C^G(X)$. How do we see $\Omega(x)$ in $NS^G(X) \cong \mathbb{Z}^{P(1)}/M'$?

For this, we need "virtual facets".

Given $D \in NS^G(X)$, say $D = \sum_{F\in P(1)} c_F D_F$, and $F' \in P(1)$, we set

$\mathcal{F}_{F',D} = P_D \cap M' \cap \{ m \in M_{\mathbb{R}} \mid \langle m, u_{F'}\rangle = -c_{F'}\}$.

Two talks ago we saw that $(X_P)^{us}(\mathcal{O}_X(D)) = \bigcup_{\substack{I \subseteq P(1) \\ \bigcap_{F'\in I} \mathcal{F}_{F',D} = \emptyset}} \bigcap_{F\in I} D_F$.

To put it more simply, $D_F \subseteq (X_P)^{us} \iff \mathcal{F}_{F,D} = \emptyset$, then

$D_{F_1} \cap D_{F_2} \subseteq (X_P)^{us} \iff \mathcal{F}_{F_1,D} \cap \mathcal{F}_{F_2,D} = \emptyset$ and so on. (containment is intended generically).

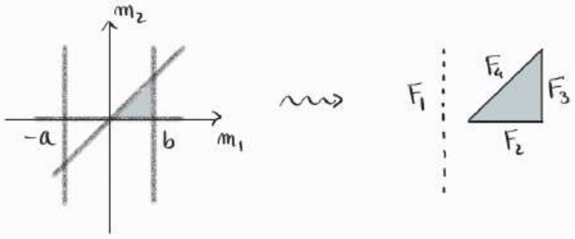
We can read this result from the point of view of a single point $x \in X$.
Given $x \in \bigcap_{F \in I} D_F$ general, we have
$\Omega(x) = \{ [\mathcal{O}_X(D)] \mid x \in X^{ss}(\mathcal{O}_X(D)) \} = \{ [\mathcal{O}_X(D)] \mid \bigcap_{F \in I} \mathcal{I}_{F,D} \neq \emptyset \} =$
$= \mathrm{Cone}\left([D_F] \mid F \in P(1), F \notin I \right).$
Let us put this in practice.

Example:

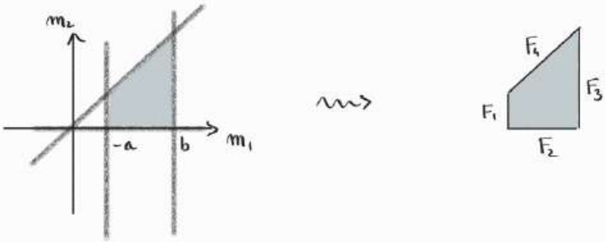
Let $M = \mathbb{Z}^4$, $M_{\mathbb{R}} = \mathbb{R}^4$, $P = [0, +\infty)^4$, so $X_P = \mathbb{A}^4$.
P has four faces, with normal vectors $u_{F_1} = (1,0,0,0)$, $u_{F_2} = (0,1,0,0)$, $u_{F_3} = (0,0,1,0)$, $u_{F_4} = (0,0,0,1)$. For all $F \in P(1)$ we have $a_F = 0$. Thus
$P = \{ m \in M_{\mathbb{R}} \mid \langle m, u_{F_i} \rangle \geq -a_{F_i} \ \forall i = 1, \dots, 4 \} = \{ m = (m_1, m_2, m_3, m_4) \in \mathbb{R}^4 \mid m_i \geq 0 \ \forall i = 1, \dots, 4 \}$
We have an action of $T = \mathbb{G}_m^4$ on X_P. Consider the subgroup G of T defined by the embedding $G = \mathbb{G}_m^2 \hookrightarrow T$, $(s,t) \mapsto (s, ts^{-1}, t, ts^{-1})$.
Let us compute $M' = \mathfrak{X}^*(T/G)$. We have an exact sequence $0 \to M' \to M \to \widehat{G} \to 0$, so M' is the kernel of the map $M \to \widehat{G}$. Given $m = (m_i)_{i=1 \dots 4} \in M$, this corresponds to the map $T \to \mathbb{G}_m$, $(t_1, \dots, t_4) \mapsto \prod_{i=1}^{4} t_i^{m_i}$. By pre-composing with $G \hookrightarrow T$, one obtains the character of G given by $G \to \mathbb{G}_m$, $(s,t) \mapsto s^{m_1} t^{m_2} s^{-m_2} t^{m_3} t^{m_4} s^{-m_4}$, so the map $M \to \widehat{G}$ is given by $(m_1, m_2, m_3, m_4) \mapsto (m_1 - m_2 - m_4, m_2 + m_3 + m_4)$.
The kernel of this map is generated by the vectors $\left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle$. $\quad m_3 = -m_1$, $m_4 = m_1 - m_2$
We have seen that $NS^G(X_P) = \mathbb{Z}^{P(1)} / M' = \mathbb{Z}^4 / \langle (1 0 -1 1), (0 1 0 -1) \rangle$. A basis for $NS^G(X_P)$ is therefore given by the image $[D_1]$ and $[D_3]$ of the vectors $(1 0 0 0)$ and $(0 0 1 0)$ in $\mathbb{Z}^{P(1)}$.
Let us compute the GIT-quotient for each point of $NS^G(X)$.
Fix $D_{a,b} = aD_1 + bD_3$. Every element of $NS^G(X)$ can be written uniquely in the form $[D_{a,b}]$ for some $a, b \in \mathbb{Z}$. We have
$P_{D_{a,b}} = \{ m \in M_{\mathbb{R}} \mid \langle m, u_{F_1} \rangle \geq -a, \langle m, u_{F_2} \geq 0, \langle m, u_{F_3} \rangle \geq -b, \langle m, u_{F_4} \rangle \geq 0 \} =$
$= \{ m \in M_{\mathbb{R}} \mid m_1 \geq -a, m_2 \geq 0, m_3 \geq -b, m_4 \geq 0 \}.$
When intersecting with M', we have $m_3 = -m_1$ and $m_4 = m_1 - m_2$. By taking coordinates (m_1, m_2) on M', we see that
$P_{D_{a,b}} \cap M' = \{ (m_1, m_2) \in M' \mid m_1 \geq -a, m_2 \geq 0, m_1 \leq b, m_1 \geq m_2 \}$

We draw all possibilities: (with virtual facets)

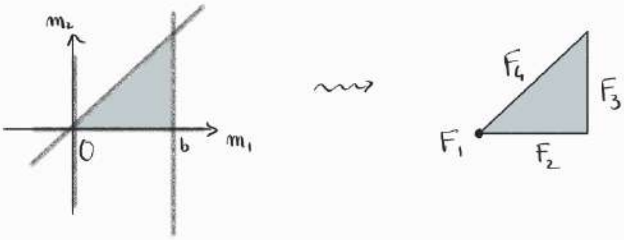
$a > 0,\ b > 0$:



$a < 0,\ b > 0$

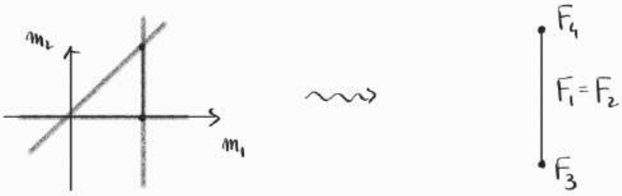


$a = 0,\ b > 0$



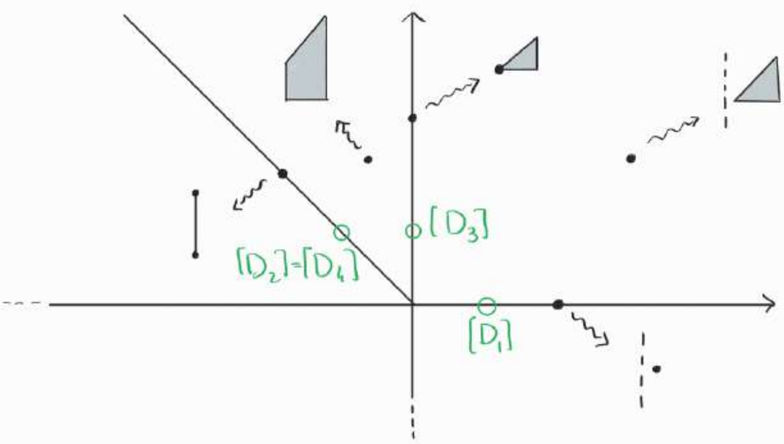
$b = 0,\ a > 0$

$b = -a$



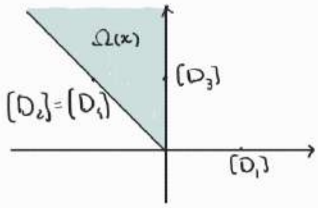
For all other choices of a and b, we obtain an empty quotient.

The overall picture of $NS^G(X)$ is:

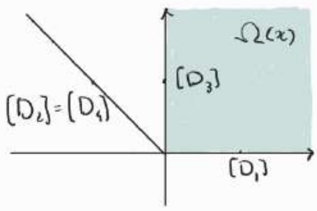


We may now describe $\Omega(x)$ for $x \in X$ by the description we have given above.

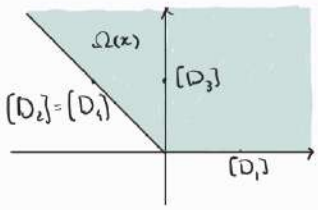
$x \in D_1$:



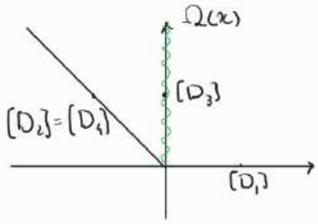
$x \in D_2 \cup D_4$:



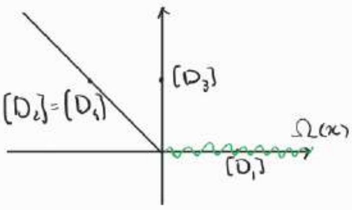
$x \in D_3$:



$x \in D_1 \cap (D_2 \cup D_4)$



$x \in D_3 \cap (D_2 \cup D_4)$



$x \in D_1 \cap D_3$