The exponential map

Let G be a connected commutative algebraic group over $k=\overline{\mathbb{Q}}$.

Then G^{an} is a connected commutative complex Lie group.

$\mathrm{Lie}(G) = \mathfrak{g}$: tangent space of G at the identity element (a k-vector space)

$\mathrm{Lie}(G^{an}) = \mathfrak{g}_{\mathbb{C}}$: tangent space of G^{an} at the identity element (a $\mathbb{C}$-vector space)

$\mathrm{Lie}(G)$ and $\mathrm{Lie}(G^{an})$ have a natural structure of Lie algebra, but, since G is commutative, the Lie brackets are trivial. Thus, we can simply regard them as vector spaces.

There is an exact functor $\mathrm{Lie}(-) : \{\text{complex Lie groups}\} \to \mathbb{C}$-Vector spaces.

<u>Lemma</u>: Given $X \in \mathfrak{g}_{\mathbb{C}}$ there is a unique morphism of Lie groups $\varphi_X : \mathbb{G}_a^{an} \to G^{an}$ such that $d\varphi_X : \mathrm{Lie}(\mathbb{G}_a^{an}) \to \mathfrak{g}_{\mathbb{C}}$ sends $\frac{d}{dt}$ to X.

<u>Def</u>: $\exp : \mathfrak{g}_{\mathbb{C}} \to G^{an}$, $X \mapsto \varphi_X(1)$. (exponential map, morphism of complex Lie groups)

Properties of exp:

$\rightsquigarrow$ $\exp : \mathfrak{g}_{\mathbb{C}} \longrightarrow G^{an}$ is the universal cover of G^{an}.

$\rightsquigarrow$ $\ker(\exp)$ is a discrete subgroup of $\mathfrak{g}_{\mathbb{C}}$ which is isomorphic to $H_1^{sing}(G^{an}, \mathbb{Z})$.

Thus, there is a short exact sequence $0 \to H_1^{sing}(G^{an}, \mathbb{Z}) \to \mathfrak{g}_{\mathbb{C}} \xrightarrow{\exp} G^{an} \to 0$.

<u>Examples</u>:

- If $G = \mathbb{G}_m$, then $G^{an} \cong \mathbb{C}^\times$, $\mathrm{Lie}(G^{an}) = \mathbb{C}$ and $\exp : \mathbb{C} \to \mathbb{C}^\times$ is the usual exponential function. Its kernel is $\Lambda = 2\pi i \cdot \mathbb{Z} \subseteq \mathbb{C}$, and $H_1^{sing}(\mathbb{C}^\times, \mathbb{Z}) \cong \mathbb{Z} \cong 2\pi i \mathbb{Z}$.
- If G is an elliptic curve, then $\mathfrak{g}_{\mathbb{C}} = \mathbb{C}$ and $\exp : \mathbb{C} \to G^{an}$ can be written in suitable projective coordinates as $z \mapsto [1 : \wp(z) : \wp'(z)]$, where $\wp$ is the Weierstraß $\wp$-function. Its kernel is a lattice in $\mathbb{C}$ generated by the periods of G.
  This realizes G^{an} as a complex torus.

Singular realization

G a connected commutative algebraic group over $k = \overline{\mathbb{Q}}$

Exponential sequence: $0 \to H_1^{sing}(G^{an}, \mathbb{Z}) \to Lie(G^{an}) \xrightarrow{exp} G^{an} \to 0$.

Def: Let $M = [L \xrightarrow{u} G]$ be a 1-motive.

Singular realization: $V_{sing}(M) = T_{sing}(M) \otimes_{\mathbb{Z}} \mathbb{Q}$ where $T_{sing}(M)$ is the fibered product (in the category of abelian groups) of the maps:

$exp: Lie(G^{an}) \to G^{an}$ and $u: L \to G^{an}$

(here we extend $u: L \to G(k)$ to $L \to G^{an}$ using $G(k) \to G(\mathbb{C})$).

Remark: the fibered product of $\varphi_1: A_1 \to B$, $\varphi_2: A_2 \to B$ in the category of abelian groups is the subgroup of $A_1 \times A_2$ given by $\{(a_1, a_2) \in A_1 \times A_2 \mid \varphi_1(a_1) = \varphi_2(a_2)\}$.

If $[0 \to G]$ is the 1-motive associated with a conn. comm. alg. grp. G, then

$$\begin{tikzcd} T_{sing}(M) \arrow[r] \arrow[d] & Lie(G^{an}) \arrow[d, "exp"] \\ 0 \arrow[r] & G^{an} \end{tikzcd}$$

implies $T_{sing}(M) = \ker(exp) = H_1^{sing}(M, \mathbb{Z})$.

Thus $V_{sing}(M) = H_1^{sing}(G^{an}, \mathbb{Z}) \otimes \mathbb{Q} \cong H_1^{sing}(G^{an}, \mathbb{Q})$.

On the other hand, if G is trivial, then $T_{sing}(M) = L$, so $V_{sing}(M) = L_{\mathbb{Q}}$.

Lemma: The functor $V_{sing}: 1\text{-}Mot_k \to \mathbb{Q}\text{-Vector spaces}$ is faithful and exact.

De Rham realisation

$\mathcal{G}$: connected commutative algebraic groups (not abelian)

$\mathcal{G}_{\mathbb{Q}}$: objects of $\mathcal{G}$, hom's tensored with $\mathbb{Q}$ (abelian)

$1\text{-}\mathrm{Mot}_k$: 1-motives over k

$1\text{-}\mathrm{MOT}_k$: 1-motives over k with hom's tensored with $\mathbb{Q}$.

The functor $\mathcal{G} \to 1\text{-}\mathrm{Mot}_k$ extends to $\mathcal{G}_{\mathbb{Q}} \to 1\text{-}\mathrm{MOT}_k$. The image of $\mathcal{G}_{\mathbb{Q}} \to 1\text{-}\mathrm{MOT}_k$ is closed under extensions, so taking $\mathrm{Ext}^1_{\mathcal{G}_{\mathbb{Q}}}$ and restricting $\mathrm{Ext}^1_{1\text{-}\mathrm{MOT}_k}$ to $\mathcal{G}_{\mathbb{Q}}$ is equivalent.

<u>Recall</u>: Given $G \in \mathcal{G}$, we have a universal vector extension: $\quad (\mathrm{Ext}^1 = \mathrm{Ext}^1_{\mathcal{G}_{\mathbb{Q}}})$

$$0 \to \mathrm{Ext}^1(G, \mathbb{G}_a)^\vee \longrightarrow G^\natural \to G \to 0.$$

Universal property: if $0 \to V \to G' \to G \to 0$ is another vector extension of G, there is a unique $G^\natural \xrightarrow{\varphi} G'$ such that
$$\begin{tikzcd} G^\natural \arrow[r] \arrow[d, "\varphi"'] & G \arrow[d, equal] \\ G' \arrow[r] & G \end{tikzcd}$$
commutes.

<u>Lemma</u>: For $M = [L \to G]$ in $1\text{-}\mathrm{Mot}_k$ there is a natural short exact sequence of k-vector spaces:
$$0 \longrightarrow \mathrm{Hom}_{ab}(L, \mathbb{G}_a) \longrightarrow \mathrm{Ext}^1_{1\text{-}\mathrm{MOT}}(M, \mathbb{G}_a) \longrightarrow \mathrm{Ext}^1_{\mathcal{G}}(G, \mathbb{G}_a) \to 0$$
In particular, these k-vector spaces are all finite dimensional.

<u>Proof</u>: Start with the short exact sequence in $1\text{-}\mathrm{MOT}_k$: $0 \to [0 \to G] \to M \to [L \to 0] \to 0$

Apply $\mathrm{Hom}_{1\text{-}\mathrm{MOT}_k}(-, [0 \to \mathbb{G}_a])$ to get a long exact sequence
$$\dots \to \mathrm{Hom}_{1\text{-}\mathrm{MOT}_k}(G, \mathbb{G}_a) \to \mathrm{Ext}^1_{1\text{-}\mathrm{MOT}_k}(L, \mathbb{G}_a) \to \mathrm{Ext}^1_{1\text{-}\mathrm{MOT}_k}(M, \mathbb{G}_a) \to \mathrm{Ext}^1_{1\text{-}\mathrm{MOT}_k}(G, \mathbb{G}_a) \to \mathrm{Ext}^2 \dots$$
Now $\mathrm{Hom}_{1\text{-}\mathrm{MOT}_k}(G, \mathbb{G}_a) = \mathrm{Hom}_{\mathcal{G}_{\mathbb{Q}}}(G, \mathbb{G}_a) = 0$ because G is semi-abelian.

Let us prove that $\mathrm{Ext}^2_{1\text{-}\mathrm{MOT}_k}(L, \mathbb{G}_a)$ is zero, i.e. the map $\mathrm{Ext}^1(M, \mathbb{G}_a) \to \mathrm{Ext}^1_{1\text{-}\mathrm{MOT}_k}(G, \mathbb{G}_a)$ is surjective. Given an extension $0 \to \mathbb{G}_a \to E \to G \to 0$, the morphism $L \to G(k)$ lifts to a morphism $L \to E(k)$ because L is free. Thus, we have a short exact sequence $0 \to [0 \to \mathbb{G}_a] \to [L \to E] \to [L \to G] \to 0$, which lies in $\mathrm{Ext}^1_{1\text{-}\mathrm{MOT}}(M, \mathbb{G}_a)$.

We conclude that we have a short exact sequence
$$0 \to \mathrm{Ext}^1_{1\text{-}\mathrm{MOT}_k}(L, \mathbb{G}_a) \to \mathrm{Ext}^1_{1\text{-}\mathrm{MOT}_k}(M, \mathbb{G}_a) \to \mathrm{Ext}^1_{1\text{-}\mathrm{MOT}_k}(G, \mathbb{G}_a) \to 0$$
The only extensions of $[L \to 0]$ by $[0 \to \mathbb{G}_a]$ are of the form
$$0 \to [0 \to \mathbb{G}_a] \to [L \to \mathbb{G}_a] \to [L \to 0] \to 0 \quad \text{for some } L \to \mathbb{G}_a(k).$$
Thus $\mathrm{Ext}^1_{1\text{-}\mathrm{MOT}_k}(L, \mathbb{G}_a) \cong \mathrm{Hom}_{ab}(L, \mathbb{G}_a(k))$. $\square$

Def: A vector extension of $M = [L \to G] \in 1\text{-}Mot_k$ is an element of $\mathcal{E}xt^1_{1\text{-}MOT_k}(V, M)$ for some vector group V.
A vector extension always has the form $0 \to [0 \to V] \to [L \to G'] \to [L \to G] \to 0$.

Fix $M \in 1\text{-}MOT_k$. The datum of a vector extension of M in $1\text{-}MOT_k$ is equivalent to giving a map $\mathcal{E}xt^1_{1\text{-}MOT_k}(M, \mathbb{G}_a)^\vee \to V$ for some vector group V.
(The argument is exactly as the one in the previous talk).
Again we have a canonical choice for $V = \mathcal{E}xt^1_{1\text{-}MOT_k}(M, \mathbb{G}_a)^\vee$, namely the identity.
This canonical vector extension is denoted by $[L \to M^\natural]$ and it is called the "universal vector extension" of the 1-motive M.

Lemma: (Universal property of the vector extension)
Fix $M = [L \to G]$. For every vector extension $[L \to G']$ of M there is a unique morphism $[L \to M^\natural] \to [L \to G']$. Moreover, $L \to M^\natural$ is injective.

Proof: The universal property follows as in the last talk. For injectivity: let $l \in L$ be in the kernel of $L \to M^\natural$. Then l also goes to zero in G. Thus, we only need to prove injectivity for $G = 0$. In this case $\mathcal{E}xt^1(M, \mathbb{G}_a) = \mathrm{Hom}(L, \mathbb{G}_a)$, so $M^\natural = \mathrm{Hom}(L, \mathbb{G}_a)^\vee$. The natural map $L \to M^\natural = \mathrm{Hom}(L, \mathbb{G}_a)^\vee$ is given by evaluation, which is injective □

Given a 1-motive $M = [L \to G]$, what is the relation between $M^\natural$ and $G^\natural$?
Since $M^\natural$ is a vector extension of $G^\natural$, we have a map $G^\natural \to M^\natural$, which fits into the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}xt^1_{\mathcal{G}_k}(G, \mathbb{G}_a)^\vee \arrow[r] \arrow[d] & G^\natural \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{E}xt^1_{1\text{-}MOT_k}(G, \mathbb{G}_a)^\vee \arrow[r] & M^\natural \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

Now we make explicit kernels and cokernels of the vertical maps:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & 0 \arrow[r] \arrow[d] & \ker\varphi \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[dddll, bend left=60] & \\
0 \arrow[r] & \mathrm{Ext}^1_{\mathcal{G}_Q}(G, G_a)^\vee \arrow[r] \arrow[d] & G^\natural \arrow[r] \arrow[d, "\varphi"] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1_{1\text{-}\mathrm{MOT}_k}(G, G_a)^\vee \arrow[r] \arrow[d] & M^\natural \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
 & \mathrm{Hom}_{ab}(L, G_a)^\vee \arrow[r] \arrow[d] & \operatorname{coker}\varphi \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}
$$

The red column is justified by one of our first lemmas. It follows that $\ker\varphi = 0$ and $\operatorname{coker}\varphi = \mathrm{Hom}_{ab}(L, G_a)^\vee$ by the snake lemma.

Overall, we get a short exact sequence

$$0 \to G^\natural \to M^\natural \to \mathrm{Hom}_{ab}(L, G_a)^\vee.$$

By the structure theorem of connected commutative algebraic group, it can be checked that this sequence splits, so $M^\natural \cong G^\natural \times \mathrm{Hom}_{ab}(L, G_a)^\vee$.

<u>Lemma:</u> The functors: $1\text{-Mot}_k \longrightarrow \mathcal{G}$, $M \mapsto M^\natural$ and $1\text{-Mot}_k \to 1\text{-MOT}_k$, $M = [L \to G] \to [L \to M^\natural]$ are faithful and exact.

<u>Proof:</u> Write $0 \to T \to G \to A \to 0$ for the canonical decomposition of G.

Since $M^\natural \cong G^\natural \times \mathrm{Hom}_{ab}(L, G_a)^\vee$, $\dim M^\natural = \mathrm{rk}(L) + \dim A^\natural + \dim T$.

For exactness, consider a short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$. We get a diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1(M_1, G_a)^\vee \arrow[r] \arrow[d] & \mathrm{Ext}^1(M_2, G_a)^\vee \arrow[r] \arrow[d] & \mathrm{Ext}^1(M_3, G_a)^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1^\natural \arrow[r] \arrow[d] & M_2^\natural \arrow[r] \arrow[d] & M_3^\natural \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G_1 \arrow[r] & G_2 \arrow[r] & G_3 \arrow[r] & 0
\end{tikzcd}
$$

To deduce exactness of the middle row is enough to compute dimensions.

For faithfulness, enough to see that if $M^\natural = 0$ then $M = 0$, which is clear $\square$

Def: Given a 1-motive $M=[L\to G]$, we define its deRham realization as
$V_{dR}(M) = Lie(M^{\natural})$.

Lemma: The functor V_{dR}: 1-$Mot_k \longrightarrow$ k-vector spaces is faithful and exact.

Proof: V_{dR} is the composition of the functors $(-)^{\natural}$ and $Lie(-)$, which are both exact. For faithfulness, it suffices to check that $V_{dR}(M)=0$ implies $M=0$. If $M^{\natural}=0$, then $G=0$, so $0=M^{\natural}=Hom(L,\mathbb{G}_a)^{\vee}$, and therefore $L=0$ $\square$

Remark: Let $G\in \mathcal{G}$ and consider its 1-motive $M=[0\to G]$. Intuitively, we would expect $V_{dR}(M) = (H^1_{dR}(G,k))^{\vee}$. Let us check this. Since $M^{\natural} = G^{\natural}\times Hom_{ab}(L,\mathbb{G}_a)^{\vee}$ and $L=0$, we have $M^{\natural}=G^{\natural}$. Thus, we have a short exact sequence

$0\to Ext^1(G,\mathbb{G}_a)^{\vee} \to Lie(G^{\natural}) \to Lie(G) \to 0$.

Since $Lie(-)$ is exact and short exact sequences split in k-vector spaces, by the structure theorem for commutative algebraic groups we only need to check this for $G=\mathbb{G}_a, \mathbb{G}_m, A$ with A an abelian variety.

If $G=\mathbb{G}_a, \mathbb{G}_m$, then $Ext^1(G,\mathbb{G}_a)^{\vee}=0$, so $Lie(G^{\natural}) = Lie(G)$.

For $G=\mathbb{G}_m$, $Lie(G)$ has dimension 1, exactly as $H^1_{dR}(G,k)^{\vee}$.

For $G=A$, we get $Lie(A^{\natural}) = H^1(A,\mathcal{O}_A)\oplus Lie(A)$. If $\dim H^1(A,\mathcal{O}_A)=\dim A$, then at least we have $\dim Lie(A^{\natural}) = 2\cdot \dim A = \dim H^1_{dR}(A,k)^{\vee}$.

For $G=\mathbb{G}_a$, there seems to be something weird going on...

I do not have a conceptual explanation for this, sorry.

The period isomorphism

There is an isomorphism of functors between $V_{sing}\otimes\mathbb{C}$ and $V_{dR}\otimes\mathbb{C}$.

Here is how it goes. Fix $M=[L\to G]\in$ 1-Mot_k and consider its universal vector extension $[L\to M^{\natural}]$. We know that the map $L\to G(k)$ factors through $L\to M^{\natural}(k)$. Moreover, we have seen that the morphism $L\to M(k)$ is injective.

By passing to analytifications, we get a commutative diagram as follows:

$$\begin{tikzcd}
 & H_1^{sing}(M^{\natural,an},\mathbb{Z}) \arrow[r] \arrow[d] & H_1^{sing}(G^{an},\mathbb{Z}) \arrow[d] \\
 & Lie(M^{\natural,an}) \arrow[r] \arrow[d] & Lie(G^{an}) \arrow[d] \\
L \arrow[r] & M^{\natural,an} \arrow[r] & G^{an}
\end{tikzcd}$$

The arrow $H_1^{sing}(M^{\natural,an},\mathbb{Z}) \longrightarrow H_1^{sing}(G^{\natural,an},\mathbb{Z})$ is an isomorphism. Indeed, $M^\natural \cong G^\natural \times Hom_{ab}(L,\mathbb{G}_a)^\vee$ is a vector bundle over $G^\natural$, which is itself a vector bundle over G via the sequence $0 \to Ext^1(G,\mathbb{G}_a)^\vee \to G^\natural \to G \to 0$. By homotopy invariance, $M^{\natural,an}$ and G^{an} have the same homology.

This implies that the pullbacks $L \times_{G^{an}} Lie(G^{an})$ and $L \times_{M^{\natural,an}} Lie(M^{\natural,an})$ are isomorphic.

Now notice that:

⇝ $L \times_{G^{an}} Lie(G^{an}) = T_{sing}(M)$

⇝ $L \times_{M^{\natural,an}} Lie(M^{\natural,an}) \longrightarrow Lie(M^{\natural,an})$ is injective, because $L \to M^{\natural,an}$ is.

Also $Lie(M^{\natural,an}) = Lie(M^\natural) \otimes \mathbb{C} = V_{dR}(M) \otimes \mathbb{C}$.

By tensoring with $\mathbb{C}$, we obtain a map $\phi_M : V_{sing}(M) \otimes \mathbb{C} \to V_{dR}(M) \otimes \mathbb{C}$.

<u>Theorem</u>: ϕ_M is an isomorphism.

<u>Proof</u>: It is enough to treat the case of vector groups, tori and abelian varieties separately.

The analytic subgroup theorem

Let G be a connected commutative algebraic group over $\overline{\mathbb{Q}}$ with Lie algebra $\mathfrak{g}$.
G^{an} is a complex Lie group, with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$.
We have a short exact sequence $0 \to H_1^{sing}(G^{an}, \mathbb{Z}) \to \mathfrak{g}_{\mathbb{C}} \to G^{an} \to 0$.
There is a correspondence $\{$ Lie subalgebras of $\mathfrak{g}_{\mathbb{C}} \} \longleftrightarrow \{$ Lie subgroups of $G^{an} \}$

$$\mathfrak{h} \longmapsto \exp(\mathfrak{h})$$

Let us start with a Lie subalgebra $\mathfrak{b} \subseteq \mathfrak{g}$ and set $\mathfrak{b}_{\mathbb{C}} = \mathfrak{b} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$. Let B denote the analytic subgroup of G^{an} which corresponds to $\mathfrak{b}$.

Question: Does B contain any algebraic point of G other than 0 ?

Set $B(\overline{\mathbb{Q}}) = B \cap G(\overline{\mathbb{Q}})$

Theorem (Analytic subgroup theorem)

The group of algebraic points $B(\overline{\mathbb{Q}})$ of B is non-trivial if and only if there is a connected algebraic subgroup H of G with Lie subalgebra $\mathfrak{h}$ such that $0 \neq \mathfrak{h} \subseteq \mathfrak{b}$.

This means: if an analytic subgroup of G contains an algebraic point, then it must contain a whole connected (!) algebraic subgroup.

Illustration: Take $G = \mathbb{G}_m \times \mathbb{G}_m$. Topologically, $G^{an} \sim S^1 \times S^1$ is a torus.
As an analytic subgroup of a torus (identified with $\mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z} \oplus \mathbb{Z}i$), we may take any real line in $\mathbb{C}$ passing through the origin. Algebraic subgroups of this torus should correspond to copies of S^1 (if they are connected and different from G), which are in turn given by lines with rational (algebraic?) slope. Lines with irrational slope give subgroups which are dense in the torus. These subgroups cannot contain an algebraic point: if they did, by the analytic subgroup theorem they would also contain a copy of S^1.

The proof of the analytic subgroup theorem heavily relies on techniques of transcendental number theory.

A refined version of the analytic subgroup theorem:
Let $\langle \cdot, \cdot \rangle$ denote the duality pairing $\mathfrak{g}^\vee \times \mathfrak{g} \to \overline{\mathbb{Q}}$. For any $u \in \mathfrak{g}_{\mathbb{C}}$ such that $\exp(u) \in G(\overline{\mathbb{Q}})$. Set $u^\perp = \{ f \in \mathfrak{g}^\vee_{\mathbb{C}} \mid f(u) = 0 \}$ and define the annihilator of u to be $\mathrm{Ann}(u) = u^\perp \cap \mathfrak{g}^\vee$ (thus, we consider only the algebraic points of $u^\perp$)

Example: Consider $G = \mathbb{G}_m^2$, so $G^{an} = (\mathbb{C}^\times)^2$, $\mathrm{Lie}(G^{an}) = \mathbb{C}^2$ with exponential map $(\exp, \exp): \mathbb{C}^2 \to (\mathbb{C}^\times)^2$. Take $u = (\log 2, \log 3) \in \mathbb{C}^2$, so $\exp(u) = (2,3)$ is an algebraic point of $\mathbb{G}_m^2$. Then $f \in u^\perp$ if and only if $f: \mathbb{C}^2 \to \mathbb{C}$, $f(x,y) = \alpha x + \beta y$ vanishes at u, i.e. $\alpha \log 2 + \beta \log 3 = 0$. Thus $\mathrm{Ann}(u) \neq 0$ if and only if there are $\alpha, \beta \in \overline{\mathbb{Q}}$ such that $\alpha \log 2 + \beta \log 3 = 0$, i.e. $\log 2$ and $\log 3$ are linearly independent over $\overline{\mathbb{Q}}$.
In general: fix a basis of $\mathfrak{g}$ and write $u = (u_1, \dots, u_n)$. Then
$\dim \mathrm{Ann}(u) = n - \dim_{\overline{\mathbb{Q}}} \mathrm{Span}_{\overline{\mathbb{Q}}} \langle u_1, \dots, u_n \rangle$.

Theorem: Let G be a connected commutative algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$, $u \in \mathfrak{g}_{\mathbb{C}}$. Assume that $\exp_G(u) \in \overline{\mathbb{Q}}$. Then there exists a connected commutative algebraic subgroup H of G defined over $\overline{\mathbb{Q}}$ with the following properties: $(\mathfrak{h} := \mathrm{Lie}(H) \subseteq \mathrm{Lie}(G))$
1) $u \in \mathfrak{h}_{\mathbb{C}}$;
2) $\mathrm{Ann}(u) = (\mathfrak{g}/\mathfrak{h})^\vee$ $(\subseteq \mathfrak{g}^\vee)$.
Moreover, H is uniquely determined by these properties.

Proof: Write $P = \exp_G(u)$, which by assumption lies in $G(\overline{\mathbb{Q}})$. If $u = 0$, then we are done with $H = 0$. If $u \neq 0$, but $P = \exp_G(u) = 0$, then we may replace u by $\frac{1}{n}u$ for any $n \in \mathbb{N}$. Since $\ker(\exp)$ is discrete, there is $n \in \mathbb{N}$ big enough such that $\exp_G(\frac{1}{n}u) \neq 0$. Moreover, $\exp_G(\frac{1}{n}u)$ is a torsion point of G, so it is still in $G(\overline{\mathbb{Q}})$. It follows that we may assume $P \neq 0$.
Consider $\mathfrak{b} = \mathrm{Ann}(u)^\perp \subseteq \mathfrak{g}$. Then $u \in \mathfrak{b}_{\mathbb{C}}$, so by the analytic subgroup theorem there is a subgroup $H \subseteq G$ (connected, algebraic) with Lie algebra $\mathfrak{h}$ such that $0 \neq \mathfrak{h} \subseteq \mathfrak{b}$. We may also assume that $u \in \mathfrak{h}_{\mathbb{C}}$. Otherwise, we apply the same argument to G/H, which has smaller dimension, and we go on like this. This process terminates, as the involved dimensions are finite, so we shall assume $u \in \mathfrak{h}_{\mathbb{C}}$.

Since $\mathfrak{h} \subseteq \mathfrak{b} = \mathrm{Ann}(u)^{\perp}$, we get $\mathrm{Ann}(u) = \mathfrak{b}^{\perp} \subseteq \mathfrak{h}^{\perp}$. Since $u \in \mathfrak{h}_{\mathbb{C}}$, we have $(\mathfrak{h}^{\perp})_{\mathbb{C}} \subseteq u^{\perp}$, so $\mathfrak{h}^{\perp} \subseteq \mathrm{Ann}(u)$. Thus, $\mathfrak{h}^{\perp} = \mathrm{Ann}(u)$.

We have exact sequences:

$$0 \to H \to G \xrightarrow{\varpi} G/H \to 0, \quad 0 \to \mathfrak{h} \to \mathfrak{g} \xrightarrow{\varpi_*} \mathfrak{g}/\mathfrak{h} \to 0, \quad 0 \to (\mathfrak{g}/\mathfrak{h})^{\vee} \xrightarrow{\varpi^*} \mathfrak{g}^{\vee} \to \mathfrak{h}^{\vee} \to 0$$

Now $\mathfrak{h}^{\perp} = \pi^*((\mathfrak{g}/\mathfrak{h})^{\vee})$. This is just linear algebra:

$\rightsquigarrow$ if $f \in \mathfrak{h}^{\perp}$, then f vanishes over $\mathfrak{h}$, so it induces a linear map $f : \mathfrak{g}/\mathfrak{h} \to \mathbb{C}$, i.e. f lies in $(\mathfrak{g}/\mathfrak{h})^{\vee}$.

$\rightsquigarrow$ since $\varpi_* \mathfrak{h} = 0$, we have $0 = \langle (\mathfrak{g}/\mathfrak{h})^{\vee}, \varpi_* \mathfrak{h} \rangle = \langle \varpi^*((\mathfrak{g}/\mathfrak{h})^{\vee}), \mathfrak{h} \rangle$, so $\varpi_*((\mathfrak{g}/\mathfrak{h})^{\vee}) \subseteq \mathfrak{h}^{\perp}$.

This proves that $\mathrm{Ann}(u) = \varpi^*((\mathfrak{g}/\mathfrak{h})^{\vee})$.

If H' is another algebraic subgroup of G with properties (1) and (2), with Lie algebra $\mathfrak{h}'$, then (2) implies that $\varpi'^*((\mathfrak{g}/\mathfrak{h}')^{\vee}) = \varpi^*((\mathfrak{g}/\mathfrak{h})^{\vee})$, so $\mathfrak{h}' = \mathfrak{h}$ and $H' = H$. $\square$