

THE PERIOD CONJECTURE FOR 1-MOTIVES

[HW18] Huber & Wüstholz, "Transcendence and linear relations of 1-periods"

Recall that in talk 8 we defined cohomological periods, formal periods and the evaluation map, and we stated the period conjecture:

$\mathcal{P} := \bigcup_{(X,Y,i) \in \text{Pair}^{\text{eff}}/\bar{\mathbb{Q}}} \text{Im}(\text{per} : H_{\text{DR}}^1(X,Y/\bar{\mathbb{Q}}) \times H_{\text{sing}}^1(X,Y;\mathbb{Q}) \rightarrow \mathbb{C})$ SPACE OF COHOMOLOGICAL PERIODS

$(w, \sigma) \longmapsto \text{comp}(w)(\sigma)$

was deduced by \mathcal{P}^{eff} . We avoid the decoration since we will consider only the effective setting.

$\text{comp} : H_{\text{DR}}^1(X,Y/\bar{\mathbb{Q}}) \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \xrightarrow{\cong} H_{\text{sing}}^1(X,Y;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$

$\tilde{\mathcal{P}} := \bar{\mathbb{Q}}$ -vector space generated by symbols $[X,Y,i,w,\sigma]$ SPACE OF FORMAL PERIODS

with $(X,Y,i) \in \text{Pair}^{\text{eff}}/\bar{\mathbb{Q}}$, $w \in H_{\text{DR}}^1(X,Y/\bar{\mathbb{Q}})$, $\sigma \in H_{\text{sing}}^1(X,Y;\mathbb{Q})$,
modulo the relations:

(A) bilinearity in w and σ

(B) induced by edges in $\text{Pair}^{\text{eff}}/\bar{\mathbb{Q}}$:

$$(B1) \quad (X',Y',i) \rightarrow (X,Y,i) \rightsquigarrow [X,Y,i, f^*w', \sigma] = [X',Y',i, w', f_*\sigma]$$

$$f : (X,Y) \rightarrow (X',Y')$$

$$(B2) \quad (Y,Z,i) \rightarrow (X,Y,i+1) \rightsquigarrow [X,Y,i+1, dw, \sigma] = [Y,Z,i, w, \partial\sigma]$$

$$d : H_{\text{DR}}^1(Y,Z/\bar{\mathbb{Q}}) \rightarrow H_{\text{DR}}^{1+1}(X,Y/\bar{\mathbb{Q}}) \quad \partial : H_{\text{sing}}^1(X,Y;\mathbb{Q}) \rightarrow H_{\text{sing}}^1(Y,Z;\mathbb{Q})$$

boundary maps of long exact sequences of the pair.

$\text{ev} : \tilde{\mathcal{P}} \rightarrow \mathbb{C}$ the $\bar{\mathbb{Q}}$ -linear map s.t. $[X,Y,i,w,\sigma] \mapsto \text{per}(w,\sigma)$

EVALUATION MAP

Conj: (PERIOD CONJECTURE)

[HW18, Conj. 13.1] | $\text{ev} : \tilde{\mathcal{P}} \rightarrow \mathbb{C}$ is injective.

The image is $\text{ev}(\tilde{\mathcal{P}}) = \mathcal{P}$. This conjecture tells that all $\bar{\mathbb{Q}}$ -linear relations between cohomological periods are given by (A), (B1) & (B2).

The aim of this talk is to prove this conjecture for "1-periods".

Plan: §1. Say what we mean by "1-periods" and state precisely the period conjecture we are going to prove.

§2. The statement of this conjecture doesn't involve motives, but we use the motivic framework to prove it!

We need to formalize the period conjecture using motives.

This formalization can be applied to different categories of motives.

Taking the category of NORI 1-MOTIVES we recover the period conjecture stated in §1.

Taking the category of DEUGNE 1-MOTIVES we obtain an equivalent conjecture, that we will prove in §3.

§3. We prove the PERIOD CONJECTURE FOR DEUGNE 1-MOTIVES.

§ 1. 1-PERIODS AND THEIR PERIOD CONJECTURE

Def: We define the **SPACE OF COHOMOLOGICAL 1-PERIODS**
 [HW18, Def. 12.2] $P^1 := \bigcup_{\substack{(X,Y,1) \in \text{Pair}^{\text{eff}}/\bar{\mathbb{Q}} \\ X,Y \text{ are alg!}}} \text{Im} \left(\text{per} : \underbrace{H_{\text{DR}}^1(X,Y/\bar{\mathbb{Q}}) \times H_1^{\text{sing}}(X,Y;\mathbb{Q})}_{=: P(X,Y,1)} \rightarrow \mathbb{C} \right)$

Rank: • $P^1 \subset P$ is a $\bar{\mathbb{Q}}$ -subvector space of \mathbb{C}

Differently from P , P^1 is not a $\bar{\mathbb{Q}}$ -algebra! We don't care about multiplicative structure in this talk.

[HW18, Lem. 12.4] For the same reason why P is:

- The sum of 2 periods of $(X,Y,1)$ and $(X',Y',1)$ can be realized as a period of $(X \amalg X', Y \amalg Y', 1)$ (see talk 8)
- Action of $\bar{\mathbb{Q}}$ is given by $\bar{\mathbb{Q}}$ -linearity of $H_{\text{DR}}^1(X,Y/\bar{\mathbb{Q}})$. \blacksquare

- P^1 contains also

$$P^0 := \bigcup_{(X,Y,0) \in \text{Pair}^{\text{eff}}/\bar{\mathbb{Q}}} \text{Im} \left(\text{per} : \underbrace{H_{\text{DR}}^0(X,Y/\bar{\mathbb{Q}}) \times H_0^{\text{sing}}(X,Y;\mathbb{Q})}_{=: P(X,Y,0)} \rightarrow \mathbb{C} \right)$$

Indeed $P^0 = \bar{\mathbb{Q}}$.

Alternatively, notice that a period of $(X,Y,0)$ is also a period of $(X \times \mathbb{A}^1, Y \times \mathbb{A}^1, 1)$ (see talk 8) \blacksquare

So, 1-PERIODS are PERIODS obtained by restricting to consider cohomological degree ≤ 1 .

Their period conjecture is obtained by restricting the PERIOD CONJECTURE to cohomological degree ≤ 1 .

More precisely, we consider

$$\text{Pair}^1/\bar{\mathbb{Q}} \subset \text{Pair}^{\text{eff}}/\bar{\mathbb{Q}}$$

the free subdiagram containing vertices of the kind

$$(X,Y,i) \text{ for } \underline{i \leq 1}.$$

Def: We define the **SPACE OF FORMAL 1-PERIODS**

$\tilde{\mathcal{P}}^1 := \bar{\mathbb{Q}}$ -vector space generated by symbols

$$[X, Y, i, w, \sigma] \quad \underline{i \leq 1}$$

modulo relations:

(A') bilinearity in w & σ

(B') induced by edges in $\text{Pair}^1 / \bar{\mathbb{Q}}$:

$$(B1') \quad \underset{i \leq 1}{(X', Y', i) \rightarrow (X, Y, i)} \rightsquigarrow [X, Y, i, f^* w', \sigma] = [X', Y', i, w', f_* \sigma] \quad \underset{i \leq 1}{}$$

$$(B2') \quad (Y, Z, 0) \rightarrow (X, Y, 1) \rightsquigarrow [X, Y, 1, dw, \sigma] = [Y, Z, 0, w, \partial \sigma]$$

We still have an **EVALUATION MAP**

$$\text{ev}^1: \tilde{\mathcal{P}}^1 \rightarrow \mathbb{C}$$

the $\bar{\mathbb{Q}}$ -linear map s.t. $[X, Y, i, w, \sigma] \mapsto \text{per}(w, \sigma) \quad \underset{i \leq 1}{}$

Thm: **(PERIOD CONJECTURE FOR 1-PERIODS)**

[HW18, Thm. B.3 (3)] | $\text{ev}^1: \tilde{\mathcal{P}}^1 \rightarrow \mathbb{C}$ is injective.

The image is $\text{ev}(\tilde{\mathcal{P}}^1) = \mathcal{P}^1$. This conjecture tells that all $\bar{\mathbb{Q}}$ -linear relations between cohomological 1-periods are given by (A'), (B1') & (B2').

This is the "conjecture for 1-periods" we are going to prove in the talk.

§ 2. THE MOTIVIC FORMULATION

Idea: A category of motives over $\bar{\mathbb{Q}}$ $\mathcal{M}(\bar{\mathbb{Q}})$ comes equipped with a de Rham and a singular realization functors, which are isomorphic extending scalars to \mathbb{C}

$$\begin{array}{ccc}
 \mathcal{M}(\bar{\mathbb{Q}}) & \xrightarrow{H_{\text{dR}}} & \bar{\mathbb{Q}}\text{-Vect} \\
 & \searrow^{H_{\text{sing}}} & \cong_{\text{comp}} \\
 & & \mathbb{Q}\text{-Vect}
 \end{array}
 \begin{array}{c}
 \xrightarrow{-\otimes_{\bar{\mathbb{Q}}}\mathbb{C}} \\
 \xrightarrow{-\otimes_{\mathbb{Q}}\mathbb{C}}
 \end{array}
 \begin{array}{c}
 \\
 \mathbb{C}\text{-Vect}
 \end{array}$$

They can be put together to give rise to a de Rham - singular realization functor

$$\mathcal{M}(\bar{\mathbb{Q}}) \longrightarrow (\bar{\mathbb{Q}}, \mathbb{Q})\text{-Vect}$$

$$M \longmapsto (H_{\text{dR}}(M), H_{\text{sing}}(M), \text{comp}_M)$$

$$\text{comp}_M: H_{\text{dR}}(M) \otimes_{\bar{\mathbb{Q}}}\mathbb{C} \xrightarrow{\cong} H_{\text{sing}}(M) \otimes_{\mathbb{Q}}\mathbb{C}$$

With this data, we can define the SPACE OF PERIODS of $\mathcal{M}(\bar{\mathbb{Q}})$

$$\begin{aligned}
 \mathcal{P}(\mathcal{M}(\bar{\mathbb{Q}})) &:= \bigcup_{M \in \mathcal{M}(\bar{\mathbb{Q}})} \underbrace{\text{Im}(\text{per}: H_{\text{dR}}(M) \times H_{\text{sing}}(M)^{\vee} \rightarrow \mathbb{C})}_{=: P(M)} \\
 (w, \sigma) &\longmapsto \text{comp}_M(w)(\sigma)
 \end{aligned}$$

and the SPACE OF FORMAL PERIODS of $\mathcal{M}(\bar{\mathbb{Q}})$

$$\begin{aligned}
 \tilde{\mathcal{P}}(\mathcal{M}(\bar{\mathbb{Q}})) &:= \bar{\mathbb{Q}}\text{-vector space generated by symbols} \\
 & [M, w, \sigma] \quad \text{with } w \in H_{\text{dR}}(M) \\
 & \quad \quad \quad \sigma \in H_{\text{sing}}(M)^{\vee}
 \end{aligned}$$

modulo the relations:

(A) bilinearity in w and σ

(B) induced by morphisms in $\mathcal{M}(\bar{\mathbb{Q}})$

We also have an EVALUATION MAP

$$\begin{aligned}
 \text{ev}: \tilde{\mathcal{P}}(\mathcal{M}(\bar{\mathbb{Q}})) &\longrightarrow \mathcal{P}(\mathcal{M}(\bar{\mathbb{Q}})) \subset \mathbb{C} \\
 (M, w, \sigma) &\longmapsto \text{comp}_M(w)(\sigma)
 \end{aligned}$$

We say that the PERIOD CONJECTURE holds for $\mathcal{M}(\bar{\mathbb{Q}})$ if ev is injective.

This idea can be easily generalized to any category (or more generally a direct diagram) with a functor (representation) into (K, L) -Vect for some $K, L \subset \mathbb{C}$ subfields. [HW18, §7.1]

However, we directly give the examples we are interested in, without entirely repeating each time the definitions exploited above.

I) $\text{Pair}^{\text{eff}} / \bar{\mathbb{Q}} \rightarrow (\bar{\mathbb{Q}}, \mathbb{Q})\text{-Vect}$

$$(X, Y, i) \mapsto (H_{\text{dn}}^i(X, Y; \bar{\mathbb{Q}}), H_{\text{sing}}^i(X, Y; \mathbb{Q}), \text{comp})$$

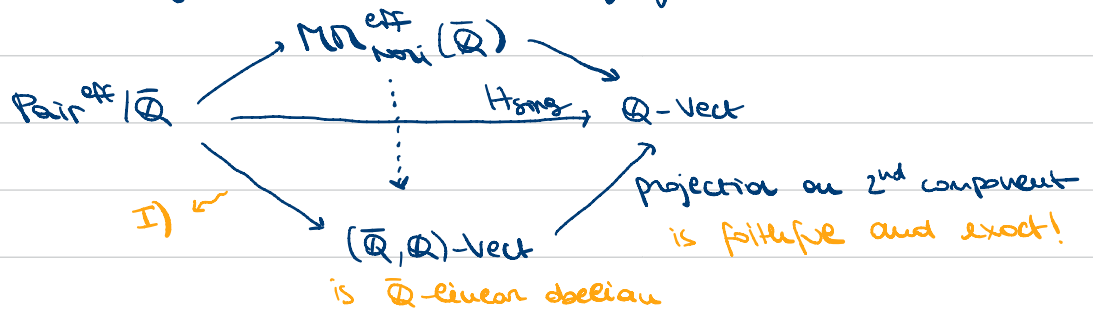
We recover exactly the notions of

$$P(\text{Pair}^{\text{eff}} / \bar{\mathbb{Q}}) = P \quad \text{SPACE OF COHOMOLOGICAL PERIODS}$$

$$\tilde{P}(\text{Pair}^{\text{eff}} / \bar{\mathbb{Q}}) = \tilde{P} \quad \text{SPACE OF FORMAL PERIODS}$$

and the PERIOD CONJECTURE! It's immediate by definition!

II) By universal property of diagram category



we obtain the functor

$$\text{MN}_{\text{noni}}^{\text{eff}}(\bar{\mathbb{Q}}) \rightarrow (\bar{\mathbb{Q}}, \mathbb{Q})\text{-Vect}$$

and hence the $\bar{\mathbb{Q}}$ -vector spaces

$$P(\text{MN}_{\text{noni}}^{\text{eff}}(\bar{\mathbb{Q}})) \quad \text{SPACE OF PERIODS OF NONI MOTIVES}$$

$$\tilde{P}(\text{MN}_{\text{noni}}^{\text{eff}}(\bar{\mathbb{Q}})) \quad \text{SPACE OF FORMAL PERIODS OF NONI MOTIVES}$$

and a PERIOD CONJECTURE FOR NONI MOTIVES.

As usual, this conjecture tells that $\bar{\mathbb{Q}}$ -linear relations between PERIODS OF NONI MOTIVES are given by

(A) bilinearity

(B) induced by morphisms in the CATEGORY OF NONI MOTIVES.

It's not so clear what it is, especially because morphisms in the category of non motives are difficult to understand. It's the big drawback of non motives!
 In fact, it is nothing new!

Prop : The period conjecture for non motives is equivalent to the period conjecture.

- Since $M_{\text{non}}^{\text{eff}}(\bar{\mathbb{Q}}) = \langle H^i(X, Y) \rangle$, \leadsto thick abelian subcategory generated (= full, closed under subquotients)

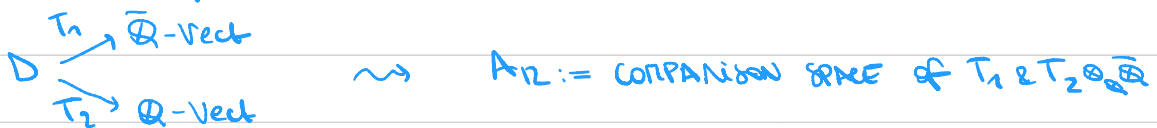
then

$$P(M_{\text{non}}^{\text{eff}}(\bar{\mathbb{Q}})) = \langle P(H^i(X, Y)) \rangle_{\bar{\mathbb{Q}}} = \langle P(X, Y, i) \rangle_{\bar{\mathbb{Q}}} = \bigcup_{(X, Y, i) \in \text{Pair}^{\text{eff}}/\bar{\mathbb{Q}}} P(X, Y, i) = P(\text{Pair}^{\text{eff}}/\bar{\mathbb{Q}})$$

easy to check \uparrow $\bar{\mathbb{Q}}$ -vector space generated

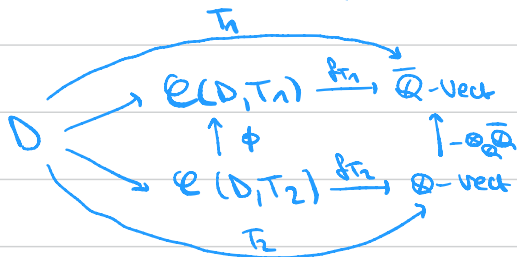
- Recall that in talk 8 we saw that:

Given 2 representations



and $\phi: T_1 \otimes_{\mathbb{Q}} \mathbb{C} \cong T_2 \otimes_{\mathbb{Q}} \mathbb{C} \rightsquigarrow P_{12} := \text{FORMAL PERIODS}$, then

- $P_{12} \cong A_{12}$ as $\bar{\mathbb{Q}}$ -vector spaces
- $A_{12} \cong A(\mathcal{E}(D, T_2), f_{T_1} \circ \phi, f_{T_2} \otimes \bar{\mathbb{Q}}) := \text{COMPANION SPACE OF } f_{T_1} \circ \phi \text{ \& } f_{T_2} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$



Applying this to $D = \text{Pair}^{\text{eff}}/\bar{\mathbb{Q}}$, $T_1 = H_{\text{dr}}$ & $T_2 = H_{\text{sing}}$, we obtain a canonical isomorphism

$$\begin{aligned}
 \tilde{P}(\text{Pair}^{\text{eff}}/\bar{\mathbb{Q}}) &\stackrel{1)}{\cong} A(\text{Pair}^{\text{eff}}, H_{\text{dr}}, H_{\text{sing}} \otimes \bar{\mathbb{Q}}) \cong \\
 &\stackrel{2)}{\cong} A(M_{\text{non}}^{\text{eff}}(\bar{\mathbb{Q}}), H_{\text{dr}}, H_{\text{sing}} \otimes \bar{\mathbb{Q}}) \cong \\
 &\stackrel{1)}{\cong} \tilde{P}(M_{\text{non}}^{\text{eff}}(\bar{\mathbb{Q}}))
 \end{aligned}$$

- We have the commutative square

$$\begin{array}{ccc}
 \tilde{P}(\text{Pair}^{\text{eff}}/\bar{\mathbb{Q}}) & \xrightarrow{\text{ev}} & P(\text{Pair}^{\text{eff}}/\bar{\mathbb{Q}}) \\
 \downarrow \cong & & \parallel \\
 \tilde{P}(M_{\text{non}}^{\text{eff}}(\bar{\mathbb{Q}})) & \xrightarrow{\text{ev}} & P(M_{\text{non}}^{\text{eff}}(\bar{\mathbb{Q}}))
 \end{array}$$

Hence PERIOD CONJECTURES for $\text{Pair}^{\text{eff}}/\bar{\mathbb{Q}}$ and $M_{\text{non}}^{\text{eff}}(\bar{\mathbb{Q}})$ are equivalent

$\subset \text{Pair}^{\text{eff}}/\bar{\mathbb{Q}} \rightarrow$

III) $\text{Pair}^1/\bar{\mathbb{Q}} \rightarrow (\bar{\mathbb{Q}}, \mathbb{Q})\text{-Vect}$

$(X, Y, i) \mapsto (H_{\text{DR}}^i(X, Y/\bar{\mathbb{Q}}), H_{\text{sing}}^i(X, Y; \mathbb{Q}), \text{comp})$
 $X, Y \text{ any}, i \leq 1$

We recover exactly the notions of

$\mathcal{P}(\text{Pair}^1/\bar{\mathbb{Q}}) = \mathcal{P}$ SPACE OF COHOMOLOGICAL 1-PERIODS
 $\tilde{\mathcal{P}}(\text{Pair}^1/\bar{\mathbb{Q}}) = \tilde{\mathcal{P}}^1$ SPACE OF FORMAL 1-PERIODS

and the PERIOD CONJECTURE FOR 1-PERIODS!

It's immediate by definition!

IV) Recall the CATEGORY OF NORA 1-NOTIVES

$d_1 \mathcal{M}_{\text{Nori}}^{\text{eff}}(\bar{\mathbb{Q}}) := \langle H^0(X, Y), H^1(X, Y) \rangle \subset \mathcal{M}_{\text{Nori}}^{\text{eff}}(\bar{\mathbb{Q}})$

thick abelian subcategory generated (= full, closed under subquotients)

Composing with inclusion, we get

$d_1 \mathcal{M}_{\text{Nori}}^{\text{eff}}(\bar{\mathbb{Q}}) \subset \mathcal{M}_{\text{Nori}}^{\text{eff}}(\bar{\mathbb{Q}}) \rightarrow (\bar{\mathbb{Q}}, \mathbb{Q})\text{-Vect}$

and hence the $\bar{\mathbb{Q}}$ -vector spaces

$\mathcal{P}(d_1 \mathcal{M}_{\text{Nori}}^{\text{eff}}(\bar{\mathbb{Q}}))$ SPACE OF PERIODS OF NORA 1-NOTIVES

$\tilde{\mathcal{P}}(d_1 \mathcal{M}_{\text{Nori}}^{\text{eff}}(\bar{\mathbb{Q}}))$ SPACE OF FORMAL PERIODS OF NORA 1-NOTIVES

and a PERIOD CONJECTURE FOR NORA 1-NOTIVES.

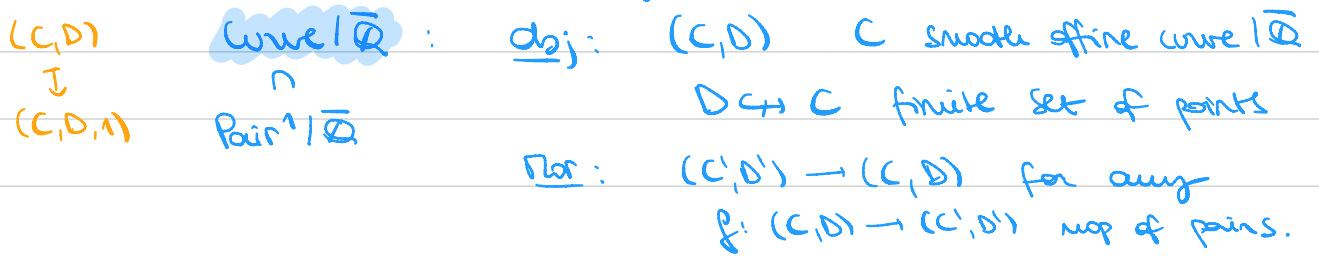
As usual, this conjecture tells that $\bar{\mathbb{Q}}$ -linear relations between PERIODS OF NORA NOTIVES are given by

(A) bilinearity

(B) induced by morphisms in the CATEGORY OF NORA 1-NOTIVES.

Lemma: $d_1 \mathcal{M}_{\text{Nori}}^{\text{eff}}(\bar{\mathbb{Q}})$ is equivalent to the diagram category of $\text{Pair}^1/\bar{\mathbb{Q}}$ with representation $H_{\text{sing}}^* : \text{Pair}^1/\bar{\mathbb{Q}} \subset \text{Pair}^{\text{eff}}/\bar{\mathbb{Q}} \xrightarrow{H_{\text{sing}}^*} \mathbb{Q}\text{-Vect}$.

Recall the direct diagram



The Theorem by Ayoub & Barbieri-Viale cited in talk 9 states that the diagram category of $\text{Cwve}/\bar{\mathbb{Q}}$ with representation

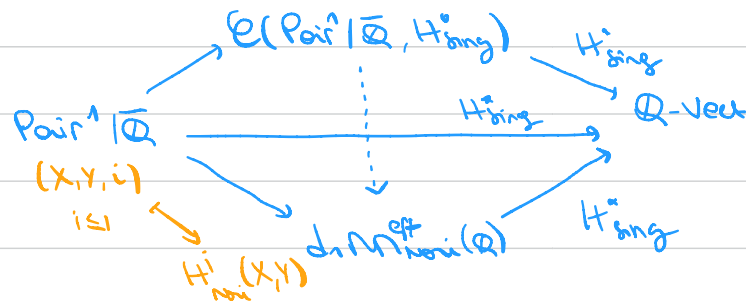
$$H_{\text{sing}}^1: \text{Cwve}/\bar{\mathbb{Q}} \subset \text{Pair}^1/\bar{\mathbb{Q}} \xrightarrow{H_{\text{sing}}^*} \mathbb{Q}\text{-Vect}$$

$$(C, D) \mapsto (C, D, 1) \mapsto H_{\text{sing}}^1(C, D; \mathbb{Q})$$

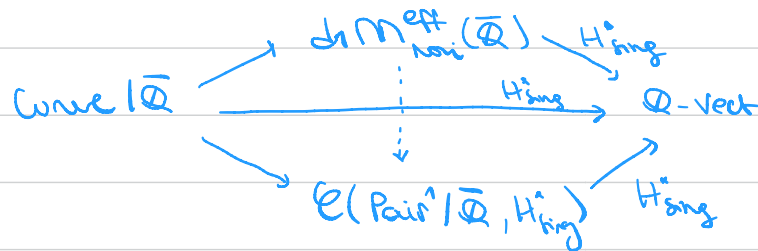
is equivalent to $d_1 M_{\text{non}}^{\text{eff}}(\bar{\mathbb{Q}})$

$$d_1 M_{\text{non}}^{\text{eff}}(\bar{\mathbb{Q}}) \simeq \mathcal{E}(\text{Cwve}/\bar{\mathbb{Q}}, H_{\text{sing}}^*)$$

By universal property of diagram category $\mathcal{E}(\text{Pair}^1/\bar{\mathbb{Q}}, H_{\text{sing}}^*)$ we have:



By universal property of diagram category $d_1 M_{\text{non}}^{\text{eff}}(\bar{\mathbb{Q}})$ we have:



By universality we conclude that dotted arrows are quasi-inverse functors. ■

Prop: | The PERIOD CONJECTURE FOR NON 1-MOTIVES is equivalent to the PERIOD CONJECTURE FOR 1-PERIODS.

It follows from the above lemma, reasoning as in the proof of the previous proposition. ■

Rmk: • We can also take the representation

diagram seen in the last lemma! $\leftarrow \text{Curve}/\bar{\mathbb{Q}} \subset \text{Pair}^1/\bar{\mathbb{Q}} \rightarrow (\bar{\mathbb{Q}}, \mathbb{Q})\text{-Vect}$

Hence, we define

$\mathbb{P}(\text{Curve}/\bar{\mathbb{Q}})$ SPACE OF PERIODS OF CWNF TYPE

$\tilde{\mathbb{P}}(\text{Curve}/\bar{\mathbb{Q}})$ SPACE OF FORMAL PERIODS OF CWNF TYPE

and a PERIOD CONJECTURE FOR PERIODS OF CWNF TYPE.

As usual, this conjecture tells that $\bar{\mathbb{Q}}$ -linear relations between PERIODS OF CWNF TYPE are given by

(A) bilinearity

(B) induced by morphisms in the diagram $\text{Curve}/\bar{\mathbb{Q}}$.

Since, as we recalled in the last lemma

$$e(\text{Curve}/\bar{\mathbb{Q}}, H_{\text{sing}}^1) \simeq \dim_{\text{off}}^{\text{off}}(\bar{\mathbb{Q}}),$$

this PERIOD CONJECTURE FOR PERIODS OF CWNF TYPE is equivalent to the PERIOD CONJECTURE FOR NON 1-MOTIVES.

So, it gives an even simpler formulation of the PERIOD CONJECTURE FOR 1-PERIODS!

b/c FORMAL PERIODS have both less generators and less relations

SLOGAN: PERIOD CONJECTURE \Leftrightarrow PERIOD CONJECTURE
for a diagram D for its diagram category $e(D, H_{\text{sing}}^1)$

this allows to give a more explicit description of the PERIOD CONJECTURE b/c morphisms are known!

\leadsto while, typically, morphisms of $e(D, H_{\text{sing}}^1)$ are difficult to describe

we mean the DEIGNE ISO-1-MOTIVES, which is abelian.

II) Finally, recall from talk 9 the CATEGORY OF DEIGNE 1-MOTIVES $1\text{-Mot}(\bar{\mathbb{Q}})$

and from talk 10 the singular and DE RHIM DEFORMATIONS

$$M \in 1\text{-Mot}(\bar{\mathbb{Q}}) \rightsquigarrow \begin{matrix} V_{\text{sing}}(\Gamma) := T_{\text{sing}}(\Gamma) \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}, \\ \text{is a } \bar{\mathbb{Q}}\text{-vector space} \end{matrix}, \quad \begin{matrix} T_{\text{sing}}(\Gamma) \rightarrow \text{Lie}(G^{\text{an}}) \\ \downarrow \quad \downarrow \text{exp} \\ L \rightarrow G^{\text{an}} \\ \searrow \quad \parallel \\ G(\bar{\mathbb{Q}}) \subset G(\mathbb{C}) \end{matrix}$$

$$V_{\text{dr}}(M) := \text{Lie}(\Gamma^{\text{an}}), \quad \text{is a } \bar{\mathbb{Q}}\text{-vector space}$$

$$0 \rightarrow [0 \rightarrow V] \rightarrow [L \rightarrow \Gamma^{\text{an}}] \rightarrow [L \rightarrow G] \rightarrow 0$$

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together with an isomorphism

$$\phi_{\Gamma}: V_{\text{sing}}(\Gamma) \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \xrightarrow{\cong} V_{\text{dr}}(\Gamma) \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$$

So, we can define the functor

$$1\text{-Mot}(\bar{\mathbb{Q}}) \rightarrow (\mathbb{Q}, \bar{\mathbb{Q}})\text{-Vect}$$

$$\Gamma \longmapsto (V_{\text{sing}}(\Gamma), V_{\text{dr}}(\Gamma), \phi_{\Gamma})$$

Notice that they are reversed w.r.t. the previous examples! This reflects the fact that DEIGNE MOTIVES are homological while NON MOTIVE cohomological!

Hence we have the $\bar{\mathbb{Q}}$ -vector spaces

- $P(1\text{-Mot}(\bar{\mathbb{Q}}))$ SPACE OF PERIODS OF DEIGNE 1-MOTIVES
- $\tilde{P}(1\text{-Mot}(\bar{\mathbb{Q}}))$ SPACE OF FORMAL PERIODS OF DEIGNE 1-MOTIVES

and a PERIOD CONJECTURE FOR DEIGNE 1-MOTIVES

As usual, this conjecture says that $\bar{\mathbb{Q}}$ -linear relations between PERIODS OF DEIGNE 1-MOTIVES are given by

- (A) bilinearity
- (B) induced by morphisms in the CATEGORY OF DEIGNE 1-MOTIVES.

Prop: | The PERIOD CONJECTURE FOR DELIGNE 1-MOTIVES is equivalent to the PERIOD CONJECTURE FOR NAMI 1-MOTIVES.

By the Theorem of Ayoub & Barbieri Viale (to be?) we have an anti-equivalence of categories between CATEGORY OF DELIGNE 1-MOTIVES and CATEGORY OF NAMI 1-MOTIVES, which is compatible with singular and de Rham realizations.

More precisely, we have the commutative diagram of functors

$$\begin{array}{ccc}
 1\text{-Mot}(\bar{\mathbb{Q}})^{\text{eff}} & \longrightarrow & (\mathbb{Q}, \bar{\mathbb{Q}})\text{-Vect}^{\text{eff}} & (V_{\mathbb{Q}}, V_{\bar{\mathbb{Q}}}, \phi) & \phi: V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\cong} V_{\bar{\mathbb{Q}}} \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \\
 \downarrow \cong & & \downarrow (-)^{\vee} & \downarrow & \downarrow \\
 d_1 \mathcal{M}_{\text{Nami}}^{\text{eff}}(\bar{\mathbb{Q}}) & \longrightarrow & (\bar{\mathbb{Q}}, \mathbb{Q})\text{-Vect} & (V_{\bar{\mathbb{Q}}}^{\vee}, V_{\mathbb{Q}}^{\vee}, \phi^{\vee}) & \phi^{\vee}: V_{\bar{\mathbb{Q}}}^{\vee} \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \xrightarrow{\cong} V_{\mathbb{Q}}^{\vee} \otimes_{\mathbb{Q}} \mathbb{C}
 \end{array}$$

which induces the commutative square

$$\begin{array}{ccc}
 \tilde{\mathcal{P}}(1\text{-Mot}(\bar{\mathbb{Q}})) & \xrightarrow{\text{ev}} & \mathcal{P}(1\text{-Mot}(\bar{\mathbb{Q}})) & \text{is an equality by the anti-equivalence + fact that } \phi \text{ and } \phi^{\vee} \text{ have representative matrices with same entries.} \\
 \downarrow \cong & & \parallel & \\
 \tilde{\mathcal{P}}(d_1 \mathcal{M}_{\text{Nami}}^{\text{eff}}(\bar{\mathbb{Q}})) & \xrightarrow{\text{ev}} & \mathcal{P}(d_1 \mathcal{M}_{\text{Nami}}^{\text{eff}}(\bar{\mathbb{Q}})) &
 \end{array}$$

is an iso by the anti-equivalence

Hence PERIOD CONJECTURES for $1\text{-Mot}(\bar{\mathbb{Q}})$ and $d_1 \mathcal{M}_{\text{Nami}}^{\text{eff}}(\bar{\mathbb{Q}})$ are equivalent. □

The last 2 propositions tell that to prove the PERIOD CONJECTURE FOR 1-PERIODS we can reduce to prove the PERIOD CONJECTURE FOR DELIGNE 1-MOTIVES.

We prove the latter in the next section.

Resume: $PC = PC(\text{Pair}^{\text{eff}}/\bar{\mathbb{Q}}) \Leftrightarrow PC(\mathcal{M}_{\text{Nami}}^{\text{eff}}(\bar{\mathbb{Q}}))$
} restricts

$$1\text{-PC} = PC(\text{Pair}^1/\bar{\mathbb{Q}}) \Leftrightarrow PC(d_1 \mathcal{M}_{\text{Nami}}^{\text{eff}}(\bar{\mathbb{Q}})) \Leftrightarrow PC(1\text{-Mot}(\bar{\mathbb{Q}})) \\
 \downarrow \\
 PC(\text{Curve}/\bar{\mathbb{Q}})$$

§ 3. PROOF OF THE PERIOD CONJECTURE FOR DELIGNE 1-MOTIVES

As we said above, the PERIOD CONJECTURE FOR DELIGNE 1-MOTIVES states that $\bar{\mathbb{Q}}$ -linear relations between PERIODS OF DELIGNE 1-MOTIVES

$$V_{\text{sing}}(\Pi) \times V_{\text{dn}}(\Pi)^{\vee} \rightarrow \mathbb{C}$$

$$(\sigma, w) \longmapsto w_{\mathbb{C}}(\phi_{\Pi}(\sigma)) =: \int_{\sigma} w \quad \rightsquigarrow \text{is only notation for us.}$$

$$w_{\mathbb{C}}: V_{\text{dn}}(\Pi) \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \rightarrow \mathbb{C}$$

$$\phi_{\Pi}: V_{\text{sing}}(\Pi) \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \rightarrow V_{\text{dn}}(\Pi) \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$$

are (A) bilinearity

(B) induced by morphisms in $1\text{-Mot}(\bar{\mathbb{Q}})$.

As anticipated, we need the ANALYTIC SUBGROUP THEOREM (AST).

Just to give an idea of how it will be used, we list the main steps of the proof:

- We start from a $\bar{\mathbb{Q}}$ -linear combination of PERIODS = 0

$$\lambda_1 \int_{\sigma_1} w_1 + \dots + \lambda_n \int_{\sigma_n} w_n = 0 \quad \lambda_i \in \bar{\mathbb{Q}}, \quad \sigma_i \in V_{\text{sing}}(\Pi_i), \quad w_i \in V_{\text{dn}}(\Pi_i)^{\vee}$$

- Using (A) & (B) we rewrite it as a single PERIOD = 0

$$\int_{\sigma} w = 0 \quad \sigma \in V_{\text{sing}}(\Pi) \quad w \in V_{\text{dn}}(\Pi)$$

- We use a motivic version of the AST to find a sub-motiv $\Pi' \subset \Pi$ st.

$$w = p^* w'' \quad \& \quad \sigma = i_* \sigma'$$

$$\text{with } 0 \rightarrow \Pi' \xrightarrow{i} \Pi \xrightarrow{p} \Pi/\Pi' \rightarrow 0, \quad \sigma' \in V_{\text{sing}}(\Pi') \quad w \in V_{\text{dn}}(\Pi/\Pi')^{\vee}$$

Then,

$$\int_{\sigma} w = \int_{i_* \sigma'} p^* w'' = \int_{\sigma'} \underbrace{i^* p^* w''}_{=0} = 0$$

that is, the relation $\int_{\sigma} w = 0$ is induced by (B)!

We start recalling the version of AST for algebraic groups from talk 10.

category of connected commutative algebraic groups / $\bar{\mathbb{Q}}$

Theorem: (ANALYTIC SUBGROUP THEOREM)

[HW18, Thm. 6.2.]

let $G \in \mathcal{G} \rightsquigarrow \mathfrak{g} := \text{lie}(G) \rightsquigarrow \mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \cong \text{lie}(G^{\text{an}})$
 $u \in \mathfrak{g}_{\mathbb{C}}$ s.t. $\exp_G(u) \in G(\bar{\mathbb{Q}})$.

Then $\exists H \subset G$ subgroup, $H \in \mathcal{G}$, s.t.

- 1) $u \in \mathfrak{h}_{\mathbb{C}}$ where $\mathfrak{h} = \text{lie}(H) \rightsquigarrow \mathfrak{h}_{\mathbb{C}} := \mathfrak{h} \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \cong \text{lie}(H^{\text{an}})$
- 2) $\text{Ann}(u) = (\mathfrak{g}/\mathfrak{h})^{\vee} \subset \mathfrak{g}^{\vee}$
 $\mathfrak{g}^{\vee} \ni \{f \in \mathfrak{g}^{\vee} \mid f(u) = 0\}$
 \hookrightarrow image of ker

First thing to do is obtain a motivic version of this theorem:

Theorem: (ANALYTIC SUBGROUP THEOREM FOR DAIGNE 1-MOTIVES)

[HW18, Thm. 9.7]

let $\Gamma \in 1\text{-Mot}(\bar{\mathbb{Q}}) \rightsquigarrow V_{\text{dr}}(\Gamma) := \text{lie}(\Gamma^{\vee})$
 $\theta \in V_{\text{sing}}(\Gamma)$.

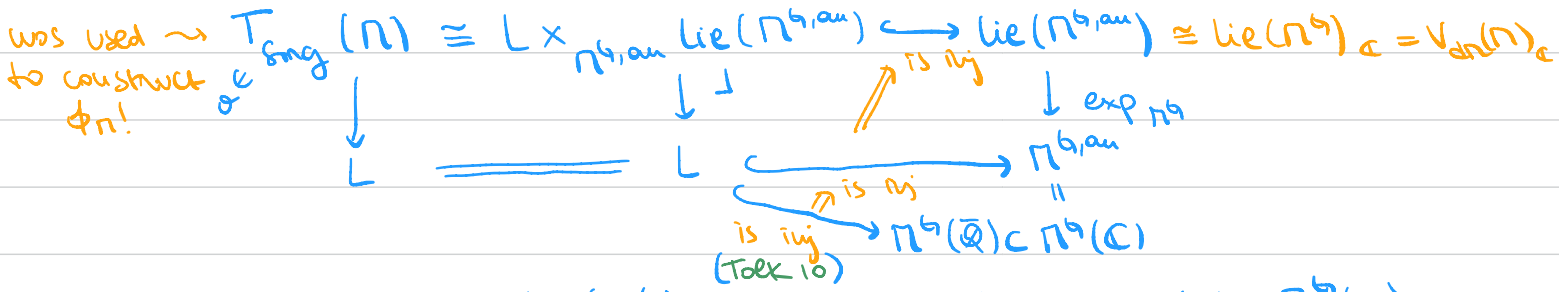
Then $\exists! \Gamma' \hookrightarrow \Gamma$ submotive, $\Gamma' \in 1\text{-Mot}(\bar{\mathbb{Q}})$, s.t.

- 1) $\theta \in i_* V_{\text{sing}}(\Gamma')$
- 2) $\text{Ann}(\theta) = p^* V_{\text{dr}}(\Gamma'/\Gamma)^{\vee} \subset V_{\text{dr}}(\Gamma)^{\vee}$, where $p: \Gamma \rightarrow \Gamma/\Gamma'$
 $\{w \in V_{\text{dr}}(\Gamma)^{\vee} \mid \int_{\theta} w = 0\}$

\square

WLOG, we can assume $\theta \in T_{\text{sing}}(\Gamma)$ otherwise we can take $n \in \mathbb{N}$ s.t. $n\theta \in T_{\text{sing}}(\Gamma)$ and work with $n\theta$

Recall we have the commutative diagram



\rightsquigarrow we can see $\theta \in \text{lie}(\Gamma^h)_C$. It is s.t. $\exp_{\Gamma^h}(\theta) \in \Gamma^h(\bar{\mathbb{Q}})$

By AST $\exists H \subset \Gamma^h$, $H \in \mathcal{G}$, s.t.

- 1) $\theta \in \text{lie}(H)_C$
- 2) $\text{Ann}(\theta) = p^* \text{lie}(\Gamma^h/H)^{\vee}$ with $p: \Gamma^h \rightarrow \Gamma^h/H$
 $\rightsquigarrow p^*: \text{lie}(\Gamma^h/H)^{\vee} \hookrightarrow \text{lie}(\Gamma^h)^{\vee}$

\hookrightarrow since $\theta \in T_{\text{sing}}(\Gamma)$ is seen as an element of $V_{\text{dr}}(\Gamma)_C$ via ϕ_n , this Ann is exactly the one in AST!

Now we need:

Lemma:
(HW18, Prop. 8.21)

Let $\pi = [L \rightarrow G] \in 1\text{-Not}(\overline{\mathbb{Q}})$.

For any $H \subset \pi^{\flat}$, $H \in \mathcal{G}$, $\exists \pi' \subset \pi$, $\pi' \in 1\text{-Not}(\overline{\mathbb{Q}})$,
with $\pi^{\flat} \hookrightarrow \pi'^{\flat}$
 $\searrow \quad \swarrow$
 $\quad H \hookrightarrow$

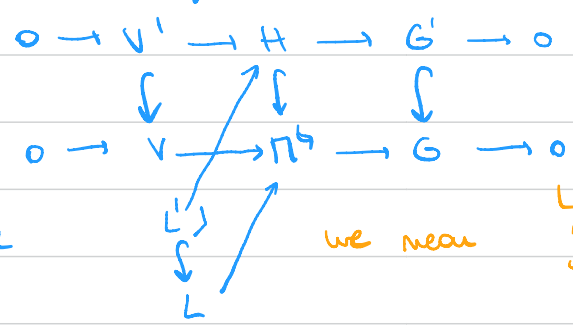
s.t. $V_{\text{sing}}(\pi) \cap \text{lie}(H)_{\mathbb{C}} = V_{\text{sing}}(\pi')$

We only see the construction of π' .

Let $0 \rightarrow V' \rightarrow H \rightarrow G' \rightarrow 0$ be the canonical decomp. in
vector group semi-abelian variety

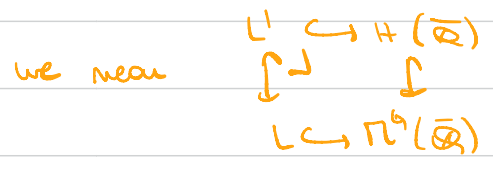
given by the STRUCTURE THEOREM.

We have the injective morphism of SES \rightarrow



We have a SES
ble on the
canonical SES
of the STRUCTURE
THEOREM.
H is reg ble
H' \hookrightarrow π^{\flat} is.

and we define



We put $\pi' := [L' \rightarrow G'] \in 1\text{-Not}(\overline{\mathbb{Q}})$.

Notice it is a

Properties can be proved. ▣

We turn back to the proposition. By the Lemma we can find

$$\pi' \xrightarrow{i} \pi \rightsquigarrow p: \pi \rightarrow \pi/\pi' =: \pi''$$

s.t. $V_{\text{sing}}(\pi') = V_{\text{sing}}(\pi) \cap \text{lie}(H)_{\mathbb{C}}$

$\rightarrow \bullet \theta \in V_{\text{sing}}(\pi') \rightarrow \theta \in i_* V_{\text{sing}}(\pi')$, which is 1)

$\bullet p^* \text{lie}(\pi''^{\flat})^{\vee} \subset \text{Ann}(\theta)$

From now on
is only
linear
algebras

Moreover the proposition gives

$$\pi''^{\flat} \twoheadrightarrow H/H' =: H''$$

$\rightarrow \text{Ann}(\theta) = \text{lie}(H'')^{\vee} \subset \text{lie}(\pi''^{\flat})^{\vee} \subset \text{Ann}(\theta)$

$\rightarrow \text{Ann}(\theta) = \text{lie}(\pi''^{\flat})^{\vee} = p_* V_{\text{lie}}(\pi'')^{\vee}$, which is 2).

! let $\pi_1 \hookrightarrow \pi$ be another fibration with same properties.

$\sim p: \pi \rightarrow \pi/\pi_1 =: \pi_2.$

Then, by 2), $p^*V_{\text{an}}(\pi_2) = \text{Ann}(\sigma) = p^*V_{\text{an}}(\pi'') \sim \pi_2^{\vee} = \pi'^{\vee}$

$\sim \pi_2 = \pi''.$

↑ $(-)^{\vee}$ is faithful



Cor: let $\pi \in 1\text{-Not}(\bar{\mathbb{Q}})$, $\sigma \in V_{\text{sing}}(\pi)$, $w \in V_{\text{an}}(\pi)^{\vee}$ s.t. $\int_{\sigma} w = 0.$

[HW18, Cor. 9.9]

Then exists a SES in $1\text{-Not}(\bar{\mathbb{Q}})$

$$0 \rightarrow \pi' \hookrightarrow \pi \xrightarrow{p} \pi'' \rightarrow 0$$

s.t. $\sigma = i_* \sigma'$ $\sigma' \in V_{\text{sing}}(\pi')$

$w = p^* w''$ $w'' \in V_{\text{an}}(\pi'')$

Apply previous Thm to π' and σ' . we get the SES

$$0 \rightarrow \pi' \hookrightarrow \pi \xrightarrow{p} \pi'' \rightarrow 0$$

\parallel
 π''

s.t. 1) $\sigma' \in i_* V_{\text{sing}}(\pi') \sim \sigma' = i_* \sigma''$

2) $\text{Ann}(\sigma') = p^* V_{\text{an}}(\pi'')^{\vee} \sim w = p^* w''$
 w'' by hp.



Thm: (PERIOD CONJECTURE FOR DELIGNE 1-MOTIVES)

[HW18, Thm 9.10] All $\bar{\mathbb{Q}}$ -linear relations between PERIODS OF DELIGNE 1-MOTIVES are:

- (A) bilinearity
- (B) induced by morphisms in $1\text{-Not}(\bar{\mathbb{Q}}).$

Proof follows steps outlined above:

• Consider a $\bar{\mathbb{Q}}$ -linear combination of PERIODS = 0

$$\lambda_1 \int_{\sigma_1} w_1 + \dots + \lambda_n \int_{\sigma_n} w_n = 0$$

where $\int_{\sigma_i} w_i$ is a PERIOD of $\pi_i \in 1\text{-Not}(\bar{\mathbb{Q}}).$

We want to "prove" the = using rules (A) & (B).

• we can see the linear combination as a single period

over $\pi := \pi_1 \oplus \dots \oplus \pi_n$ b/c, if $\pi_i \hookrightarrow \pi \xrightarrow{\pi_i} \pi_i$

$$\lambda_1 \int_{\sigma_1} w_1 + \dots + \lambda_n \int_{\sigma_n} w_n \stackrel{(B)}{=} \sum \lambda_i \int_{i_* \sigma_i} \pi_i^* w_i \stackrel{(A)}{=} \int_{\sigma} w \quad \sigma = \sum i_* \sigma_i$$

$$w = \sum \lambda_i w_i$$

- Since $\int_{\sigma} w = 0$, there we can apply the previous Cor. and obtain a SES in $1-\text{Nat}(\mathbb{Q})$

$$0 \rightarrow N' \xrightarrow{i} N \xrightarrow{p} N'' \rightarrow 0 \text{ st. } \sigma = i_* \sigma' \text{ and } w = p^* w''.$$

$$\text{So } \int_{\sigma} w = \int_{i_* \sigma'} p^* w'' \stackrel{(b)}{=} \int_{\sigma'} \underbrace{i^* p^* w''}_{=0} = 0 \quad \blacksquare$$