THE PERIOD CONJECTURE FOR 1-MOTIVES

[HW18] Huber & Wüstholz, "Transcendence and linear relations of 1-periods"

Recall that in talk 8 we defined COHOMOLOGICAL PERIODS, FORMAL PERIODS and the EVALUATION MAP, and we stated the PERIOD CONJECTURE:

$\mathcal{P} := \bigcup_{(X,Y,i)\in \mathrm{Pair}^{\mathrm{eff}}/\overline{\mathbb{Q}}} \mathrm{Im}\left(\mathrm{per}: H^i_{dR}(X,Y|\overline{\mathbb{Q}}) \times H_i^{\mathrm{sing}}(X,Y;\mathbb{Q}) \longrightarrow \mathbb{C}\right)$ SPACE OF COHOMOLOGICAL PERIODS

$(\omega, \sigma) \longmapsto \mathrm{comp}(\omega)(\sigma)$

$\mathrm{comp}: H^i_{dR}(X,Y|\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\simeq} H^i_{\mathrm{sing}}(X,Y;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$

was denoted by $\mathcal{P}^{\mathrm{eff}}$. We avoid the decoration since we will consider only the effective setting.

$\widetilde{\mathcal{P}} := \overline{\mathbb{Q}}$-vector space generated by symbols $[X,Y,i,\omega,\sigma]$ SPACE OF FORMAL PERIODS

with $(X,Y,i)\in \mathrm{Pair}^{\mathrm{eff}}/\overline{\mathbb{Q}}$, $\omega\in H^i_{dR}(X,Y|\overline{\mathbb{Q}})$, $\sigma \in H_i^{\mathrm{sing}}(X,Y;\mathbb{Q})$,

modulo the relations:

(A) bilinearity in ω and σ

(B) induced by edges in $\mathrm{Pair}^{\mathrm{eff}}/\overline{\mathbb{Q}}$:

(B1) $(X',Y',i) \to (X,Y,i) \rightsquigarrow [X,Y,i,f^*\omega',\sigma] = [X',Y',i,\omega',f_*\sigma]$

$f:(X,Y)\to(X',Y')$

(B2) $(Y,Z,i) \to (X,Y,i+1) \rightsquigarrow [X,Y,i+1,d\omega,\sigma] = [Y,Z,i,\omega,\partial\sigma]$

$d: H^i_{dR}(Y,Z|\overline{\mathbb{Q}}) \to H^{i+1}_{dR}(X,Y|\overline{\mathbb{Q}})$ $\quad \partial: H^{\mathrm{sing}}_{i+1}(X,Y;\mathbb{Q}) \to H^{\mathrm{sing}}_{i+1}(Y,Z;\mathbb{Q})$

boundary maps of long exact sequences of the pair.

$\mathrm{ev}: \widetilde{\mathcal{P}} \to \mathbb{C}$ the $\overline{\mathbb{Q}}$-linear map s.t. $[X,Y,i,\omega,\sigma] \longmapsto \mathrm{per}(\omega,\sigma)$

EVALUATION MAP

Conj.: (PERIOD CONJECTURE)

[HW18, Conj. 13.1] | $\mathrm{ev}: \widetilde{\mathcal{P}} \to \mathbb{C}$ is injective.

The image is $\mathrm{ev}(\widetilde{\mathcal{P}}) = \mathcal{P}$. This conjecture tells that all $\overline{\mathbb{Q}}$-linear relations between COHOMOLOGICAL PERIODS are given by (A), (B1) & (B2).

The aim of this talk is to prove this conjecture for "1-periods".

Plan: §1. Say what we mean by "1-periods" and state precisely the period conjecture we are going to prove.

§2. The statement of this conjecture doesn't involve motives, but we use the motivic framework to prove it!
We need to formalize the period conjecture using motives.
This formalization can be applied to different categories of motives.
Taking the category of NORI 1-MOTIVES we recover the period conjecture stated in §1.
Taking the category of DELIGNE 1-MOTIVES we obtain an equivalent conjecture, that we will prove in §3.

§3. We prove the PERIOD CONJECTURE FOR DELIGNE 1-MOTIVES.

§ 1. 1-PERIODS AND THEIR PERIOD CONJECTURE

Def: We define the SPACE OF COHOMOLOGICAL 1-PERIODS
[HW18, Def. 12.2]

$$\mathcal{P}^1 := \bigcup_{(X,Y,1)\in \mathrm{Pair}^{\mathrm{eff}}/\overline{\mathbb{Q}}} \underbrace{\mathrm{Im}\left(\mathrm{per}: H^1_{dR}(X,Y/\overline{\mathbb{Q}}) \times H_1^{\mathrm{sing}}(X,Y;\mathbb{Q}) \longrightarrow \mathbb{C}\right)}_{=:\mathcal{P}(X,Y,1)}$$

X, Y are any!

Rmk: • $\mathcal{P}^1 \subset \mathcal{P}$ is a $\overline{\mathbb{Q}}$-subvector space of $\mathbb{C}$

Differently from $\mathcal{P}$, $\mathcal{P}^1$ is not a $\overline{\mathbb{Q}}$-algebra! We don't care about multiplicative structure in this talk.

[HW18, Lem. 12.4] For the same reason why $\mathcal{P}$ is:

- The sum of 2 periods of $(X,Y,1)$ and $(X',Y',1)$ can be realized as a period of $(X \sqcup X', Y \sqcup Y', 1)$ (see talk 8)
- Action of $\overline{\mathbb{Q}}$ is given by $\overline{\mathbb{Q}}$-linearity of $H^1_{dR}(X,Y/\overline{\mathbb{Q}})$. ∎

• $\mathcal{P}^1$ contains also

$$\mathcal{P}^0 := \bigcup_{(X,Y,0)\in \mathrm{Pair}^{\mathrm{eff}}/\overline{\mathbb{Q}}} \underbrace{\mathrm{Im}\left(\mathrm{per}: H^0_{dR}(X,Y/\overline{\mathbb{Q}}) \times H_0^{\mathrm{sing}}(X,Y;\mathbb{Q}) \longrightarrow \mathbb{C}\right)}_{=:\mathcal{P}(X,Y,0)}$$

Indeed $\mathcal{P}^0 = \overline{\mathbb{Q}}$.

Alternatively, notice that a period of $(X,Y,0)$ is also a period of $(X \times \mathbb{A}^1, Y \times \mathbb{A}^1, 1)$ (see talk 8) ∎

So, 1-PERIODS are PERIODS obtained by restricting to consider cohomological degree ≤ 1.
Their period conjecture is obtained by restricting the PERIOD CONJECTURE to cohomological degree ≤ 1.
More precisely, we consider

$$\mathrm{Pair}^1/\overline{\mathbb{Q}} \subset \mathrm{Pair}^{\mathrm{eff}}/\overline{\mathbb{Q}}$$

the full subdiagram containing vertices of the kind

$$(X,Y,i) \text{ for } i \leq 1.$$

Def: We define the SPACE OF FORMAL 1-PERIODS

$\tilde{\mathcal{P}}^1 :=$ $\overline{\mathbb{Q}}$-vector space generated by symbols

$$[X, Y, i, \omega, \sigma] \qquad i \leq 1$$

modulo relations:

(A') bilinearity in ω & σ

(B') induced by edges in $\mathrm{Pair}^1/\overline{\mathbb{Q}}$:

$$(B1') \quad (X', Y', i) \to (X, Y, i) \rightsquigarrow [X, Y, i, f^*\omega', \sigma] = [X', Y', i, \omega', f_*\sigma]$$

$$i \leq 1 \qquad\qquad\qquad i \leq 1$$

$$(B2') \quad (Y, Z, 0) \to (X, Y, 1) \rightsquigarrow [X, Y, 1, d\omega, \sigma] = [Y, Z, 0, \omega, \partial\sigma].$$

We still have an EVALUATION MAP

$$ev^1 : \tilde{\mathcal{P}}^1 \to \mathbb{C}$$

the $\overline{\mathbb{Q}}$-linear map s.t. $\underset{i \leq 1}{[X, Y, i, \omega, \sigma]} \longmapsto \mathrm{per}(\omega, \sigma)$.

Thm: (PERIOD CONJECTURE FOR 1-PERIODS)

[HW18, Thm. 13.3 (3)] | $ev^1 : \tilde{\mathcal{P}}^1 \to \mathbb{C}$ is injective.

The image is $ev(\tilde{\mathcal{P}}^1) = \mathcal{P}^1$. This conjecture tells that all $\overline{\mathbb{Q}}$-linear relations between COHOMOLOGICAL 1-PERIODS are given by (A'), (B1') & (B2').

This is the "conjecture for 1-periods" we are going to prove in the talk.

§ 2. THE MOTIVIC FORMALIZATION

Idea: A category of motives over $\bar{\mathbb{Q}}$ $\mathcal{M}(\bar{\mathbb{Q}})$ comes equipped with a de Rham and a singular realization functors, which are isomorphic extending scalars to $\mathbb{C}$

$$\begin{tikzcd} & \bar{\mathbb{Q}}\text{-Vect} \arrow[dr, "-\otimes_{\bar{\mathbb{Q}}}\mathbb{C}"] & \\ \mathcal{M}(\bar{\mathbb{Q}}) \arrow[ur, "H_{dR}"] \arrow[dr, "H_{sing}"'] & \cong_{comp} & \mathbb{C}\text{-Vect} \\ & \mathbb{Q}\text{-Vect} \arrow[ur, "-\otimes_{\mathbb{Q}}\mathbb{C}"'] & \end{tikzcd}$$

They can be put together to give rise to a de Rham-singular realization functor

$$\mathcal{M}(\bar{\mathbb{Q}}) \longrightarrow (\bar{\mathbb{Q}}, \mathbb{Q})\text{-Vect}$$
$$M \longmapsto (H_{dR}(M), H_{sing}(M), comp_M)$$

$comp_M : H_{dR}(M) \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \xrightarrow{\simeq} H_{sing}(M) \otimes_{\mathbb{Q}} \mathbb{C}$

With this data, we can define the <u>SPACE OF PERIODS</u> of $\mathcal{M}(\bar{\mathbb{Q}})$

$$P(\mathcal{M}(\bar{\mathbb{Q}})) := \bigcup_{M \in \mathcal{M}(\bar{\mathbb{Q}})} \overbrace{\mathrm{Im}\,(per : H_{dR}(M) \times H_{sing}(M)^\vee \longrightarrow \mathbb{C})}^{=: P(M)}$$
$$(\omega, \sigma) \longmapsto comp_M(\omega)(\sigma)$$

and the <u>SPACE OF FORMAL PERIODS</u> of $\mathcal{M}(\bar{\mathbb{Q}})$

$\tilde{P}(\mathcal{M}(\bar{\mathbb{Q}})) :=$ $\bar{\mathbb{Q}}$-vector space generated by symbols $[M, \omega, \sigma]$ with $\omega \in H_{dR}(M)$, $\sigma \in H_{sing}(M)^\vee$

modulo the relations:

(A) bilinearity in ω and σ

(B) induced by morphisms in $\mathcal{M}(\bar{\mathbb{Q}})$

We also have an <u>EVALUATION MAP</u>

$$ev : \tilde{P}(\mathcal{M}(\bar{\mathbb{Q}})) \twoheadrightarrow P(\mathcal{M}(\bar{\mathbb{Q}})) \subset \mathbb{C}$$
$$(M, \omega, \sigma) \longmapsto comp_M(\omega)(\sigma)$$

We say that the <u>PERIOD CONJECTURE</u> holds for $\mathcal{M}(\bar{\mathbb{Q}})$ if ev is injective.

This idea can be easily generalized to any category (or more generally a direct diagram) with a functor (representation) into (K,L)-Vect for some $K, L \subset \mathbb{C}$ subfields. [HW18, §7.1]
However, we directly give the examples we are interested in, without entirely repeating each time the definitions explained above.

I) $\boxed{\mathrm{Pair}^{\mathrm{eff}}/\overline{\mathbb{Q}} \longrightarrow (\overline{\mathbb{Q}}, \mathbb{Q})\text{-Vect}}$

$$(X, Y, i) \longmapsto \left(H^i_{dR}(X, Y/\overline{\mathbb{Q}}),\ H^i_{sing}(X, Y; \mathbb{Q}),\ \mathrm{comp}\right)$$

We recover exactly the notions of

$P(\mathrm{Pair}^{\mathrm{eff}}/\overline{\mathbb{Q}}) = \mathcal{P}$ SPACE OF COHOMOLOGICAL PERIODS

$\widetilde{P}(\mathrm{Pair}^{\mathrm{eff}}/\overline{\mathbb{Q}}) = \widetilde{\mathcal{P}}$ SPACE OF FORMAL PERIODS

and the PERIOD CONJECTURE! It's immediate by definition!

II) By universal property of diagram category

$$\begin{tikzcd}
 & \mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}(\overline{\mathbb{Q}}) \arrow[dr] \arrow[dd, dotted] & \\
\mathrm{Pair}^{\mathrm{eff}}/\overline{\mathbb{Q}} \arrow[ur] \arrow[rr, "H_{sing}"] \arrow[dr, "\mathrm{I})"'] & & \mathbb{Q}\text{-Vect} \\
 & (\overline{\mathbb{Q}}, \mathbb{Q})\text{-Vect} \arrow[ur, "\text{projection on } 2^{nd} \text{ component}"'] &
\end{tikzcd}$$

is faithful and exact!

is $\overline{\mathbb{Q}}$-linear abelian

we obtain the functor

$$\boxed{\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}(\overline{\mathbb{Q}}) \longrightarrow (\overline{\mathbb{Q}}, \mathbb{Q})\text{-Vect}}$$

and hence the $\overline{\mathbb{Q}}$-vector spaces

$P(\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}(\overline{\mathbb{Q}}))$ SPACE OF PERIODS OF NORI MOTIVES

$\widetilde{P}(\mathcal{MM}^{\mathrm{eff}}_{\mathrm{Nori}}(\overline{\mathbb{Q}}))$ SPACE OF FORMAL PERIODS OF NORI MOTIVES

and a PERIOD CONJECTURE FOR NORI MOTIVES.
As usual, this conjecture tells that $\overline{\mathbb{Q}}$-linear relations between PERIODS OF NORI MOTIVES are given by
(A) bilinearity
(B) induced by morphisms in the CATEGORY OF NORI MOTIVES.

It's not so clear what it is, especially because morphisms in the CATEGORY OF NORI MOTIVES are difficult to understand.
In fact, it is nothing new! It's the big drawback of NORI MOTIVES!

Prop : The PERIOD CONJECTURE FOR NORI MOTIVES is equivalent to the PERIOD CONJECTURE.

- Since $\mathcal{M}^{eff}_{Nori}(\bar{\mathbb{Q}}) = \langle H^i(X,Y) \rangle$, ⤳ thick abelian subcategory generated (= full, closed under subquotients)
then

$$P(\mathcal{M}^{eff}_{Nori}(\bar{\mathbb{Q}})) = \langle P(H^i(X,Y)) \rangle_{\bar{\mathbb{Q}}} = \langle P(X,Y,i) \rangle_{\bar{\mathbb{Q}}} = \bigcup_{(X,Y,i) \in Pair^{eff}/\bar{\mathbb{Q}}} P(X,Y,i) = P(Pair^{eff}/\bar{\mathbb{Q}})$$

easy to check; $\bar{\mathbb{Q}}$-vector space generated

- Recall that in talk 8 we saw that:

Given 2 representations

$$\begin{tikzcd} D \arrow[r, "T_1"] \arrow[dr, "T_2"'] & \bar{\mathbb{Q}}\text{-Vect} \\ & \mathbb{Q}\text{-Vect} \end{tikzcd} \rightsquigarrow A_{12} := \text{COMPANION SPACE of } T_1 \,\&\, T_2 \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$$

and $\varphi : T_1 \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \xrightarrow{\cong} T_2 \otimes_{\mathbb{Q}} \mathbb{C}$ ⤳ P_{12} := FORMAL PERIODS, then

1) $P_{12} \cong A_{12}$ as $\bar{\mathbb{Q}}$-vector spaces

2) $A_{12} \cong A(\mathcal{C}(D,T_2), f_{T_1} \circ \phi, f_{T_2} \otimes \bar{\mathbb{Q}})$:= COMPANION SPACE of $f_{T_1} \circ \phi$ & $f_{T_2} \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}$

$$\begin{tikzcd} & \mathcal{C}(D,T_1) \arrow[r, "f_{T_1}"] & \bar{\mathbb{Q}}\text{-Vect} \\ D \arrow[ur] \arrow[dr] \arrow[urr, bend left, "T_1"] \arrow[drr, bend right, "T_2"'] & & \\ & \mathcal{C}(D,T_2) \arrow[r, "f_{T_2}"] \arrow[uu, "\phi"] & \mathbb{Q}\text{-Vect} \arrow[uu, "-\otimes_{\mathbb{Q}}\bar{\mathbb{Q}}"'] \end{tikzcd}$$

Applying this to $D = Pair^{eff}/\bar{\mathbb{Q}}$, $T_1 = H_{dR}$ & $T_2 = H_{sing}$, we obtain a canonical isomorphism

$$\tilde{P}(Pair^{eff}/\bar{\mathbb{Q}}) \overset{1)}{\cong} A(Pair^{eff}, H_{dR}, H_{sing} \otimes \bar{\mathbb{Q}}) \cong$$
$$\overset{2)}{\cong} A(\mathcal{M}^{eff}_{Nori}(\bar{\mathbb{Q}}), H_{dR}, H_{sing} \otimes \bar{\mathbb{Q}}) \cong$$
$$\overset{1)}{\cong} \tilde{P}(\mathcal{M}^{eff}_{Nori}(\bar{\mathbb{Q}}))$$

- We have the commutative square

$$\begin{tikzcd} \tilde{P}(Pair^{eff}/\bar{\mathbb{Q}}) \arrow[r, "ev"] \arrow[d, "\wr"] & P(Pair^{eff}/\bar{\mathbb{Q}}) \arrow[d, equal] \\ \tilde{P}(\mathcal{M}^{eff}_{Nori}(\bar{\mathbb{Q}})) \arrow[r, "ev"] & P(\mathcal{M}^{eff}_{Nori}(\bar{\mathbb{Q}})) \end{tikzcd}$$

Hence PERIOD CONJECTURES for $Pair^{eff}/\bar{\mathbb{Q}}$ and $\mathcal{M}^{eff}_{Nori}(\bar{\mathbb{Q}})$ are equivalent ▨

III) $\mathrm{Pair}^1/\bar{\mathbb{Q}} \subset \mathrm{Pair}^{\mathrm{eff}}/\bar{\mathbb{Q}} \longrightarrow (\bar{\mathbb{Q}},\mathbb{Q})\text{-Vect}$

$(X,Y,i) \longmapsto \left(H^i_{dR}(X,Y/\bar{\mathbb{Q}}),\ H^i_{sing}(X,Y;\mathbb{Q}),\ \mathrm{comp}\right)$

X,Y any, $i\leq 1$

We recover exactly the notions of

$P(\mathrm{Pair}^1/\bar{\mathbb{Q}}) = P$ SPACE OF COHOMOLOGICAL 1-PERIODS

$\tilde{P}(\mathrm{Pair}^1/\bar{\mathbb{Q}}) = \tilde{P}^1$ SPACE OF FORMAL 1-PERIODS

and the PERIOD CONJECTURE FOR 1-PERIODS!

It's immediate by definition!

IV) Recall the CATEGORY OF NORI 1-MOTIVES

$d_1 MM^{\mathrm{eff}}_{Nori}(\bar{\mathbb{Q}}) := \langle H^0(X,Y),\ H^1(X,Y)\rangle \subset MM^{\mathrm{eff}}_{Nori}(\bar{\mathbb{Q}})$

thick abelian subcategory generated (= full, closed under subquotients)

Composing with inclusion, we get

$d_1 MM^{\mathrm{eff}}_{Nori}(\bar{\mathbb{Q}}) \subset MM^{\mathrm{eff}}_{Nori}(\bar{\mathbb{Q}}) \longrightarrow (\bar{\mathbb{Q}},\mathbb{Q})\text{-Vect}$

and hence the $\bar{\mathbb{Q}}$-vector spaces

$P(d_1 MM^{\mathrm{eff}}_{Nori}(\bar{\mathbb{Q}}))$ SPACE OF PERIODS OF NORI 1-MOTIVES

$\tilde{P}(d_1 MM^{\mathrm{eff}}_{Nori}(\bar{\mathbb{Q}}))$ SPACE OF FORMAL PERIODS OF NORI 1-MOTIVES

and a PERIOD CONJECTURE FOR NORI 1-MOTIVES.

As usual, this conjecture tells that $\bar{\mathbb{Q}}$-linear relations between periods of Nori motives are given by

(A) bilinearity

(B) induced by morphisms in the CATEGORY OF NORI 1-MOTIVES.

Lemma: $d_1 MM^{\mathrm{eff}}_{Nori}(\bar{\mathbb{Q}})$ is equivalent to the diagram category of $\mathrm{Pair}^1/\bar{\mathbb{Q}}$ with representation

$H^*_{sing}: \mathrm{Pair}^1/\bar{\mathbb{Q}} \subset \mathrm{Pair}^{\mathrm{eff}}/\bar{\mathbb{Q}} \xrightarrow{H^*_{sing}} \mathbb{Q}\text{-Vect}.$

Recall the direct diagram

$\mathrm{Curve}/\bar{\mathbb{Q}}$: <u>obj</u>: (C,D) C smooth affine curve $/\bar{\mathbb{Q}}$

$\cap$ $\qquad D \hookrightarrow C$ finite set of points

$\mathrm{Pair}^1/\bar{\mathbb{Q}}$ $\qquad$ <u>Mor</u>: $(C',D') \to (C,D)$ for any $f: (C,D) \to (C',D')$ map of pairs.

$(C,D) \mapsto (C,D,1)$

The Theorem by Ayoub & Barbieri-Viale cited in talk 9 states that the diagram category of $\mathrm{Curve}|\overline{\mathbb{Q}}$ with representation

$$H^1_{sing} : \mathrm{Curve}|\overline{\mathbb{Q}} \subset \mathrm{Pair}^1|\overline{\mathbb{Q}} \xrightarrow{H^*_{sing}} \mathbb{Q}\text{-Vect}$$
$$(C,D) \mapsto (C,D,1) \longmapsto H^1_{sing}(C,D;\mathbb{Q})$$

is equivalent to $d_1\mathcal{M}^{eff}_{Nori}(\overline{\mathbb{Q}})$

$$d_1\mathcal{M}^{eff}_{Nori}(\overline{\mathbb{Q}}) \simeq \mathcal{C}(\mathrm{Curve}|\overline{\mathbb{Q}}, H^*_{sing}).$$

By universal property of diagram category $\mathcal{C}(\mathrm{Pair}^1|\overline{\mathbb{Q}}, H^*_{sing})$ we have:

$$\begin{tikzcd}
 & \mathcal{C}(\mathrm{Pair}^1|\overline{\mathbb{Q}}, H^*_{sing}) \arrow[dr, "H^*_{sing}"] \arrow[dd, dotted] & \\
\mathrm{Pair}^1|\overline{\mathbb{Q}} \arrow[ur] \arrow[rr, "H^*_{sing}"] \arrow[dr] & & \mathbb{Q}\text{-Vect} \\
 & d_1\mathcal{M}^{eff}_{Nori}(\overline{\mathbb{Q}}) \arrow[ur, "H^*_{sing}"'] &
\end{tikzcd}$$

(X,Y,i), $i\leq 1$ $\longmapsto H^i_{Nori}(X,Y)$

as

By universal property of diagram category $d_1\mathcal{M}^{eff}_{Nori}(\overline{\mathbb{Q}})$ we have:

$$\begin{tikzcd}
 & d_1\mathcal{M}^{eff}_{Nori}(\overline{\mathbb{Q}}) \arrow[dr, "H^*_{sing}"] \arrow[dd, dotted] & \\
\mathrm{Curve}|\overline{\mathbb{Q}} \arrow[ur] \arrow[rr, "H^*_{sing}"] \arrow[dr] & & \mathbb{Q}\text{-Vect} \\
 & \mathcal{C}(\mathrm{Pair}^1|\overline{\mathbb{Q}}, H^*_{sing}) \arrow[ur, "H^*_{sing}"'] &
\end{tikzcd}$$

By universality we conclude that dotted arrows are quasi-inverse functors. ∎

Prop: The period conjecture for Nori 1-motives is equivalent to the period conjecture for 1-periods.

It follows from the above lemma, reasoning as in the proof of the previous proposition. ∎

RMK: • We can also take the representation

diagram seen in the last lecture! ← $\mathrm{Curve}/\bar{\mathbb{Q}} \subset \mathrm{Pair}^1/\bar{\mathbb{Q}} \longrightarrow (\bar{\mathbb{Q}}, \mathbb{Q})\text{-Vect}$

Hence, we obtain

$\mathcal{P}(\mathrm{Curve}/\bar{\mathbb{Q}})$ SPACE OF PERIODS OF CURVE TYPE

$\tilde{\mathcal{P}}(\mathrm{Curve}/\bar{\mathbb{Q}})$ SPACE OF FORMAL PERIODS OF CURVE TYPE

and a PERIOD CONJECTURE FOR PERIODS OF CURVE TYPE.

As usual, this conjecture tells that $\bar{\mathbb{Q}}$-linear relations between PERIODS OF CURVE TYPE are given by

(A) bilinearity

(B) induced by morphisms in the diagram $\mathrm{Curve}/\bar{\mathbb{Q}}$.

Since, as we recalled in the last lecture

$$\mathcal{C}(\mathrm{Curve}/\bar{\mathbb{Q}}, H^1_{sing}) \simeq \mathrm{d}_1\mathcal{M}^{eff}_{Nori}(\bar{\mathbb{Q}}),$$

then PERIOD CONJECTURE FOR PERIODS OF CURVE TYPE is equivalent to the PERIOD CONJECTURE FOR NORI 1-MOTIVES.

So, it gives an even simpler formulation of the PERIOD CONJECTURE FOR 1-PERIODS!

b/c FORMAL PERIODS have both less generators and less relations

SLOGAN: PERIOD CONJECTURE for a diagram D $\iff$ PERIOD CONJECTURE for its diagram category $\mathcal{C}(D, H^*_{sing})$

this allows to give a more explicit description of the PERIOD CONJECTURE b/c morphisms are known! ⇝ while, typically, morphisms of $\mathcal{C}(D, H^*_{sing})$ are difficult to describe we mean the DELIGNE ISO-1-MOTIVES, which is abelian.

V) Finally, recall from talk 9 the CATEGORY OF DELIGNE 1-MOTIVES
$1\text{-Mot}(\overline{\mathbb{Q}})$

and from talk 10 the SINGULAR and DE RHAM REALIZATIONS

$$M \in 1\text{-Mot}(\overline{\mathbb{Q}}) \rightsquigarrow V_{sing}(M) := T_{sing}(M) \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad \begin{array}{ccc} T_{sing}(M) & \longrightarrow & \mathrm{Lie}(G^{an}) \\ \downarrow & \lrcorner & \downarrow \exp \\ L & \longrightarrow & G^{an} \end{array}$$

$M = [L \to G]$

is a $\mathbb{Q}$-vector space

$G^{an} = G(\mathbb{C})$, $G(\overline{\mathbb{Q}}) \subset G(\mathbb{C})$

$$V_{dR}(M) := \mathrm{Lie}(M^{\natural}),$$

is a $\overline{\mathbb{Q}}$-vector space

$$0 \to [0 \to V] \to [L \to M^{\natural}] \to [L \to G] \to 0$$

UNIVERSAL VECTOR EXTENSION of M

together with an isomorphism

$$\phi_M : V_{sing}(M) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} V_{dR}(M) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$$

So, we can define the functor

$$1\text{-Mot}(\overline{\mathbb{Q}}) \longrightarrow (\mathbb{Q}, \overline{\mathbb{Q}})\text{-Vect}$$
$$M \longmapsto (V_{sing}(M), V_{dR}(M), \phi_M)$$

Notice that they are reversed w.r.t. the previous examples! This reflects the fact that DELIGNE MOTIVES are homological while NORI MOTIVE cohomological!

Hence we have the $\overline{\mathbb{Q}}$-vector spaces

$\mathcal{P}(1\text{-Mot}(\overline{\mathbb{Q}}))$ SPACE OF PERIODS OF DELIGNE 1-MOTIVES

$\widetilde{\mathcal{P}}(1\text{-Mot}(\overline{\mathbb{Q}}))$ SPACE OF FORMAL PERIODS OF DELIGNE 1-MOTIVES

and a PERIOD CONJECTURE FOR DELIGNE 1-MOTIVES

As usual, this conjecture tells that $\overline{\mathbb{Q}}$-linear relations between PERIODS OF DELIGNE 1-MOTIVES are given by

(A) bilinearity

(B) induced by morphisms in the CATEGORY OF DELIGNE 1-MOTIVES.

Prop: The PERIOD CONJECTURE FOR DELIGNE 1-MOTIVES is equivalent to the PERIOD CONJECTURE FOR NORI 1-MOTIVES.

By the Theorem of Ayoub & Barbieri Viale (talk 9) we have an anti-equivalence of categories between CATEGORY OF DELIGNE 1-MOTIVES and CATEGORY OF NORI 1-MOTIVES, which is compatible with singular and de Rham realizations.

More precisely, we have the commutative diagram of functors

$$\begin{tikzcd} 1\text{-}\mathrm{Mot}(\overline{\mathbb{Q}})^{op} \arrow[r] \arrow[d, "\wr"] & (\mathbb{Q},\overline{\mathbb{Q}})\text{-}\mathrm{Vect}^{op} \arrow[d, "(\_)^\vee"] \\ d_1\mathcal{M}^{eff}_{Nori}(\overline{\mathbb{Q}}) \arrow[r] & (\overline{\mathbb{Q}},\mathbb{Q})\text{-}\mathrm{Vect} \end{tikzcd}$$

$$\begin{tikzcd} (V_{\mathbb{Q}}, V_{\overline{\mathbb{Q}}}, \phi) \arrow[d, mapsto] & \phi : V_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{C} \xrightarrow{\simeq} V_{\overline{\mathbb{Q}}}\otimes_{\overline{\mathbb{Q}}}\mathbb{C} \arrow[d, phantom, "\wr"] \\ (V^\vee_{\overline{\mathbb{Q}}}, V^\vee_{\mathbb{Q}}, \phi^\vee) & \phi^\vee : V^\vee_{\overline{\mathbb{Q}}}\otimes_{\overline{\mathbb{Q}}}\mathbb{C} \xrightarrow{\simeq} V^\vee_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{C} \end{tikzcd}$$

which induces the commutative square

is an iso by the anti-equivalence

$$\begin{tikzcd} \widetilde{\mathcal{P}}(1\text{-}\mathrm{Mot}(\overline{\mathbb{Q}})) \arrow[r, "ev"] \arrow[d, "\wr"] & \mathcal{P}(1\text{-}\mathrm{Mot}(\overline{\mathbb{Q}})) \arrow[d, equal] \\ \widetilde{\mathcal{P}}(d_1\mathcal{M}^{eff}_{Nori}(\overline{\mathbb{Q}})) \arrow[r, "ev"] & \mathcal{P}(d_1\mathcal{M}^{eff}_{Nori}(\overline{\mathbb{Q}})) \end{tikzcd}$$

is an equality by the anti-equivalence + fact that ϕ and $\phi^\vee$ have representative matrices with same entries.

Hence PERIOD CONJECTURES for $1\text{-}\mathrm{Mot}(\overline{\mathbb{Q}})$ and $d_1\mathcal{M}^{eff}_{Nori}(\overline{\mathbb{Q}})$ are equivalent. ■

The last 2 propositions tell that to prove the PERIOD CONJECTURE FOR 1-PERIODS we can reduce to prove the PERIOD CONJECTURE FOR DELIGNE 1-MOTIVES.

We prove the latter in the next section.

Résumé:

$$\begin{tikzcd} PC = PC(\mathrm{Pair}^{eff}/\overline{\mathbb{Q}}) \arrow[r, Leftrightarrow] \arrow[d, phantom, "\wr\ \text{restricts}"] & PC(\mathcal{M}^{eff}_{Nori}(\overline{\mathbb{Q}})) & \\ 1\text{-}PC = PC(\mathrm{Pair}^1/\overline{\mathbb{Q}}) \arrow[r, Leftrightarrow] & PC(d_1\mathcal{M}^{eff}_{Nori}(\overline{\mathbb{Q}})) \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & PC(1\text{-}\mathrm{Mot}(\overline{\mathbb{Q}})) \\ & PC(\mathrm{Curve}/\overline{\mathbb{Q}}) & \end{tikzcd}$$

§3. PROOF OF THE PERIOD CONJECTURE FOR DELIGNE 1-MOTIVES

As we said above, the PERIOD CONJECTURE FOR DELIGNE 1-MOTIVES states that $\bar{\mathbb{Q}}$-linear relations between PERIODS of DELIGNE 1-MOTIVES

$$V_{sing}(M) \times V_{dR}(M)^{\vee} \longrightarrow \mathbb{C}$$
$$(\sigma, \omega) \longmapsto \omega_{\mathbb{C}}(\phi_M(\sigma)) =: \int_\sigma \omega$$

$\rightsquigarrow$ is only notation for us.

$\omega_{\mathbb{C}} : V_{dR}(M) \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \to \mathbb{C}$

$\phi_M : V_{sing}(M) \otimes_{\mathbb{Q}} \mathbb{C} \to V_{dR}(M) \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$

are (A) bilinearity

(B) induced by morphisms in $1\text{-Mot}(\bar{\mathbb{Q}})$.

As anticipated, we need the ANALYTIC SUBGROUP THEOREM (AST). Just to give an idea of how it will be used, we list the main steps of the proof:

- We start from a $\bar{\mathbb{Q}}$-linear combination of PERIODS $= 0$
$$\lambda_1 \int_{\sigma_1} \omega_1 + \dots + \lambda_n \int_{\sigma_n} \omega_n = 0 \qquad \lambda_i \in \bar{\mathbb{Q}}, \quad \sigma_i \in V_{sing}(M_i), \quad \omega_i \in V_{dR}(M_i)^{\vee}$$
- Using (A) & (B) we rewrite it as a single PERIOD $= 0$
$$\int_\sigma \omega = 0 \qquad \sigma \in V_{sing}(M) \qquad \omega \in V_{sing}(M)$$
- We use a motivic version of the AST to find a sub-motive $M' \subset M$ s.t.
$$\omega = p^* \omega'' \quad \& \quad \sigma = i_* \sigma'$$
with $0 \to M' \xrightarrow{i} M \xrightarrow{p} M/M' \to 0$, $\sigma' \in V_{sing}(M')$ $\omega \in V_{dR}(M/M')^{\vee}$.
Then,
$$\int_\sigma \omega = \int_{i_*\sigma'} p^* \omega'' = \int_{\sigma'} \underbrace{i^* p^* \omega''}_{=0} = 0$$

That is, the relation $\int_\sigma \omega = 0$ is induced by (B)!

We start recalling the version of AST for algebraic groups from talk 10.

Thm: (ANALYTIC SUBGROUP THEOREM) — category of connected commutative algebraic groups $/\overline{\mathbb{Q}}$

[HW18, Thm. 6.2.]

Let $G \in \mathcal{G}$ $\rightsquigarrow$ $\mathfrak{g} := \mathrm{Lie}(G) \rightsquigarrow \mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \cong \mathrm{Lie}(G^{an})$

$u \in \mathfrak{g}_{\mathbb{C}}$ s.t. $\exp_G(u) \in G(\overline{\mathbb{Q}})$.

Then $\exists\, H \subset G$ subgroup, $H \in \mathcal{G}$, s.t.

1) $u \in \mathfrak{h}_{\mathbb{C}}$ where $\mathfrak{h} = \mathrm{Lie}(H) \rightsquigarrow \mathfrak{h}_{\mathbb{C}} := \mathfrak{h} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \cong \mathrm{Lie}(H^{an})$

2) $\mathrm{Ann}(u) = (\mathfrak{g}/\mathfrak{h})^\vee \subset \mathfrak{g}^\vee$

$\quad\;\; \| $

$\mathfrak{g}^\vee \cap \{ f \in \mathfrak{g}_{\mathbb{C}}^\vee \mid f(u) \} = 0$

$\hookrightarrow$ image of the

First thing to do is obtain a motivic version of this theorem:

Thm: (ANALYTIC SUBGROUP THEOREM FOR DELIGNE 1-MOTIVES)

[HW18, Thm 9.7]

Let $M \in 1\text{-Mot}(\overline{\mathbb{Q}}) \rightsquigarrow V_{dR}(M) := \mathrm{Lie}(M^{G})$

$\theta \in V_{sing}(M)$.

Then $\exists!\, M' \overset{i}{\hookrightarrow} M$ submotive, $M' \in 1\text{-Mot}(\overline{\mathbb{Q}})$, s.t.

1) $\theta \in i_* V_{sing}(M')$

2) $\mathrm{Ann}(\theta) = p^* V_{dR}(M/M')^\vee \subset V_{dR}(M)^\vee$, where $p: M \twoheadrightarrow M/M'$

$\qquad\quad \|$

$\{ w \in V_{dR}(M)^\vee \mid \int_\theta w = 0 \}$

$\boxed{\exists}$

WLOG, we can assume $\theta \in T_{sing}(M)$. Otherwise we can take $n \in \mathbb{N}$ s.t. $n\theta \in T_{sing}(M)$ and work with $n\theta$.

Recall we have the commutative diagram

was used to construct ϕ_M!

$$\begin{tikzcd}
T_{sing}(M) \cong L \times_{M^{G,an}} \mathrm{Lie}(M^{G,an}) \arrow[d] \arrow[r] & L \times_{M^{G,an}} \mathrm{Lie}(M^{G,an}) \arrow[d] \arrow[r, hook] & \mathrm{Lie}(M^{G,an}) \cong \mathrm{Lie}(M^G)_{\mathbb{C}} = V_{dR}(M)_{\mathbb{C}} \arrow[d, "\exp_{M^G}"] \\
L \arrow[r, equal] & L \arrow[r, hook] \arrow[dr, hook, "\text{is inj}"'] & M^{G,an} \\
& & M^G(\overline{\mathbb{Q}}) \subset M^G(\mathbb{C}) \arrow[u, equal]
\end{tikzcd}$$

$\theta \in$; is inj; is inj; is inj (Talk 10)

$\rightsquigarrow$ we can see $\theta \in \mathrm{Lie}(M^G)_{\mathbb{C}}$. It is s.t. $\exp_{M^G}(\theta) \in M^G(\overline{\mathbb{Q}})$

By AST $\exists\, H \subset M^G$, $H \in \mathcal{G}$, s.t.

1) $\theta \in \mathrm{Lie}(H)_{\mathbb{C}}$

2) $\mathrm{Ann}(\theta) = p^* \mathrm{Lie}(M^G/H)^\vee$ with $p: M^G \twoheadrightarrow M^G/H$ $\rightsquigarrow$ $p^*: \mathrm{Lie}(M^G/H)^\vee \hookrightarrow \mathrm{Lie}(M^G)^\vee$

$\hookrightarrow$ since $\theta \in T_{sing}(M)$ is seen as an element of $V_{dR}(M)_{\mathbb{C}}$ via ϕ_M, this Ann is exactly the one in AST!

Now we need:

Lemma (HW18, Prop. 8.2.1): Let $M=[L\to G]\in 1\text{-Mot}(\bar{\mathbb{Q}})$. For any $H\subset M^\natural$, $H\in\mathcal{G}$, $\exists\, M'\subset M$, $M'\in 1\text{-Mot}(\bar{\mathbb{Q}})$, with

$$\begin{tikzcd} M'^\natural \arrow[r, hookrightarrow] \arrow[dr, hookrightarrow] & M^\natural \\ & H \arrow[u, hookrightarrow] \end{tikzcd}$$

s.t. $V_{sing}(M)\cap \mathrm{Lie}(H)_{\mathbb{C}} = V_{sing}(M')$.

We only see the construction of M'.

Let $0\to \underset{\text{vector group}}{V'} \to H \to \underset{\text{semi-abelian variety}}{G'} \to 0$ the canonical decomp. in given by the structure theorem.

We have the injective morphism of SES ⇝

we have a SES b/c one the canonical SES of the structure theorem. H is inj b/c $H' \hookrightarrow M^\natural$ is.

$$\begin{tikzcd} 0 \arrow[r] & V' \arrow[r] \arrow[d, hookrightarrow] & H \arrow[r] \arrow[d, hookrightarrow] & G' \arrow[r] \arrow[d, hookrightarrow] & 0 \\ 0 \arrow[r] & V \arrow[r] & M^\natural \arrow[r] & G \arrow[r] & 0 \\ L' \arrow[uurr] \arrow[d, hookrightarrow] & & & & \\ L \arrow[uurr] & & & & \end{tikzcd}$$

and we define L'. we mean

$$\begin{tikzcd} L' \arrow[r, hookrightarrow] \arrow[d, hookrightarrow] \arrow[dr, phantom, "\lrcorner"] & H(\bar{\mathbb{Q}}) \arrow[d, hookrightarrow] \\ L \arrow[r, hookrightarrow] & M^\natural(\bar{\mathbb{Q}}) \end{tikzcd}$$

We put $M' := [L' \to G'] \in 1\text{-Mot}(\bar{\mathbb{Q}})$.

Notice it is a

Properties can be proved. ■

We turn back to the proposition. By the Lemma we can find

$$M' \overset{i}{\hookrightarrow} M \quad ⇝ \quad p: M \twoheadrightarrow M/M' =: M''$$

s.t. $V_{sing}(M') = V_{sing}(M)\cap \mathrm{Lie}(H)_{\mathbb{C}}$

⇝ • $\theta \in V_{sing}(M')$ ⇝ $\theta \in i_* V_{sing}(M')$, which is 1)

• $p^*\,\mathrm{Lie}(M''^\natural)^\vee \subset \mathrm{Ann}(\theta)$

Moreover the proposition gives

$$M''^\natural \twoheadrightarrow H/H' =: H''$$

From now on is only linear algebra

⇝ $\mathrm{Ann}(\theta) = \mathrm{Lie}(H'')^\vee \subset \mathrm{Lie}(M''^\natural)^\vee \subset \mathrm{Ann}(\theta)$

⇝ $\mathrm{Ann}(\theta) = \mathrm{Lie}(M''^\natural) = p_* V_{dR}(M'')^\vee$, which is 2).

$\boxed{!}$ let $M_1 \overset{i}{\hookrightarrow} M$ be another submotive with same properties.

$\rightsquigarrow$ $p: M \longrightarrow M/M_1 =: M_2$.

Then, by 2), $p^* V_{dR}(M_2) = Ann(\sigma) = p^* V_{dR}(M'')$ $\rightsquigarrow$ $M_2^G = M''^G$

$\rightsquigarrow$ $M_2 = M''$.

$\uparrow$ $(-)^G$ is faithful ∎

Cor: [HW18, Cor. 9.9] Let $M \in 1\text{-Mot}(\bar{\mathbb{Q}})$, $\sigma \in V_{sing}(M)$, $\omega \in V_{dR}(M)^\vee$ s.t. $\int_\sigma \omega = 0$. Then exists a SES in $1\text{-Mot}(\bar{\mathbb{Q}})$

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$$

s.t. $\sigma = i_* \sigma'$ $\quad \sigma' \in V_{sing}(M')$

$\omega = p^* \omega''$ $\quad \omega'' \in V_{dR}(M'')$

Apply previous Thm to M and σ. We get the SES

$$0 \to M' \overset{i}{\hookrightarrow} M \xrightarrow{p} \underset{\substack{\| \\ M''}}{M/M'} \to 0$$

s.t. 1) $\sigma \in i_* V_{sing}(M)$ $\rightsquigarrow$ $\sigma = i_* \sigma'$

2) $\underset{\substack{\Psi \\ \omega}}{Ann(\sigma)} = p^* V_{dR}(M'')^\vee$ $\rightsquigarrow$ $\omega = p^* \omega''$

by hp. ∎

Thm: (PERIOD CONJECTURE FOR DELIGNE 1-MOTIVES)

[HW18, Thm 9.10] All $\bar{\mathbb{Q}}$-linear relations between periods of Deligne 1-motives are:

(A) bilinearity

(B) induced by morphisms in $1\text{-Mot}(\bar{\mathbb{Q}})$.

Proof follows steps outlined above:

- Consider a $\bar{\mathbb{Q}}$-linear combination of periods $= 0$

$$\lambda_1 \int_{\sigma_1} \omega_1 + \ldots + \lambda_n \int_{\sigma_n} \omega_n = 0$$

where $\int_{\sigma_i} \omega_i$ is a period of $M_i \in 1\text{-Mot}(\bar{\mathbb{Q}})$.

We want to "prove" the $=$ using rules (A) & (B).

- We can see the linear combination as a single period over $M := M_1 \oplus \ldots \oplus M_n$ b/c, if $M_i \overset{\iota_i}{\hookrightarrow} M \overset{\pi_i}{\twoheadrightarrow} M_i$

$$\lambda_1 \int_{\sigma_1} \omega_1 + \ldots + \lambda_n \int_{\sigma_n} \omega_n \overset{(B)}{=} \sum \lambda_i \int_{\iota_{i*}\sigma_i} \pi_i^* \omega_i \overset{(A)}{=} \int_\sigma \omega$$

$\sigma = \sum \iota_{i*} \sigma_i$

$\omega = \sum \lambda_i \omega_i$

- Since $\int_\theta \omega = 0$, then we can apply the previous Cor. and obtain a SES in 1-Mot($\overline{\mathbb{Q}}$)

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0 \quad \text{s.t.} \quad \theta = i_* \theta' \quad \text{and} \quad \omega = p^* \omega''.$$

So $\displaystyle\int_\theta \omega = \int_{i_* \theta'} p^* \omega'' \overset{(B)}{=} \int_{\theta'} \underbrace{i^* p^*}_{=0} \omega'' = 0$

∎