§ $(K, \mathbb{Q})$-Vector Spaces

- $K \hookrightarrow \mathbb{C}$ subfield

Def: $(K,\mathbb{Q})$-vector space: It's a triple $(V_K, V_{\mathbb{Q}}, \phi_{\mathbb{C}})$

- V_K K-vect. space, finite dimensional
- $V_{\mathbb{Q}}$ $\mathbb{Q}$-vect. space, fin. dim.
- $\phi_{\mathbb{C}} : V_K \otimes_K \mathbb{C} \xrightarrow{\sim} V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ $\mathbb{C}$-linear iso (i.e. iso of $\mathbb{C}$-vect. spaces).

Def: Category $(K,\mathbb{Q})$-Vect

- Objects: $(K,\mathbb{Q})$-vector spaces
- Maps$_{\mathrm{Vect}_{(K,\mathbb{Q})}}((V_K, V_{\mathbb{Q}}, \phi_{\mathbb{C}}), (V_K', V_{\mathbb{Q}}', \phi_{\mathbb{C}}')) = \left\{ \begin{array}{l} f_K : V_K \to V_K' \text{ in } \mathrm{Vect}_K \\ f_{\mathbb{Q}} : V_{\mathbb{Q}} \to V_{\mathbb{Q}}' \text{ in } \mathrm{Vect}_{\mathbb{Q}} \end{array} \right.$

such that

$$\left.\begin{tikzcd} V_K \otimes_K \mathbb{C} \arrow[r, "f_K \otimes \mathbb{C}"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\circlearrowleft"] & V_K' \otimes_K \mathbb{C} \arrow[d, "\simeq"] \\ V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \arrow[r, "f_{\mathbb{Q}} \otimes \mathbb{C}"] & V_{\mathbb{Q}}' \otimes_{\mathbb{Q}} \mathbb{C} \end{tikzcd}\right\}$$

Rmk: ~~map~~ $f = (f_K, f_{\mathbb{Q}})$ morph. in $\mathrm{Vect}_{(K,\mathbb{Q})}$ is iso iff $f_{\mathbb{Q}}$ is iso.

Properties of $\mathrm{Vect}_{(\mathbb{R},\mathbb{Q})}$:

- It is abelian: elementary linear algebra checks.
- It is ~~then~~ a tensor category (with the stupid notion of $\otimes$ componentwise), and $\mathbb{Q}$-linear (enriched over $\mathrm{Vect}_{\mathbb{Q}}$).
- It is rigid: all object $(V_{\mathbb{R}}, \mathbb{Q} V_{\mathbb{Q}}, \phi_c)$ has dual $(V_{\mathbb{R}}^*, \mathbb{Q}^*, \phi_c^{*-1})$.
- It is tannakian ... What does that mean?

Tannakian: ~~A cat~~ Let $\mathcal{C}$ ~~is tannakian~~ be a category over K ~~if there exist~~ that is K-linear ~~ab~~ abelian tensor rigid ~~sm~~ (symmetric monoidal) such that $\mathrm{End}(\mathbb{1}) \simeq K$. $\mathcal{C}$ is Tannakian over K iff if $\exists$ $L \supseteq K$ extension & a K-linear functor ~~F:C~~ $F: \mathcal{C} \to \mathrm{Vect}^{f.d.}_{L}$ exact & faithful. $\lrcorner$

$\mathrm{Vect}_{(\mathbb{R},\mathbb{Q})}$ is Tannakian over $\mathbb{Q}$ with fibre functor $(V_{\mathbb{R}}, V_{\mathbb{Q}}, \phi_c) \to V_{\mathbb{Q}}$.

- The invertible objects ($\otimes$-invertible, i.e. dualizable with δ, ev isomorphisms) are those where $V_{\mathbb{R}}, V_{\mathbb{Q}}$ have dimension 1. They are: $(\mathbb{R}, \mathbb{Q}, z)$, where $z \in \mathbb{C}^\times$ ($\phi_c(x) := x \cdot z$) up to iso.

We denote this object by $\mathcal{L}(z)$

B § Cohomological $(\mathbb{R}, \mathbb{Q})$-vector-Complexes

Def. Cohomological $(\mathbb{R}, \mathbb{Q})$-Vector complex is the data of:

- $K^\cdot_{\mathbb{R}}$ b.b. complex of ~~filt~~ $\mathbb{R}$-vect. spaces with fin. dim coh.
- $K^\cdot_{\mathbb{Q}}$ b.b. complex of ~~filt~~ $\mathbb{Q}$ vector spaces with fin. dim. coh
- $K^\cdot_{\mathbb{C}}$ b.b. complex of $\mathbb{C}$-vect. spaces with f.d. coh.
- $\phi^\cdot_{\mathbb{R},\mathbb{C}} : K_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \to K^\cdot_{\mathbb{C}}$ q.-iso
- $\phi^\cdot_{\mathbb{Q},\mathbb{C}} : K_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \to K^\cdot_{\mathbb{C}}$ q.-iso.

Def: $C^+_{(\mathbb{R},\mathbb{Q})}$ category with:

objects: $(\mathbb{R}, \mathbb{Q})$-vect. complexes

Maps := triples of morphism of complexes that make the obvious diagrams commute.

$K^+_{(\mathbb{R},\mathbb{Q})}$:= homotopy category of $(\mathbb{R},\mathbb{Q})$-Vect = Category of $(\mathbb{R},\mathbb{Q})$-vector complexes up to ~~q-isomorphism~~ homotopy.

$D^+_{(\mathbb{R},\mathbb{Q})}$ = Derived cat. of $(\mathbb{R},\mathbb{Q})$-vect complexes = category $(\mathbb{R},\mathbb{Q})$-vect. complexes up to q.-isos = $K^+_{(\mathbb{R},\mathbb{Q})}$ localized at q.-isos.

$D^+_{(h,\alpha)}$ is a triangulated category in the usual way (that is $D^{+\geq 0}_{(h,\alpha)} := \{(V_n, \alpha_n, \phi^i_\sigma) \mid H^i(-) = 0 \ \forall\, i<0\}$ & $D^{-}_{(h,\alpha)}$ obvious) with $D^{+\heartsuit}_{(h,\alpha)} = \mathrm{Vect}_{(h,\alpha)}$.

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The tensor product of complexes gives a tensor product on $C^+_{(h,\alpha)}$, by taking complex-wise ~~componentwise~~ (h, α, σ) tensor products.

[Complexwise: For two complexes $F^\cdot$ & $G^\cdot$ we define $F^\cdot \otimes G^\cdot$ by: $(F^\cdot \otimes G^\cdot)^n = \bigoplus_{i+j=n} F^i \otimes G^j$, and ~~$d_{F\otimes G} = id_F \otimes d_G$~~ $d^i_{F\otimes G} = id_F \otimes d^i_G + (-1)^i d_F \otimes id_G$.]

It is compatible with the comparison isomorphism

This tensor product induces a tensor product in $K^+_{(h,\alpha)}$ & $D^+_{(h,\alpha)}$.

<u>Lemma</u>: ~~On~~ $C^+_{(h,\alpha)}$, $K^+_{(h,\alpha)}$ & $D^+_{(h,\alpha)}$ ~~we have~~ are associative & commutative tensor categories. And, for $X^\cdot, Y^\cdot \in D^+_{(h,\alpha)}$:

- $H^*(X^\cdot \otimes Y^\cdot) \cong H^*(X^\cdot) \otimes H^*(Y^\cdot)$
- $(H^*(\,) \otimes H^*(\,)) \otimes H^*(\,) \simeq H^*(\,) \otimes (H^*(\,) \otimes H^*(\,))$
- $H^p(X^\cdot) \otimes H^q(Y^\cdot) = (-1)^{pq} H^q(Y^\cdot) \otimes H^p(X^\cdot)$.

§ Period isomorphism

$k \hookrightarrow \mathbb{C}$ subfield $\quad X \in Sm_k$.

Let's review some isomorphism that we have already seen:

- $H^*_{dR}(X) \otimes_k \mathbb{C} \to H^*_{dR}(X \times_k \mathbb{C})$ change of coeff.

[There is a natural quasi-isomorphism $R\Gamma_{dR}(X) \otimes_k K \to R\Gamma_{dR}(X_K)$ given by functoriality of Godement resolution. This induces that iso taking cohomology]

- $H^*_{dR}(X \times_k \mathbb{C}) \to H^*_{dR}(X^{an})$ (GAGA principle)
- $H^*_{Sing}(X^{an}) \to H^*_{dR}(X^{an}, \mathbb{C})$ (classical de Rham for holomorphic varieties, also called Poincaré lemma)
- $H^*_{Sing}(X^{an}, \mathbb{C}) \to H^*_{Sing}(X^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$.

This is the natural change of coefficients, since every variety in $\mathbb{C}$ is a variety in $\mathbb{Q}$.

By taking the composition of all these isomorphism (the third being taken in the inverse direction), we obtain:

<u>Def</u>: [Period isomorphism].

$$\mathrm{Per}: H^*_{dR}(X) \otimes_k \mathbb{C} \xrightarrow{\sim} H^*_{sing}(X^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

· [Period pairing]:

$$H^*_{dR}(X) \times H_*^{sing}(X^{an}, \mathbb{Q}) \to \mathbb{C}$$

$$(\omega, \gamma) \longmapsto (\mathrm{per}\,\omega)(\gamma)$$

~~Def: H: Sm_k → (H*_dR(X), H*_sing(X, Q), Per)~~

<u>Def</u>: $H: Sm_k \to \mathbb{Q}\mathrm{Vect}_{(k, \mathbb{Q})}$

$$X \mapsto (H^*_{dR}(X), H^*_{sing}(X, \mathbb{Q}), \mathrm{Per})$$

is a functor.

(the functoriality holds by construction: cohomology is a functor).

<u>Lemma</u>: ~~H is closed under~~ direct ~~sums~~ & ~~tensor product~~

~~Pf: Take the disjoint~~

<u>Lemma</u>: $H(X) \otimes H(Y) \cong H(X \times Y)$

<u>Pf</u>: Singular cohomology & de Rham cohomology have both Kunneth isomorphism, and all the ~~identification~~ maps in the definition of per are compatible with the product structure.

Lemma: The image ~~is~~ of H is closed under direct sums & tensor product

Pf: Take disjoint union as direct sums & product as tensor product.

Thm: $X \in Sm_k$ affine. $[\omega] \in H^i_{dR}(X)$. ~~c~~ ~~$c = \sum a_i \gamma$~~ $c = \sum a_i \gamma_i \in H_i^{sing}(X^{an}, \mathbb{Q})$, $a_i \in \mathbb{Q}$, $\gamma_i : \Delta_i \to X^{an}$ diff. cycles

Then $\langle [\omega], c \rangle = \sum a_i \int_{\Delta_i} \gamma^*(\omega)$

Pf: $A^i(X_{an})$ grp of C^∞-differential forms (complex) with associated sheaf $\mathcal{A}^i_{X^{an}}$. $\mathcal{A}^\bullet_{X^{an}}$ complex. ~~We obtain a~~ two complexes $\mathcal{A}^\bullet_{X^{an}}(X^{an}) \hookleftarrow \Omega^\bullet_{X^{an}}(X^{an})$ that are quasi-isomorphic, because both compute singular cohomology in the affine case ($\Omega_{X^{an}}(X^{an})$ because of Poincaré lemma, the other one by its C^∞-analogue). This implies that, in the period isomorphism per$[\omega]$ can be considered as a C^∞-form instead of just an analytic form. The formula then comes from classical C^∞ de Rham. $\square$

The functor H gives a notion of cohomology in the $(k, \mathbb{Q})$-theoretical setting.

Exmp: What is $H^*(\mathbb{P}^n_k)$ in the $(k, \mathbb{Q})$-theoretical setting?

$H^{2j}_{dR}(\mathbb{P}^n_k) = k$, $H^{2j}_{B}(\mathbb{P}^n_k) = \mathbb{Q}$. Then $\mathrm{comp}_{B,dR}$ is iso of 1-dimensional complex vector spaces.

$$H^{2j}(\mathbb{P}^n_k) = (k, \mathbb{Q}, x \mapsto \xi \cdot x). \quad \text{Let's compute } \xi \in \mathbb{C}.$$

- Case $n=1$.

Mayer-Vietoris w.r.t. the standard affine cover of $\mathbb{P}^1 \simeq S^1$ gives $H^2_{dR}(\mathbb{P}^n) \simeq H^1_{dR}(\mathbb{G}_m)$. ~~$\mathbb{G}_m$ is affine~~

$\mathbb{G}_m$ is affine $\Rightarrow$ we can use the last theorem to make the period isomorphism explicit.

$H^1_{dR}(\mathbb{G}_m) = k[\frac{d}{dt}]$. $\quad H_1^{sing}(\mathbb{G}_m, \mathbb{C}) = \mathbb{C}\cdot[\gamma]$

$\gamma = S^1$ oriented counterclockwise

$$\langle [\tfrac{d}{dt}], \gamma \rangle = \int_\gamma \frac{d}{dt} = 2\pi i$$

$$\Rightarrow \xi = 2\pi i$$

- General n. Čech complex argument gives: $H^{2j}_{dR}(\mathbb{P}^n) \simeq H^j_{dR}(\mathbb{G}_m^{\times n})$

Similarly as before, $\xi = \int_{\partial \Delta^{j+1}} \left(\frac{dt_1}{t_1} \wedge \dots \wedge \frac{dt_j}{t_j}\right) = (2\pi i)^j$

We conclude: $\boxed{H^{2j}(\mathbb{P}^n_k) = L((2\pi i)^j)}$

How to generalize it to non-smooth varieties

Use ~~h~~ h'-topology (from Luca's talk)

Recall: It is the smallest Grothendieck topology on (An/X), (category of analytic spaces over X) such that proper surjective morphism & open covers are covering.

· We have h'-differentials $\Omega^p_{h'/X} := (Y \mapsto \Omega^p_Y)^{\sim}_{h'}$

Def: Relative h'-de Rham cohomology:

$i : Z \hookrightarrow X$ closed subspace.

$$\Omega^p_{h'/(X,Z)} := \mathrm{Ker}(\Omega^p_{h'/X} \to i_* \Omega^p_{h'/Z}).$$

in $(An/X)_{h'}$.

$$H^p_{dRam}(X_{h'}, Z_{h'}) = H^p_{h'}((An/X)_{h'}, \Omega^*_{h'/(X,Z)}).$$

· There are ~~relative~~ analogues of excision, Kunneth formula ecc. for h'-de Rham relative cohomology.

· There is a Gaga theorem for relative de Rham cohomology

· There is a Poincaré lemma for relative de Rham cohomology

$$\cancel{H^i_{dR}(X_{h'}, Z_{h'}) \cong H^i} \quad H^i_{sing}(X, Z, \mathbb{C}) \xrightarrow{\sim} H^i_{dR}(X_{h'}, Z_{h'}). \leftarrow$$

We obtain the period isomorphism:

$$\mathrm{per}: H^*_{dR}(X,Z)\otimes_k \mathbb{C} \xrightarrow{\sim} H^*_{sing}(X,Z;\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$$

by:

- $H^*_{dR}(X,Z)\otimes_k \mathbb{C} \xrightarrow{\sim} H^*_{dR}(X\times_k \mathbb{C}, Z\times_k \mathbb{C})$ change of ~~relative~~ ~~de Rham~~ coefficients.

- $H^*_{dR}(X\times_k \mathbb{C}, Z\times_k \mathbb{C}) \xrightarrow{\sim} H^*_{dR^{an}}(X^{an}_{h'}, Z^{an}_{h'})$ relative GAGA

  here we have h-topology

- $H^*_{dR^{an}}(X^{an}_{h'}, Z^{an}_{h'}) \xleftarrow{\sim} H^*_{sing}(X^{an}, Z^{an}, \mathbb{C})$ relative Poincaré

- Change of coefficients $H^*_{sing}(X,Z,\mathbb{C}) \xrightarrow{\sim} H^*_{sing}(X,Z,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$.

- Functoriality, Kunneth formula, direct sum as disjoint union, tensor product as product hold in this setting.

- One recovers $H^*_{dR}(X)$ and $H^*_{sing}(X^{an},\mathbb{Q})$ by taking $Z=\emptyset$, with now X being ~~just~~ possibly singular.

§ Pure Hodge Structures

Motivation:

Classical Hodge thm: M compact Kähler mfd (e.g. smooth proj. var/$\mathbb{C}$).

Then, there is a direct sum decomposition:

$$H^n(M, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} H^{p,q}(M)$$

~~with a conjugation~~ where the complex conjugation action on the LHS trasforms $H^{p,q}(M)$ in $H^{q,p}(M)$ (they say: the decomposition $H^{p,q}$ satisfies Hodge Symmetry).

Idea: $H^{p,q}(M) :=$ subspace of $H^n(M)$ of cohomology classes represented by C^∞-differential forms of type (p,q). That is, if $dz_1, \dots, dz_d$ basis for $H^n(M)$, a form of type (p,q) is a form that can be written as: $\sum_{I,J} f_{IJ}\, dz_{i_1} \wedge \dots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$.

with sums running over subsets $I = \{i_1, \dots, i_p\}$, $J = \{j_1, \dots, j_q\}$.

~~Trasp~~ Transpose Hodge thm in a more abstract context $\rightsquigarrow$

$\rightsquigarrow$ A pure Hodge structure is a vector space whose complexification has a ~~decom~~ bigraded decomposition that satisfies Hodge symmetry:

Def ['Classical' pure Hodge structure]:

A pure Hodge structure of weight m is the data of:

a f.d. $\mathbb{Q}$-vect. sp. H with a bigraded decomposition

$$H_{\mathbb{C}} := H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\substack{p+q=m \\ (p,q)\in\mathbb{Z}^2}} H^{p,q} \qquad \text{satisfying } \overline{(H^{p,q})} = H^{q,p}$$

Hodge type of H $:= \{(p,q)\in\mathbb{Z}^2 \mid H^{p,q} \neq 0\}$.

Lemma: H f.d. $\mathbb{Q}$-vect. space

H carries a p.H.s. $\iff$ $\exists$ exhaustive decreasing filtration F on $H_{\mathbb{C}}$

s.t., $\forall p$ $\quad H_{\mathbb{C}} = F^p \oplus \overline{F^{m+1-p}}$.

F is called "the Hodge Filtration of the Hodge structure".

Pf: 1) H has a Hodge structure: $H_{\mathbb{C}} = \bigoplus H^{p,q}$.

Set $F^p := \bigoplus_{r\geq p} H^{r,s}$. F^p is exhaustive and decreasing.

$$\overline{F^{m+1-p}} = \bigoplus_{r\geq m+1-p} \overline{H^{r,s}} = \bigoplus_{r\geq m+1-p} H^{s,r} \overset{s+r=m}{=} \bigoplus_{s\leq p-1} H^{s,r}$$

then $F^p \oplus \overline{F^{m+1-p}} = \bigoplus_{r\geq p} H^{r,s} \oplus \left(\bigoplus_{r\leq p-1} H^{r,s}\right) = H$.

2) F Hodge filtration on $H_{\mathbb{C}}$.

$$H^{p,m-p} = F^p \cap \overline{F^{m-p}}.$$

$$\overline{H^{p,q}} = \overline{F^p \cap \overline{F^q}} = F^q \cap \overline{F^p} = H^{q,p}.$$

We want: $H_{\mathbb{C}} = \bigoplus_{p+q=m} H^{p,q}$

By induction on $n = \max\{p \mid F^p \neq 0\}$

If $n < m/2$, $F^{\lceil \frac{m}{2} \rceil} = \underset{\overline{F^{\lceil \frac{m}{2} \rceil}}}{\underset{\parallel}{F^{m+1-\lceil \frac{m}{2} \rceil}}} \overset{\text{by def. of } n}{=} 0$. Then $H_{\mathbb{C}} = 0$

the claim is obvious (F empty filtration).

• $n \geq m/2$. (*) $H_{\mathbb{C}} = \overline{F^n} \oplus \overset{H'_{\mathbb{C}}}{\overset{\parallel}{(F^{m+1-n} \cap \overline{F^{m+1-n}})}} \oplus F^n$

easy: $\overline{F^n} = \overline{F^{m+1-n}} - (\overline{F^{m+1-n}} \cap F^{m+1-n})$

But $F^n = H^{n,m-n}$, $\overline{F^n} = H^{m-n,n}$.

$H^{n,m-n} = F^n \cap \overline{F^{m-n}}$
but $H_{\mathbb{C}} = F^{n+1} \oplus \overline{F^{m-n}}$
$= \overline{F^{m-n}}$

$H'_{\mathbb{C}}$ has a real structure. Use induction hp:

$$H'_{\mathbb{C}} = \bigoplus_{p=n+1}^{m-n-1} H^{p,m-p}.$$

By (*) we get $H_{\mathbb{C}} = \bigoplus H^{p,m-p}$ □

In ~~our~~ the setting we are interested in:

$M = X(\mathbb{C})$, $X \in Var_k$, $k \hookrightarrow \mathbb{C}$. $H^n(M, \mathbb{C}) \cdot H^n_{dR}(X) \otimes_k \mathbb{C}$.

We know that $H^*_{dR}(X)$ already has a Hodge filtration. Thus, ~~we want to accept~~ it is important for us to have a definition of pure Hodge structure that keeps track of the comparison isomorphism (and of the fact that we already have a filtration on H_{dR}). Let's change the definition then:

<u>Def</u> ["refined" pure Hodge structure]:

$k \hookrightarrow \mathbb{C}$ subfield.

A pure ~~a~~ Hodge structure (of weight n) is the datum of:

$H = (H_B, (H_{dR}, F), comp_{B,dR})$, with:
- H_B f. d. $\mathbb{Q}$-v.sp.
- (H_{dR}, F) f. d. k-v.sp. with an exhaustive decreasing filtration
- $comp_{B,dR} : H_{dR} \otimes_k \mathbb{C} \xrightarrow{\sim} H_B \otimes_{\mathbb{Q}} \mathbb{C}$ iso of complex. v.sp.'s

on the complexification

such that, the induced filtration F_B on $H_{\mathbb{C}} := H_B \otimes_{\mathbb{Q}} \mathbb{C}$ satisfying the property:

$\exists\, n \in \mathbb{Z}$ ~~si~~ such that: $\forall p$ $H_{\mathbb{C}} = F^p_B H_{\mathbb{C}} \oplus \overline{F_B^{n-p+1} H_{\mathbb{C}}}$.

Def: HS(k) category with

- objects: pure Hodge structures over k (of any weight)
- For H, H' two pure Hodge structure, $f: H \to H'$ is:

$$f = (f_B, f_{dR}) = \begin{cases} f_B : H_B \to H'_B & \mathbb{Q}\text{-linear} \\ f_{dR} : H_{dR} \to H'_{dR} & k\text{-linear} \end{cases}$$ such that

1) $f_{dR}(F^p H_{dR}) \subseteq F^p H'_{dR} \quad \forall p$

2)
$$\begin{tikzcd} H_{dR} \otimes_k \mathbb{C} \arrow[r, "\mathrm{comp}_{B,dR}"] \arrow[d, "f_{dR} \otimes id_{\mathbb{C}}"] & H_B \otimes_{\mathbb{Q}} \mathbb{C} \arrow[d, "f_B \otimes id_{\mathbb{C}}"] \\ H'_{dR} \otimes_k \mathbb{C} \arrow[r, "\mathrm{comp}'_{B,dR}"] & H'_B \otimes_{\mathbb{Q}} \mathbb{C} \end{tikzcd}$$
commutes

Rmk: If H has weight n & H' has weight m, $f \neq 0 \Rightarrow$
$\Rightarrow n = m$.

extension of scalars: $k \subseteq L \subseteq \mathbb{C}$ subfields, the assignment
$(-) \otimes_k L : HS(k) \to HS(L)$ is a functor, and

- $(H \otimes_k L)_B = H_B \otimes_k L$, $(H \otimes_k L)_{dR} = H_{dR} \otimes_k L$

Exmp [Hodge-Tate structure]:

$\mathbb{Q}(m) := (\mathbb{Q}, (\mathbb{Q}, F), \mathrm{Comp}_{B,dR})$, where

F is the trivial filtration $\mathbb{Q} = F^{-m}\mathbb{Q} \supseteq F^{-m+1}\mathbb{Q} = \{0\}$.

and $\mathrm{Comp}_{B,dR} : \mathbb{C} \to \mathbb{C}$
$z \mapsto (2\pi i)^{-m} z$.

$\mathbb{Q}(m)$ is a one-dimensional pure Hodge structure of weight $-2m$ over $\mathbb{Q}$.

The refined version of the classical Hodge thm is:

Thm [Algebraic Hodge thm]: $k \hookrightarrow \mathbb{C}$ subfield, $X \in \mathrm{Var}_k^{sm}$
X sm. proper variety$/k$. $\exists$ pure Hodge structure of weight m over k $\quad H^m(X) = (H^m_B(X), (H^m_{dR}(X), F), \mathrm{Comp}_{B,dR})$, where $\mathrm{Comp}_{B,dR}$ is the period isomorphism, and F is the Hodge filtration $F^p H^m_{dR}(X) = \mathrm{Im}(\mathbb{H}^m(X, \Omega_X^{\geq p}) \to \mathbb{H}^m(X, \Omega_X^{*}))$, where $(-)^{\geq p}$ is the stupid truncation.

$\boxed{\text{slogan: if } X \text{ sm pr. } k\text{. the } m\text{th cohomology of } X \text{ is a pure Hodge structure of weight } m.}$

Tate twist: H Hodge structure of weight n, $m \in \mathbb{Z}$.
$H(m) :=$ pure Hodge structure of weight $n-2m$ with the same underlying vector spaces, filtration shifted by m and comp iso multiplied by $(2\pi i)^{-m}$.

Similar to the case of $(\mathbb{R}, \mathbb{Q})$-vector spaces, $\otimes_{\mathbb{R}}$ & $\otimes_{\mathbb{Q}}$ give a tensor product on $HS(\mathbb{R})$, and we have $H(m) = H \otimes \mathbb{Q}(m)$.

Tate twist is the twist in the definition of the "Hodge-structure-theoretical" definition of the Gysin map:

$Z \hookrightarrow X$ closed subvariety $\rightsquigarrow$ Gysin map $H^m(Z) \to H^{m+2c}(X)$, where $c = \mathrm{codim}_X(Z)$.

"Hodge-structure-theoretical": $H^{m+2c}(X)$ & $H^m(Z)$ are two Hodge structure, and the Gysin map is ~~a~~ the map:

$$H^m(Z)(-c) \to H^{m+2c}(X).$$

~~Pro~~ Example ~~complex~~ projective space: Hodge theoretical description of the cohomology of ~~complex~~ projective space:

$$H^j(\mathbb{P}^n, \mathbb{Q}) \underset{\text{As Hodge structure}}{=} \begin{cases} \mathbb{Q}(-j/2) & \text{if } 0 \le j \le 2n \ \& \ j \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

$$H^2(\mathbb{P}^n) = \mathbb{Q}(-1) \simeq H^1(\mathbb{G}_m)$$

$$\mathbb{G}_m = \operatorname{Spec}(\mathbb{Q}[t, t^{-1}])$$

In $HS(\mathbb{R})$, the (-1)-twist is the multiplication by $H^2(\mathbb{P}^1_{\mathbb{R}})$.

§ Mixed Hodge Structures

Preliminary notions:

k field, (V,F) & (V', F') two filtered vector spaces. A morphism $f: V \to V'$ is <u>filtered</u> if $f(F^p V) \subseteq F'^p V'$, and it is <u>strict w.r.t. F</u> if, in addition, $f(F^p V) = F^p V' \cap \mathrm{Im}(f)$

The motivation for mixed Hodge structure is:
Alg. Hodge thm says that the n-th cohomology of a smooth proper variety carries a a pure Hodge structure. That's not true in general (e.g. $H^1(\mathbb{G}_m)$ is one-dimensional, then it cannot carry a Hodge structure of weight one)

We want our category of Hodge structure to be a prototype of the category of motives, then we want a Hodge structure in the cohomology of any variety. We need a generalization.

<u>Thm</u> [Deligne's remake of Algebraic Hodge thm]:

$X \in \mathrm{Var}_{\mathbb{C}}$.

1) $\exists$ increasing filtration $0 = W_{-1} \subseteq W_0 \subseteq \dots \subseteq W_{2n} = H^n(X)$ and a decreasing one $F^0 = H^n(X, \mathbb{C}) \supseteq F^1 \supseteq \dots \supseteq F^{n+1} = 0$ s.t.

the second (F) induces a pure Hodge structure of weight m on the m-th graded pieces of the first ($\forall m$)

$$Gr_m^W H^n(X) = W^m H^n(X) \big/ W^{m-1} H^n(X)$$

2) If $f: X \to Y$ morph. of complex varieties $\rightsquigarrow$ $f^*: H^n(Y) \to H^n(X)$ is filtered with respect to both ~~filtra~~ filtrations.

3) If X is smooth, $Gr_m^W H^n(X) = 0$ $\forall$ $m < n$, and, if X is proper, $Gr_m^W H^n(X) = 0$ for $m > n$.

<u>Def.</u>: $k \subseteq \mathbb{C}$ subfield. A <u>mixed Hodge structure</u> is a triple

$H = ((H_B, W^B), (H_{dR}, F, W^{dR}), comp_{B,dR})$ with:

- H_B f.d. $\mathbb{Q}$-vector space, with an incr. filtration W^B,
- H_{dR} f.d. k-vect. space with an incr. filtr. W^{dR} and a decreasing one F,
- An iso of complex vector spaces $comp_{B,dR}: H_{dR} \otimes_k \mathbb{C} \xrightarrow{\sim} H_B \otimes_{\mathbb{Q}} \mathbb{C}$, filtered with respect the increasing filtrations W^{dR}, W^B, i.e. $\boxed{comp_{B,dR}(W^{dR} \otimes_k \mathbb{C}) = W^B \otimes_{\mathbb{Q}} \mathbb{C}}$

and all these data verify the condition:

$\forall m$ $Gr_m^W H = (Gr_m^{W^B} H_B, (Gr_m^{W^{dR}} H_{dR}, F), comp_{B,dR})$ is a pure Hodge structure of weight m.

Def [Morph. of mixed Hodge structure]:

$f: H \to H'$ morph. of mixed Hodge structures over k is a pair (f_B, f_{dR}) where

- $f_B: H_B \to H'_B$ morph. of $\mathbb{Q}$-vect. spaces
- $f_{dR}: H_{dR} \to H'_{dR}$ morph. of k-vect. spaces &

$f_B(W^B H_B) \subseteq W^B H'_B$, $f_{dR}(F H_{dR}) \subseteq F H'_{dR}$,

$f_{dR}(W^{dR} H_{dR}) \subseteq W^{dR} H'_{dR}$, $f_{dR} \circ \mathrm{comp}_{B,dR} = \mathrm{comp}'_{B,dR}(f_B \otimes \mathrm{Id}_{\mathbb{C}})$.

$[(f_B \otimes_k \mathrm{Id}_{\mathbb{C}}) \circ \mathrm{Comp}_{B,dR} = \mathrm{Comp}'_{B,dR} \circ (f_{dR} \otimes_k \mathbb{C})]$

Def: $MHS(k) :=$ category of mixed Hodge structures over k.

Def:
$$\omega_B : MHS(k) \to \mathrm{Vect}_{\mathbb{Q}}, \quad H \mapsto H_B \qquad \text{Betti fiber functor}$$

$$\omega_{dR} : MHS(k) \to \mathrm{Vect}_k, \quad H \mapsto H_{dR} \qquad \text{de Rham fiber functor}$$

Thm [Deligne]: $MHS(k)$ is abelian

This is proved in [Deligne, théorie de Hodge: II]: it's an application of the more general statement:

$\mathcal{A}$ abelian cat., $\mathcal{A}'$ cat of objects of $\mathcal{A}$ with three filtrations $W, F, \bar{F}$ ($\bar{F}$ decreasing), and as morphisms, the ones of $\mathcal{A}$ compatible with these three filtrations:

Thm 1.2.10, (i).

The proof is pure linear algebra.

Prop: $f: H \to H'$ morph. of Hodge structures, then:

~~Also~~ $\omega_B(f)$ iso $\iff$ f iso $\iff$ $\omega_{dR}(f)$ iso.

The key ingredient for the proof is ~~the~~ this property

Prop ["Deligne's Splitting"]: $f: H \to H'$ morphism of mixed Hodge structures Then, f_B is strict with respect to the weight filtration (W^B, increasing), and f_{dR} is strict with respect to the weight filtration (W^{dR}) and the Hodge filtration (F).

Pf: Is a consequence of ~~[illegible]~~ [PS08, Lemm.-Def 3.4]. $\square$

Proof of $\omega_B(f)$ iso $\Leftrightarrow f$ iso $\Leftrightarrow \omega_{dR}(f)$ iso.

By the comp. iso $\mathrm{comp}_{B,dR}$ seen in the setting of $(K,\mathbb{Q})$-vector spaces, we already know that

$\omega_B(f)$ iso $\Leftrightarrow \omega_{dR}(f)$ iso. We want f iso $\Leftrightarrow \omega_{dR}(f)$ iso

(This is not true in general for filtered vector spaces, but MHS are more than filtered vector spaces).

But, by Deligne's splitting, we already know that

$f_{dR} : (H, W, F) \to (H', W', F')$ is strict with respect to both filtrations:

if ~~f iso~~ $\omega_{dR}(f)$ is iso

$$f_{dR}(F^p H_{dR}) = F^p H'_{dR} \cap \mathrm{Im}(f_{dR}) = F^p H'_{dR}$$

$\boxed{f_{dR} = \omega_{dR}(f)}$

$$f_{dR}(W_m H_{dR}) = W_m H'_{dR} \cap \mathrm{Im}(f_{dR}) = W_m H'_{dR}$$

if $\omega_{dR}(f)$ is iso

Then, if $\omega_{dR}(f)$ is iso, f_{dR} is iso and $\omega_B(f)$ is iso, then f_B is iso, then f is iso.

The other: If f is iso, $\omega_{dR}(f)$ is obviously an iso.

$\square$