Nori's Rigidity Criterion

- Recall: $\mathcal{D}$ a diagram, $T : \mathcal{D} \longrightarrow R\text{-Mod}$ a rep'
- Nori: $\exists$

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\tilde{T}"] \arrow[dr] & \mathcal{C}(\mathcal{D},T) \arrow[r, "f_T"] \arrow[d, dashed, "\exists!"] & R\text{-Mod} \\ & \mathcal{A} \arrow[ur] & \end{tikzcd}$$

$\mathcal{C}(\mathcal{D},T)$: R-lin. abelian; f_T: faithful, exact

Ex: $\mathcal{D} = \mathrm{Var}_K$, $T = H^*_{sing}(-^{an}, \mathbb{Z})$, $\mathcal{A} = \mathbb{Z}\text{-MHS}$
or $T = H^*_{ét}(-, \mathbb{Z}_\ell)$, $\mathcal{A} = \mathrm{Rep}_{\mathbb{Z}_\ell} \mathrm{Gal}(\bar{K}/K)$

- From now on: R = a field or a Dedekind domain

Thm: Suppose T is valued in proj. (!) R-modules. Then $A(\mathcal{D},T) := \varinjlim_F \mathrm{End}(T|_F)^\vee$ is a coalgebra &

$$\mathcal{C}(\mathcal{D},T) \simeq A(\mathcal{D},T)\text{-Comod}$$

Q: What is a comodule over a coalgebra?

Def.: An R-**bialgebra** A is an R-algebra which is also an R-coalgebra and s.t. comultiplication $\Delta : A \longrightarrow A \otimes_R A$ (and counit $\varepsilon : A \longrightarrow R$) are ring homomorphisms

Prop: Let A be a commutative, unital (and counital) R-bialgebra. Then $M := \mathrm{Spec}\, A$ is a (unital) monoid scheme. Moreover,

$$\underset{\substack{\uparrow \\ \text{on fin. gen. } R\text{-modules}}}{\mathrm{Rep}_M} \simeq \underset{\substack{\uparrow \\ \text{fin. gen. } /R}}{A\text{-Comod}}$$

Proof:
- Since A is a com. unital ring, $\mathrm{Spec}(A)$ makes sense
- The other statements are obtained simply by 'turning all arrows around' $\square$

Goal 1: Enrich $A = A(\mathcal{D},T)$ with a bialgebra structure!

Goal 2: Find conditions s.t. $M = \mathrm{Spec}(A)$ is a group scheme

§1 Multiplicative structures for $T:\mathcal{D}\to R\text{-Proj}$

Def: $\mathcal{D}_1,\mathcal{D}_2$ diagrams. We define a diagram $\mathcal{D}_1\times\mathcal{D}_2$: vertices $=V(\mathcal{D}_1)\times V(\mathcal{D}_2)$

edges $=E(\mathcal{D}_1)\times\{id\}\cup\{id\}\times E(\mathcal{D}_2)$

Def: A grading of $\mathcal{D}$ is a map $|\cdot|: V(\mathcal{D})\longrightarrow\mathbb{Z}/2\mathbb{Z}$; set $|\gamma|:=|v|-|v'|$ $\forall\gamma: v\to v'$

Ex: $\mathcal{D}$ graded ⇝ Get $|\cdot|:\mathcal{D}\times\mathcal{D}\longrightarrow\mathbb{Z}/2\mathbb{Z}$, $|(v,w)|:=|v|+|w|$

Def: A graded commutative, (unital) multiplicative structure on $\mathcal{D}$ is

(i) a map $\times:(\mathcal{D}\times\mathcal{D},|\cdot|)\longrightarrow(\mathcal{D},|\cdot|)$ (+ vertex $\mathbb{1}$, $|\mathbb{1}|=0$)

(ii) choices of edges $\alpha_{v,w}: v\times w\longrightarrow w\times v$

$\beta_{v,w,u}: v\times(w\times u)\longrightarrow(v\times w)\times u$, $\beta'_{v,w,u}:(v\times w)\times u\longrightarrow v\times(w\times u)$

($u_v: v\longrightarrow\mathbb{1}\times v$) for all vertices $v,w,u\in\mathcal{D}$

Def: A graded com. multiplicative rep $T:(\mathcal{D},|\cdot|,\times)\longrightarrow R\text{-Proj}$ is

(iii) choices of iso's $\tau_{v,w}: T(v\times w)\xrightarrow{\cong}T(v)\otimes T(w)$

(+ $R\xrightarrow{\cong}T(\mathbb{1})$)

s.t. (1) $\forall\gamma: v\to v'$:

$$\begin{tikzcd} T(v\times w) \arrow[r, "T(\gamma\times id)"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\circlearrowright"] & T(v'\times w) \arrow[d, "\tau"] \\ T(v)\otimes T(w) \arrow[r, "(-1)^{|\gamma||w|}T(\gamma)\otimes id"'] & T(v')\otimes T(w) \end{tikzcd}
\qquad
\begin{tikzcd} T(w\times v) \arrow[r, "T(id\times\gamma)"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\circlearrowright"] & T(w\times v') \arrow[d, "\tau"] \\ T(w)\otimes T(v) \arrow[r, "id\otimes T(\gamma)"'] & T(w)\otimes T(v') \end{tikzcd}$$

(2)
$$\begin{tikzcd} T(v\times(w\times u)) \arrow[r, "\beta_{v,w,u}"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\circlearrowright"] & T((v\times w)\times u) \arrow[d, "\tau"] \\ T(v)\otimes(T(w)\otimes T(u)) \arrow[r, "\cong"'] & (T(v)\otimes T(w))\otimes T(u) \end{tikzcd}$$

(4)
$$\begin{tikzcd} T(v\times w) \arrow[r, "\alpha"] \arrow[dr, phantom, "\circlearrowright"] & T(w\times v) \\ T(v)\otimes T(w) \arrow[u, "\tau"] \arrow[r, "(-1)^{|v|\cdot|w|}"'] & T(w)\otimes T(v) \arrow[u, "\tau"'] \end{tikzcd}$$

(3) $T(\beta_{v,w,u})=T(\beta'_{v,w,u})^{-1}$

(5)
$$\begin{tikzcd} T(v) \arrow[r, "T(u_v)"] \arrow[d, phantom, "\cong"] & T(\mathbb{1}\times v) \arrow[dl, "\tau"] \\ T(\mathbb{1})\otimes T(v) & \end{tikzcd}$$

Ex: Define $\mathbb{N}_0$ by • $V(\mathbb{N}_0)=\{0,1,2,3,\ldots\}$

• $E(\mathbb{N}_0)=\{id_a\times\alpha_{v,w}\times id_b: a+v+w+b\longrightarrow a+w+v+b \mid a,b,v,w\in\mathbb{N}_0\}$ only self-edges !

• $|n|\equiv 0$ $\forall n\in\mathbb{N}_0$

⇝ $V:\mathbb{N}_0\longrightarrow\mathbb{Q}\text{-Mod}$, $n\longmapsto V^{\otimes n}$, $id_a\times\alpha_{v,w}\times id_b\mapsto(V^{\otimes a}\otimes V^{\otimes v}\otimes V^{\otimes w}\otimes V^{\otimes b}\xrightarrow{\sim}V^{\otimes a}\otimes V^{\otimes w}\otimes V^{\otimes v}\otimes V^{\otimes b})$

• $m\times n:=m+n$, $\mathbb{1}:=0$, $\alpha_{v,w}:=id_0\times\alpha_{v,w}\times id_0$, $\beta_{u,v,w}=\beta'_{u,v,w}=id$, $u_v=id$, $V^{\otimes(n+m)}\underset{\tau}{\cong}V^{\otimes n}\otimes V^{\otimes m}$

Prop: $T \longrightarrow R\text{-Proj}$ graded commutative unital multiplicative rep'.

(1) $A(\mathcal{D},T)$ is a commutative bialgebra

(2) $\mathcal{C}(\mathcal{D},T) = A(\mathcal{D},T)\text{-Comod}$ is a (associative, commutative & unital) $\otimes$-category and $\mathcal{C}(\mathcal{D},T) \longrightarrow R\text{-Mod}$ is a $\otimes$-functor

(3) $\mathcal{C}(\mathcal{D},T) \simeq \mathrm{Rep}_R(\mathrm{Spec}\, A(\mathcal{D},T))$

flat, unital monoid scheme $/R$

Proof: Clearly (1) $\Rightarrow$ (3) $\Rightarrow$ (2); for simplicity: say $\mathcal{D}$ is finite

Set $\mu^*: \mathrm{End}(T) \longrightarrow \mathrm{End}(T) \otimes \mathrm{End}(T)$, $(e_p)_{p\in\mathcal{D}} \longmapsto (e_{p\times p'})_{pp'\in\mathcal{D}}$

$$\mathrm{End}(T) \subseteq \prod_{p\in\mathcal{D}} \mathrm{End}(T_p), \qquad \mathrm{End}(T)\otimes\mathrm{End}(T) \subseteq \prod_{pp'\in\mathcal{D}} \underbrace{\mathrm{End}(T_p \otimes T_{p'})}_{\overset{\tau}{\cong} \mathrm{End}(T(p\times p'))}$$

$\overset{(-)^*}{\rightsquigarrow}$ $\mu: A(\mathcal{D},T) \otimes A(\mathcal{D},T) \longrightarrow A(\mathcal{D},T)$

Can check: this defines a (assoc., com., unital) ring structure on $A(\mathcal{D},T)$. $\square$

Ex: $A(\mathbb{N}_0, V) = \varinjlim_n \mathrm{End}(V|_{\{0,\dots,n\}})^\vee \overset{\text{only loops}}{=} \bigoplus_n \mathrm{End}(V|_{\{n\}})^\vee = \mathrm{Sym}^\bullet(\mathrm{End}_R(V)^\vee)$

- Usual algebra structure
- Co-algebra str. induced by composition of endomorphisms on $\mathrm{End}(V|_{\{n\}})$

In particular, $\mathrm{Spec}(A(\mathbb{N}_0, T)) = (\mathrm{End}(V), \circ)$ ← seen as an alg. monoid over R with multiplication

Problem: $H^*(X,\mathbb{Z}) \notin \mathbb{Z}\text{-Proj}$ in general

Def: $p \in \mathcal{D}$ is **T-projective** if $T(p)$ is proj.; write $p \in \mathcal{D}^{\mathrm{proj}}$

We will show that in our set-up, $\mathcal{C}(\mathcal{D},T) \simeq \mathcal{C}(\mathcal{D}^{\mathrm{proj}},T)$, inducing the $\otimes$-structure

§2 Nori's Rigidity Criterion

Notation: • $\mathcal{C}$ R-linear abelian $\otimes$-category
• $T: \mathcal{C} \longrightarrow R\text{-Mod}$ faithful exact unital $\otimes$-functor

Think: $\mathcal{C} = \mathcal{C}(D, T)$

Def.: Let $S \subseteq \mathcal{C}$ be a set of T-projective objects. We say $\mathcal{C}$ is **generated by S (relative to T) as abelian $\otimes$-category** if

$$\mathcal{C}(\langle S \rangle^{\otimes, psc\otimes}, T) \hookrightarrow \mathcal{C}(\mathcal{C}, T) \simeq \mathcal{C}$$

is an equivalence of categories

Ex: $T: D \longrightarrow R\text{-Proj}$ with graded commutative product structure
$\Rightarrow \mathcal{C}(D, T)$ is generated as abelian $\otimes$-category by $\{T_p \mid p \in D\} =: S$

Proof: • S is closed under $\otimes$ and $1 \in S \Rightarrow \langle S \rangle^{\otimes, psc\otimes} = \langle S \rangle^{psc\otimes}$
• $(D \to \langle S \rangle^{psc\otimes} \longrightarrow \mathcal{C}(D,T)) \rightsquigarrow (\mathcal{C}(D,T) \to \mathcal{C}(\langle S \rangle^{psc\otimes}, T) \to \mathcal{C}(\mathcal{C}(D,T), T) \simeq \mathcal{C}(D,T))$
$\Rightarrow \mathcal{C}(\langle S \rangle^{psc\otimes}, T) \to \mathcal{C}(D,T)$ is full + ess. surj; faithful: everything sits faithfully in Mod_R $\lrcorner$

Def.: $\mathcal{C}$ is **rigid** if every $V \in \mathcal{C}^{proj}$ admits a **strong dual $V^\vee$**, i.e. there exist iso's
$\mathrm{Hom}(- \otimes V, -) \cong \mathrm{Hom}(-, V^\vee \otimes -)$, $\mathrm{Hom}(-, V \otimes -) \cong \mathrm{Hom}(- \otimes V^\vee, -)$

Thm: (Nori) Assume
- $\mathcal{C}$ R-linear abelian $\otimes$-category, $T: \mathcal{C} \to R\text{-Mod}$ faithful, exact, R-linear $\otimes$-functor
- $\mathcal{C}$ is gen. as abelian $\otimes$-category by $S \subseteq \mathcal{C}^{proj}$
- $\forall V \in S$: $\exists W \in \mathcal{C}^{proj}$ and $q: V \otimes W \longrightarrow 1$ s.t. $T(q): T(V) \otimes T(W) \longrightarrow R$ is a perfect pairing of proj. R-modules

Then (1) $\mathcal{C}$ is rigid
(2) $G := \mathrm{Spec}(A(\langle S \rangle^{\otimes, psc\otimes}, T))$ is a flat (pro-linear algebraic) group scheme
(3) $\mathcal{C} = \mathrm{Rep}\, G$

Rmk: WLOG: $W = V$ (replace S by $S' := \{V \oplus W \mid V \in S\}$)

<u>Proof</u>: * Clearly, (2)+(3) $\Rightarrow$ (1)

* We already know that

$$\underset{\substack{\uparrow \\ S \text{ generates } \mathcal{C}}}{\mathcal{C} \simeq} \mathcal{C}(\langle S\rangle^{\otimes,\mathrm{psd}}, T) \underset{\substack{\uparrow \\ \S 1}}{\simeq} \mathrm{Rep}\, M \quad \text{with} \quad M := \mathrm{Spec}(A(\langle S\rangle^{\otimes,\mathrm{psd}}, T))$$

* Note: $A(\langle S\rangle^{\otimes,\mathrm{psd}}, T) \cong \varinjlim_{V \in \langle S\rangle^{\otimes,\mathrm{psd}}} A(\langle V\rangle^{\otimes,\mathrm{psd}}, T)$

$$\Rightarrow \mathrm{Spec}(A(\langle S\rangle^{\otimes,\mathrm{psd}}, T)) \cong \varprojlim_V \mathrm{Spec}(A(\langle V\rangle^{\otimes,\mathrm{psd}}, T)) =: \varprojlim_V G_V$$

<u>Remains to show:</u> G_V is a linear algebraic group $\forall V \in S$

$\rightsquigarrow$ The map of diagrams

$\mathbb{N}_0 \longrightarrow \langle V\rangle^{\otimes,\mathrm{psd}}$, $n \longmapsto V^{\otimes n}$, $id_a \times \alpha_{v,w} \times id_b \longmapsto (V^{\otimes a} \otimes V^{\otimes v} \otimes V^{\otimes w} \otimes V^{\otimes b} \xrightarrow{\sim} V^{\otimes a} \otimes V^{\otimes w} \otimes V^{\otimes v} \otimes V^{\otimes b})$

induces injections

$$\mathrm{End}(T|_{\langle 1, V, V^{\otimes 2}, \dots, V^{\otimes n}\rangle^{\mathrm{psd}}}) \hookrightarrow \mathrm{End}(TV|_{\{1,\dots,n\}}) = \prod_{k=0}^{n} \mathrm{Sym}^k(\mathrm{End}_R(TV))$$

subalgebra of endomorphisms commuting with all hom's in $\langle 1, V, \dots, V^{\otimes n}\rangle^{\mathrm{psd}}$ e.g. q $\subseteq$ $\prod_{k=0}^{n} \mathrm{End}_R(TV^{\otimes k})$ $\cong$

$\rightsquigarrow$ Get $A(\mathbb{N}_0, TV) \twoheadrightarrow A(\langle V\rangle^{\otimes,\mathrm{psd}}, T)$ surjective

$\rightsquigarrow$ $G_V = \mathrm{Spec}(A(\langle V\rangle^{\otimes,\mathrm{psd}}, T)) \hookrightarrow (\mathrm{End}(V), \cdot)$ closed immersion

$\searrow$ $O(T_q) \subseteq$ $\leftsquigarrow$ lin. alg. group

isometries w.r.t. q

<u>Fact</u>: M an alg. monoid, G a lin. alg. group, $M \hookrightarrow G$ closed immersion

$\Rightarrow M$ is an alg. group

<u>Proof</u>: Choose $S \in \mathrm{Alg}_R$, $g \in M(S)$; to show: $g^{-1} \in M(S)$

WLOG: $S \in \mathrm{Alg}_R^{\text{fin.gen}} \Rightarrow S$ is Noetherian

Consider $G_S \supseteq M_S \supseteq gM_S \supseteq g^2 M_S \supseteq \dots$

$\xrightarrow{\text{Noetherianess}} \exists m \geq 1$: $g^m M_S = g^{m+1} M_S \overset{g^{-(m+1)}}{\Rightarrow} M_S = g^{-1} M_S \Rightarrow g^{-1} \in M(S)$ $\square$