§ 1-Motives.

· <u>Last time</u>: $k \subseteq \mathbb{C}$.

$$\begin{tikzcd} \mathrm{Pairs} \arrow[r, bend left, "H^\bullet_{sing}"] \arrow[r, bend right, "H^\bullet_{dR}"'] & k\text{-v.s.} \end{tikzcd}$$

we constructed a scheme $X_{12} = \operatorname{Spec} A_{12}$ s.t. $X_{12}(S) = \operatorname{Hom}(H^\bullet_{sing} \otimes S, H^\bullet_{dR} \otimes S)$.
Since we have a comparison isom. $\mathrm{per}: H^\bullet_{sing} \otimes \mathbb{C} \xrightarrow{\sim} H^\bullet_{dR} \otimes \mathbb{C}$, this gives a point in $X_{12}(\mathbb{C})$, and thus a morphism

$$ev: A_{12} \longrightarrow \mathbb{C}.$$

We also showed that A_{12} is the algebra of <u>formal periods</u>, given by symbols $(\underline{X}, \underline{Y}, \omega, \gamma)$, modulo the relations given by:

in Pairs ! in $H^{sing}_i(X,Y)$

in $H^j_{dR}(X,Y)$

- ·) linearity in ω, γ.
- ·) pullback along $f: \underline{X} \to \underline{X}'$.
- ·) Stokes' theorem.

· <u>Period conjecture</u>: the map ev is injective.

⌜ $ev(A_{12})$ are the <u>periods</u>. If ev is injective, then it would mean that all the relations between periods come from these relations on A_{12}. ⌟

§1. Commutative algebraic groups and Ext^1.

· From now on, set $k = \overline{\mathbb{Q}}$. Let $\mathcal{G}$ be the category of commutative connected algebraic groups$/k$ (note that they are automatically smooth).

→ comm., geom. reduced, proper k-gp. scheme

· Examples: $\mathbb{G}_a, \mathbb{G}_m, A \in \mathcal{G}$, for A any AV/k.

· Def: $G \in \mathcal{G}$ is a vector group if it is isom. to a power of $\mathbb{G}_a$, and it is a torus if it is isom. to a power of $\mathbb{G}_m$.

· Note that $\mathcal{G}$ is not abelian, since the kernel of a morphism is not necessarily connected. For example, if E/k is an elliptic curve, the kernel of mult by an integer $n>1$ is

$$E[n] \cong \underbrace{\left(\mathbb{Z}/n\mathbb{Z}\right)^2}$$

↳ constant gp. scheme$/k$, not connected

· Def: the isogeny category of $\mathcal{G}$ is the category $\mathcal{G}_{\mathbb{Q}}$ defined by:

·) Objects: same as $\mathcal{G}$.

·) Morphisms: $\mathrm{Hom}_{\mathcal{G}_{\mathbb{Q}}}(G,H) := \mathrm{Hom}_{\mathcal{G}}(G,H) \otimes_{\mathbb{Z}} \mathbb{Q}$.

· Remark: $\mathcal{G}_{\mathbb{Q}}$ is an abelian category.

· Another important fact is that there are no non-trivial morphisms between vector groups, tori, and abelian varieties.

· <u>Thm.</u> (Structure theorem). Let $G \in \mathcal{S}$. Then, there is a canonical s.e.s.

$$0 \to L \to G \to A \to 0,$$

L is a subgp. of GL_n

where A is an ab. var. and L is a linear comm. comm. alg. gp. Moreover, there is a canonical split short exact sequence

$$0 \to U \to L \to T \to 0,$$

where U is a vector group and T is a torus.

· <u>Corollary</u>: an object of $\mathcal{S}$ is simple if and only if it is isomorphic to $\mathbb{G}_a$, $\mathbb{G}_m$, or a simple abelian variety.

· <u>Def</u>: $G \in \mathcal{S}$ is a <u>semiabelian variety</u> if it is an extension of an abelian variety by a torus (equivalently, its vector part is trivial).

For any $G \in \mathcal{S}$, we can find a canonical s.e.s.

$$0 \to U \to G \to G^{sa} \to 0$$

with U a vector group and G^{sa} semi-abelian.

· In order to introduce a concept of duals for tori and vector groups, note that we have the following functors:

$$\begin{array}{ccc} (\text{vector groups}) & \longrightarrow & (\text{fin. dim. } k\text{-v.s}) \\ U & \stackrel{F_1}{\longmapsto} & \mathrm{Hom}_{\mathcal{S}}(U, \mathbb{G}_a) \\ F_2(W) & \stackrel{F_2}{\longleftarrow\!\!\!\shortmid} & W. \end{array}$$

given on points by $F_2(W)(T) := \mathrm{Hom}(W, \mathbb{G}_a(T))$.

$$\begin{array}{ccc} (\text{tori}) & \longrightarrow & (\text{free ab. gps. of finite rank}) \\ T & \longmapsto & X(T) := \mathrm{Hom}_S(T, \mathbb{G}_m) \\ \underbrace{\mathbb{G}_m(\Gamma)}_{\text{given on points by } \mathbb{G}_m(\Gamma)(S) := \mathrm{Hom}(\Gamma, \mathbb{G}_m(S))} & \longleftarrow\!\shortmid & \Gamma . \end{array}$$

· <u>Def:</u>

(1) Let V be a vector group. Its <u>dual</u> is given as $V^\vee := F_2(\underbrace{F_1(V)^\vee}_{\text{dual as } k\text{-v.s.}})$.

(2) Let A be an ab.var. Its <u>dual ab.var.</u> is $A^\vee := \overbrace{\mathrm{Pic}^0}^{\text{rel. Picard funct.}}(A)$.

(3) Let T be a torus. Then $X(T)$ is the <u>character group of</u> T.

(4) Let Γ be a free ab. gp. of finite rank. Its <u>dual torus</u> is $\mathbb{G}_m(\Gamma)$.

· <u>Prop</u>. Let V be a vector group, A an ab.var. and T a torus. Then are canonical isomorphisms:

$$V^{\vee,\vee} \cong V, \quad A^{\vee,\vee} \cong A, \quad \mathbb{G}_m(X(T)) \cong T.$$

<u>Proof</u>

·) V: linear algebra and using that F_1, F_2 give an equiv. of categories.

·) A: using the Poincaré bundle.

·) T: the functors $X(\cdot)$ and $\mathbb{G}_m(\cdot)$ are clearly adjoint:

$$\mathrm{Hom}(T, \mathbb{G}_m(\Gamma)) \overset{\text{by def.}}{=} \mathrm{Hom}(\Gamma, \mathbb{G}_m(T)) \overset{\text{by def. of } X(T)}{=} \mathrm{Hom}(\Gamma, X(T)),$$

and thus we get a unit $T \to \mathbb{G}_m(X(T))$.

By naturality it is enough to check that this map is an iso. for $T = \mathbb{G}_m$, but one sees that $X(\mathbb{G}_m) = \mathbb{Z}$, $\mathbb{G}_m(\mathbb{Z}) = \mathbb{G}_m$, it is immediate. $\boxtimes$

· Now, we discuss extensions on $\mathcal{G}$.

· <u>Def:</u> The functor $\mathrm{Ext}^1 : \mathcal{G} \times \mathcal{G} \longrightarrow (\mathrm{AbGp})$ is given for any $A, B \in \mathcal{G}$ as

$$\mathrm{Ext}^1(A,B) = R^1(\mathrm{Hom}_{\mathcal{G}}(A,-))(B) = R^1(\mathrm{Hom}_{\mathcal{G}}(-,B))(A).$$

· One can understand the elements of $\mathrm{Ext}^1(A,B)$ as <u>extensions</u>: triples (C, ι, π) with $C \in \mathcal{G}$, $\iota \in \mathrm{Hom}(B,C)$, $\pi \in \mathrm{Hom}(C,A)$ s.t.

$$0 \longrightarrow B \xrightarrow{\iota} C \xrightarrow{\pi} A \longrightarrow 0$$

is exact. We say C is an <u>extension</u> of A by B. Furthermore, two extensions (C, ι, π) and (C', ι', π') of A by B are <u>equivalent</u> if there exists a map $\gamma : C \to C'$ s.t.

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "\iota"] \arrow[d, equal] & C \arrow[r, "\pi"] \arrow[d, "\gamma"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "\iota'"'] & C' \arrow[r, "\pi'"'] & A \arrow[r] & 0
\end{tikzcd}$$

$\rightsquigarrow$ γ is an iso. by the five lemma

commutes. One can show that for all A, B there are (functorial) isoms. of abelian groups

$$\mathrm{Ext}^1(A,B) \xrightarrow{\sim} \{(C, \iota, \pi) \text{ ext. of } A \text{ by } B\}/\sim .$$

We write $[C]$ for the equiv. class of (C, ι, π).

Moreover, the zero element corresponds to $[(B \times A, i_B, p_A)]$.

$\mapsto p_A : B \times A \to A$

$i_B : B \hookrightarrow B \times A$

· Another crucial property is that Ext^1 is additive in both variables:

$$\mathrm{Ext}^1(A_1 \times A_2, B) = \mathrm{Ext}^1(A_1, B) \times \mathrm{Ext}^1(A_2, B),$$
$$\mathrm{Ext}^1(A, B_1 \times B_2) = \mathrm{Ext}^1(A, B_1) \times \mathrm{Ext}^1(A, B_2).$$

· Prop. Let A be an ab. var. and L, L' linear connected comm. alg. gps. Then:

(1) $\mathrm{Ext}^1(A, \mathbb{G}_a) = H^1(A, \mathcal{O}_A)$.

(2) $\mathrm{Ext}^1(A, \mathbb{G}_m) = A^\vee(k) = \mathrm{Pic}^0(A)(k) \subseteq \mathrm{Pic}(A)(k) = H^1(A, \mathcal{O}_A^\times)$.

(3) $\mathrm{Ext}^1(L, L') = 0$.

Proof |

For (1) and (2), one uses the fact that if L is a linear group, then

$$\mathrm{Ext}^1(G, L) = \underbrace{\mathrm{Ext}^1(G, L)_*}$$

extensions which admit
a rational section

When $G = A$ is an abelian variety, there is a map

$$\mathrm{Ext}^1(A, L)_* \longrightarrow H^1(A, B_A),$$

when B_A is the sheaf of germs of regular maps $A \to B$. This map is injective, and:

·) If $L = \mathbb{G}_a$, then $B_A = \mathcal{O}_A$ and this is an isomorphism.

·) If $L = \mathbb{G}_m$, then $B_A = \mathcal{O}_A^\times$ and the map has as image the line bundles of A which are algebraically equivalent to 0 (i.e. $\mathrm{Pic}^0(A)(k)$).

For (3), the structure theorem shows that all linear connected commutative algebraic groups are split, and thus there are no non-trivial extensions.

$\boxtimes$

· <u>Remark</u>: The identification in (2) is induced by the <u>Poincaré bundle</u> $\mathcal{P}$ on $A \times A^\vee$, for a point $x \in A^\vee(k)$, the pullback $(id_A, x)^* \mathcal{P}$ to A is a line bundle of deg. 0. If one removes the zero section, it is a $\mathbb{G}_m$-bundle, and in fact a semi-ab. var. in $\mathrm{Ext}^1(A, \mathbb{G}_m)$.

· <u>Corollary 1</u>: Let G be a semi-ab. var. with ab. part A, and $\mathcal{V}$ a vector group. Then the natural map

$$\mathrm{Ext}^1(G, \mathcal{V}) \cong \mathrm{Ext}^1(A, \mathcal{V})$$

is an isom. In particular, $\mathrm{Ext}^1(G, \mathcal{V})$ is finite dimensional.

<u>Proof</u> | By definition, we can write a s.e.s.

$$0 \to T \to G \to A \to 0$$

with T a torus. If we apply the long exact seq. for $\mathrm{Hom}(-, \mathcal{V})$, we get:

$$\cdots \to \underbrace{\mathrm{Hom}(T, \mathcal{V})}_{\substack{=0\text{, since there}\\ \text{are no maps between}\\ \text{tori and vector gps.}}} \to \mathrm{Ext}^1(A, \mathcal{V}) \to \mathrm{Ext}^1(G, \mathcal{V}) \to \underbrace{\mathrm{Ext}^1(T, \mathcal{V})}_{=0 \text{ by the prop.}} \to \cdots$$

For finite dimensionality, we see that, if $\dim_k(\mathcal{V}) =: s$,

$$\mathrm{Ext}^1(A, \mathcal{V}) \cong \mathrm{Ext}^1(A, \mathbb{G}_a)^s \cong H^1(A, \mathcal{O}_A)^s .$$

$\boxtimes$

· Consider again a semi-abelian variety G and a s.e.s.

$$0 \to T \to G \xrightarrow{\pi} A \to 0,$$

with T a torus and A an abelian variety. Applying the l.e.s. associated to $\mathrm{Hom}(-, \mathbb{G}_m)$ we get

$$\cdots \to \underbrace{\mathrm{Hom}(T, \mathbb{G}_m)}_{X(T)} \xrightarrow{d_G} \underbrace{\mathrm{Ext}^1(A, \mathbb{G}_m)}_{A^\vee(k)} \xrightarrow{\pi^*} \mathrm{Ext}^1(G, \mathbb{G}_m) \to \cdots$$

which gives a map $\operatorname{Ext}^1(A, T) \longrightarrow \operatorname{Hom}(X(T), A^\vee(k))$

$$[G] \longmapsto d_G .$$

· Corollary 2. The induced map

$$\operatorname{Ext}^1(A, T) \longrightarrow \operatorname{Hom}(X(T), A^\vee(k))$$

is an isomorphism.

Proof | Since $T \cong \mathbb{G}_m^r$ and both sides are natural in T, we may just consider $T = \mathbb{G}_m$. Then, $X(T) = \mathbb{Z}$ and thus

$$\operatorname{Hom}(\mathbb{Z}, A^\vee(k)) = A^\vee(k) \cong \operatorname{Ext}^1(A, \mathbb{G}_m).$$

$\boxtimes$

· The image of $[G]$ in $\operatorname{Hom}(X(T), A^\vee(k))$ is the classifying map of G.

· We can repeat a similar argument for vectorial extensions. If we have a s.e.s. of the form

$$0 \longrightarrow V \longrightarrow G \xrightarrow{\pi} A \longrightarrow 0$$

with V a vector group and A an abelian variety, then we may take the l.e.s. associated to $\operatorname{Hom}(-, \mathbb{G}_a)$:

$$\cdots \longrightarrow \operatorname{Hom}(V, \mathbb{G}_a) \xrightarrow{d_G} \operatorname{Ext}^1(A, \mathbb{G}_a) \xrightarrow{\pi^*} \operatorname{Ext}^1(G, \mathbb{G}_a) \longrightarrow \cdots$$

And we get a hom. $\operatorname{Ext}^1(A, V) \longrightarrow \operatorname{Hom}(F_1(V), \operatorname{Ext}^1(A, \mathbb{G}_a))$

$$[G] \longmapsto d_G .$$

· Cor. 3. The induced map

$$\operatorname{Ext}^1(A, V) \longrightarrow \operatorname{Hom}(F_1(V), H^1(A, \mathcal{O}_A)).$$

is an isomorphism.

Proof/ As before, we may just prove it for $V = \mathbb{G}_a$. Then, since $F_1(\mathbb{G}_a) \cong k$,

$$\mathrm{Hom}(k, H^1(A, \mathcal{O}_A)) \cong H^1(A, \mathcal{O}_A) \cong \mathrm{Ext}^1(A, \mathbb{G}_a).$$

$\boxtimes$

· Lastly, we may apply Cor. 1. to obtain the following:

· Cor. 4. Let G be semi-ab., V a vector group. Then the natural map

$$\mathrm{Ext}^1(G, V) \longrightarrow \mathrm{Hom}(F_1(V), \mathrm{Ext}^1(G, \mathbb{G}_a))$$

is an isom.

Proof/ By Corollary 1, we may replace

$$\mathrm{Ext}^1(G, V) \cong \mathrm{Ext}^1(A, V) \text{ and}$$

$$\mathrm{Ext}^1(G, \mathbb{G}_a) \cong \mathrm{Ext}^1(A, \mathbb{G}_a),$$

when A is the abelian part of G. Then we apply Cor. 3. $\boxtimes$

§2. Semi-ab. varieties and universal vector extensions.

· By Cor. 2, the datum of a semi-abelian variety G over k is equivalent to the datum of a homom. $X(T) \to A^\vee(k)$.

In fact, this is a functorial construction: if $\alpha: G_1 \to G_2$ is a morphism of semi-ab. vars., then it induces a comm. diagram

$$\begin{tikzcd} X(T_1) \arrow[d, "d_{G_1}"'] & X(T_2) \arrow[l, "X(p)"'] \arrow[d, "d_{G_2}"] \\ A_1^\vee(k) & A_2^\vee(k) \arrow[l, "q^\vee"] \end{tikzcd}$$

where we are using that a morphism of exts. α is of the form:

$$\begin{tikzcd} 0 \arrow[r] & T_1 \arrow[r] \arrow[d, "p"] & G_1 \arrow[r] \arrow[d, "\alpha"] & A_1 \arrow[r] \arrow[d, "q"] & 0 \\ 0 \arrow[r] & T_2 \arrow[r] & G_2 \arrow[r] & A_2 \arrow[r] & 0 \end{tikzcd}$$

· Prop. The assignment

$$G \overset{F}{\longmapsto} [X(T) \to A^\vee(k)]$$

yields an equivalence of categories between

$$\{\text{semi-ab. vars.}/k\} \overset{\sim}{\longleftrightarrow} \left\{ (X, A, \varphi) \,\middle|\, \begin{array}{l} X \text{ free ab. gp of finite rank} \\ A \text{ ab. var}/k \\ \varphi: X \to A^\vee(k) \text{ group hom.} \end{array} \right\}.$$

Morphisms in the latter category are morphisms of complexes

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "G"] & A^\vee(k) \arrow[d] \\ X' \arrow[r, "\varphi'"'] & A'^\vee(k) \end{tikzcd}$$

Proof)

·) F is faithful: let $f: G \to G'$ be s.t. $F(f) = 0$. In particular, if it induces maps $p: T \to T'$, $q: A \to A'$ on the tors and ab.var. part respectively, then $X(p): X(T') \to X(T)$ and $q^\vee: A'^\vee(k) \to A^\vee(k)$ vanish.

$\Rightarrow p, q = 0$.

Then, the composition $G \xrightarrow{f} G' \to A'$ vanishes, as it factors as

$$\begin{tikzcd} G \arrow[r] \arrow[d, "f"'] & A \arrow[d, "q"] \\ G' \arrow[r] & A' \end{tikzcd}$$

$\rightsquigarrow$ zero map

It follows that $f(G) \subseteq T'$. By the same reasoning, since $p=0$, then $f|_T$ vanishes, and thus we get an induced map $\tilde{f}: A \to T'$. But all of these maps are trivial, so $\tilde{f}=0$.

·) F is full: let G_1, G_2 be semi-ab.vars. of the form

$$0 \to T_i \to G_i \to A_i \to 0,$$

and consider a comm. diagram

$$\begin{tikzcd} X(T_1) \arrow[d, "d_{G_1}"'] & X(T_2) \arrow[l, "p'"'] \arrow[d, "d_{G_2}"] \\ A_1^\vee(k) & A_2^\vee(k) \arrow[l, "q'"] \end{tikzcd} \quad (*)$$

We now claim that $d_{G_2 \times_{A_2} A_1} = q' \circ d_{G_2}$. Firstly, $G_2 \times_{A_2} A_1$ gives an element in $\mathrm{Ext}^1(A_1, T_2)$ as:

$$\begin{tikzcd} 0 \arrow[r] & T_2 \arrow[r] \arrow[d, equal] & G_2 \times_{A_2} A_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_1 \arrow[r] \arrow[d, "q'^{,\vee}"] & 0. \\ 0 \arrow[r] & T_2 \arrow[r] & G_2 \arrow[r] & A_2 \arrow[r] & 0. \end{tikzcd}$$

The corresponding comm. diagram is of the form

$$\begin{tikzcd} X(T_2) \arrow[r, "d_{G_2 \times_{A_2} A_1}"] & A_1^\vee(k) \\ X(T_2) \arrow[u, equal] \arrow[r, "d_{G_2}"'] \arrow[ur, phantom, "\circlearrowleft"] & A_2^\vee(k) \arrow[u, "q'"'] \end{tikzcd},$$

which shows our claim. Then, $d_{G_2 \times_{A_2} A_1} = q' \circ d_{G_2} = d_{G_1} \circ p'$, and by Corollary 2, one can see that

$$\mathrm{Ext}^1(A_1, T_2) \longrightarrow \mathrm{Hom}(X(T_2), A_1^\vee(k))$$
$$[G_2 \times_{A_1} A_2] \longmapsto d_{G_1} \circ p'$$
$$[G_1 \times T_2 / T_1] \longmapsto d_{G_1} \circ p'$$

pushout of

$$\begin{tikzcd} T_1 \arrow[r] \arrow[d] & T_2 \arrow[d] \\ G_1 \arrow[r] & \cdot \end{tikzcd}$$

which means that $G_2 \times_{A_2} A_1 \cong G_1 \times T_2 / T_1$. We finally obtain a map $G_1 \to G_2$ via

$$
\begin{tikzcd}
0 \arrow[r] & T_1 \arrow[r] \arrow[d, "\mathbb{G}_m(p')"'] & G_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & A_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T_2 \arrow[r] \arrow[dd, equal] & G_1 \times T_2 / T_1 \arrow[r] \arrow[d, phantom, "\wr"] & A_1 \arrow[r] \arrow[dd, "\varphi'^{\vee}"] & 0 \\
 & & G_2 \times_{A_2} A_1 \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & \\
0 \arrow[r] & T_2 \arrow[r] & G_2 \arrow[r] & A_2 \arrow[r] & 0
\end{tikzcd}
$$

and this composition gives back $\circledast$.

• F is full on objects: let (X, A, φ) be a triple as before, and let $e_1, \dots, e_n$ be a basis of X. Then, each $\varphi(e_i) \in A^\vee(k) \cong \mathrm{Ext}^1(A, \mathbb{G}_m)$ defines extensions

$$0 \to \mathbb{G}_m \to G_i \to A \to 0,$$

and we define

$$G := G_1 \times_A G_2 \times_A \cdots \times_A G_n.$$

By construction, $F(G) = (X, A, \varphi)$. $\boxtimes$

• **Remark:**

(i) $X(T) \to A^\vee(k)$ is zero iff $G \cong A \times T$.

(ii) Given maps $s_i : X(T_i) \to A^\vee(k)$ $i = 1,2$ corresponding to semi-ab. var. G_1, G_2, we can construct the sum $s_1 \oplus s_2 : X(T_1) \oplus X(T_2) \to A^\vee(k)$, which is associated to the semi-ab. var. G given as the pullback

$$
\begin{tikzcd}
G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G_1 \times G_2 \arrow[d] \\
A \arrow[r, "\Delta"'] & A \times A,
\end{tikzcd}
$$

and its torus part is $T_1 \times T_2$. If $s_1 = 0$, then the maps

$$G \longrightarrow G_1 \cong A \times T_1 \longrightarrow T_1,$$
$$G \longrightarrow G_2,$$
give an iso. $G \cong T_1 \times G_2$.

· <u>Def</u>: the category of <u>semi-abelian vars. up to isogeny</u> has the same objects as the cat. of semi-ab. vars. but with morphisms tensored by $\mathbb{Q}$.

· <u>Remark</u>: By the previous prop., the cat. of semi-ab. vars. up to isogeny is equivalent to the category with objects

$$\left\{ (X_{\mathbb{Q}}, A, \varphi) \;\middle|\; \begin{array}{l} X_{\mathbb{Q}}: \text{fin. dim. } \mathbb{Q}\text{-v.s.} \\ A/k: \text{ab. var.} \\ \varphi: X_{\mathbb{Q}} \longrightarrow A^{\vee}(k)_{\mathbb{Q}} \text{ a } \mathbb{Q}\text{-lin. map} \end{array} \right\}$$

and morphisms comm. diagrams as before. We often write objects as $X \longrightarrow A^{\vee}(k)_{\mathbb{Q}}$, with X a free ab. gp. of finite rank.

· <u>Cor</u>: Let G be a semi-ab. var., T a torus and $G \to T$ a surj. morphism of algebraic groups with kernel G'. Then $G \cong T \times G'$ up to isogeny.

<u>Proof</u> Since tori are semi-simple, there exists an inj. hom. $T \to G$ with image in the torus part of G. By the univ. prop. of the direct product, we get an iso. $G \cong T \times G'$. $\boxtimes$

· In the remainder of the section we discuss an analogue to the previous equivalence of categories in the case of vectorial extensions. Note that we consider instead the dual map.

· Prop. The assignment

dual map of $F_1(T) \to Ext^1(G,\mathbb{G}_a)^\vee$

$$G' \longmapsto [Ext^1(G,\mathbb{G}_a)^\vee \longrightarrow F_1(T)^\vee]$$

yields an equiv. between the category of vector extensions of semi-ab. varieties over k (here, G' is a vector extension of G) and the category with objects being triples (W, A, φ) satisfying:

·) W is a fin. dim. k-v.s.

·) A is an ab. var. $/k$.

·) $\varphi : Ext^1(A,\mathbb{G}_a)^\vee \longrightarrow W$ is a k-linear map.

Its morphisms are morphisms of complexes $[Ext^1(A,\mathbb{G}_a)^\vee \to W]$.

Proof / Same as before, but using Cor. 3.

Recall that if G is a semi-ab. var. with ab. part A,

$$Ext^1(G,\mathbb{G}_a) = Ext^1(A,\mathbb{G}_a).$$ ⊠

· Since the vector space $Ext^1(G,\mathbb{G}_a)$ is itself finite dimensional, we can give the following definition.

· Def: Let G be a semi-abelian variety. The universal vector extension of G is the extension

$$0 \longrightarrow F_2(Ext^1(G,\mathbb{G}_a)) \longrightarrow G^{\natural} \longrightarrow G \longrightarrow 0$$

associated to $[id: Ext^1(G,\mathbb{G}_a)^\vee \longrightarrow Ext^1(G,\mathbb{G}_a)^\vee]$ via the previous prop.

Recall that our isom. was

$$Ext^1(G,T) \cong Hom(F_1(T), Ext^1(G,\mathbb{G}_a)),$$ so

since $F_1(T)^\vee = Ext^1(G,\mathbb{G}_a)^\vee \Rightarrow T \cong F_2(F_1(T)) = F_2(Ext^1(G,\mathbb{G}_a))$.

· From this definition, it follows that $G^{\natural}$ satisfies the following universal property:

· **Prop.** Let G be a semi-ab. var. For any vector extension of G,

$$0 \to V \to G' \to G \to 0,$$

there is a unique morphism $G^\natural \to G'$ compatible with the projection to G.

Proof | Under the prev. equiv. of categories, G' corresponds to the tuple $(F_1(V)^\vee, G, \varphi : \mathrm{Ext}^1(G, \mathbb{G}_a)^\vee \to F_1(V)^\vee)$, and $G^\natural$ corresponds to $(\mathrm{Ext}^1(G,\mathbb{G}_a)^\vee, A, \mathrm{id} : \mathrm{Ext}^1(G,\mathbb{G}_a)^\vee \to \mathrm{Ext}^1(G,\mathbb{G}_a)^\vee)$.

Again via the correspondence, a morphism $G' \to G^\natural$ is associated with a linear map $\psi : \mathrm{Ext}^1(G,\mathbb{G}_a)^\vee \to F_1(V)^\vee$ s.t. the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Ext}^1(G,\mathbb{G}_a)^\vee \arrow[r, "\varphi"] \arrow[d, equal] & F_1(V)^\vee \\
\mathrm{Ext}^1(G,\mathbb{G}_a)^\vee \arrow[r, equal] & \mathrm{Ext}^1(G,\mathbb{G}_a)^\vee \arrow[u, "\psi"']
\end{tikzcd},$$

and thus the only option for φ is ψ. ⊠

· **Remark:** Let G be a semi-ab.var., A its abelian part. We can write explicitly

$$0 \to F_2(H^1(A, \mathcal{O}_A)) \to A^\natural \to A \to 0.$$

Moreover, by Cor. 1, $\mathrm{Ext}^1(G,\mathbb{G}_a) \cong \mathrm{Ext}^1(A, \mathbb{G}_a)$, implying that

$$G^\natural \cong A^\natural \times_A G.$$

· **Cor:** The univ. vector ext. $G^\natural$ of a semi-ab. variety G also satisfies the univ. property of a vector ext. in the isogeny cat. $\mathcal{S}a$.

Proof: $\mathrm{Hom}_k(G,T)$ and $\mathrm{Ext}^1(G,T)$ are already k-v.s., so they do not change when one replaces G by $G_{\mathbb{Q}}$. $\boxtimes$

§3. 1-Motives.

· This discussion motivates the following definition.

· **Def:** A 1-motive $M = [L \to G]$ over k is the datum given by a semi-ab. group G/k, a free abelian group L of finite rank, and a group hom. $L \to G(k)$. Morphisms of 1-motives are morphisms of complexes $L \to G$.
We define the category of iso-1-motives $1\text{-}\mathrm{Mot}_k$ as having the same objects but morphisms tensored by $\mathbb{Q}$.

· Remark: we can identify semi-ab. varieties with 1-motives via

$$G \mapsto [0 \to G].$$

· As it turns out, these 1-motives are related back to Nori motives.

· **Def:** for $n \geq 0$, let $d_n MM_{Nori} \subseteq MM_{Nori}$ be the thick abelian subcategory (i.e. full and closed under extensions and subquotients) generated by the objects $H^i_{Nori}(X, D)$ with X a k-variety, $D \subseteq X$ a closed subvariety and $i \leq n$.

· Thm. (Ayoub, Barbieri-Viale).

(1) The inclusion

$$d_1 MM_{Nori} \longrightarrow MM_{Nori}$$

has a left adjoint.

(2) There is an anti-equivalence of categories

$$1\text{-}Mot_k \longrightarrow d_1 MM_{Nori}.$$

(3) The abelian category $d_1 MM_{Nori}$ is equal to the diagram category $\mathcal{C}(D_1, H^*_{sing})$, where D_1 is the diagram with vertices given by pairs (C, D), where C is a smooth affine curve and D a collection of points in C, and with edges given by morphisms of pairs.