Babyseminar: condensed mathematics

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The category of topological abelian groups is, despite the name of its objects, not abelian. The issue here is that, for example, the map $\mathbf{R}^\delta \to \mathbf{R}$, where $\mathbf{R}$ is the topological group of real numbers and δ means with the discrete topology, has zero kernel and cokernel while not being an isomorphism. An abelian category of locally compact abelian groups (LCA) has been an ellusive object in quite some while, with many authors suggesting some ad-hoc definitions of $D^b(LCA)$ and continuous group cohomologies.

In 2019, Scholze and Claussen proposed an ingenious solution based on the theory of pro-étale cohomology of rigid-spaces and schemes (which already was known to capture the right continuous cohomology groups with an intuitive definition).

The idea is to consider profinite sets as test spaces. If X is a compact (separated) space then there are enough maps $S \to X$ with S profinite as will be made precise in the seminar. As a concrete and instructive example the "binary expansion map"

$$\prod_{\mathbf{N}}\{0,1\} \twoheadrightarrow [0,1], \quad (a_n) \mapsto 0,a_1a_2\cdots \in \mathbf{R}$$

presents the compact line as a quotient of the Cantor set.

Definition. Let ProfSet be the category of profinite sets. A *condensed set* is a functor

$$X : \mathrm{ProfSet}^\circ \to \mathsf{Set}$$

which satisfies a sheaf theoretic condition. A condensed abelian group, ring, module is defined by changing the codomain.

Using this definition one shows that each topological space X defines a condensed set (also denoted X) by

$$X(S) = \mathrm{Hom}_{\mathsf{Top}}(S, X)$$

and that, in fact, (compactly generated[1]) topological spaces embbed into condensed sets this way. (Note that the underlying set of X can be recovered as $X(*)$).

We therefore obtain a commutative diagram

$$\begin{tikzcd} \mathsf{Top} \arrow[r, hook] & \{\text{condensed sets}\} \\ \{\text{locally compact abelian groups}\} \arrow[u] \arrow[r, hook] & \{\text{condensed abelian groups}\} \arrow[u] \end{tikzcd}$$

and, in fact, the category of condesed abelian groups has fantastic properties: It is abelian, and in fact even (AB6) and (AB4*). In particular there are enough projectives. There is also an internal notion of cohomology which one shows that often agrees with the usual ad-hoc definition of continuous cohomology using continuous cochains.

Our goal in this seminar would be to follow somewhat closely Scholze's first course on condensed mathematics (and other variations eg. the research seminar on the topic given some terms ago). It sketches the basics of the theory with one application in mind: the existence of a six-functor formalism on schemes which extends quasi-coherent sheaves.

Recall that for each scheme X one defines a derived category $D_{\mathrm{qc}}(X)$ of quasi-coherent sheaves on X. This can be seen as a category of coefficients for the coherent cohomology of X, but, differently of what one might expect from other such formalisms, no functor $f_!$ adjoint to $f^!$ can be defined.

In fact if $X = \operatorname{Spec} A$ and $U = \operatorname{Spec} A[1/f]$ and $i\colon U \hookrightarrow X$ is the canonical open immersion then the functor $i^! = i^*$ cannot have a left adjoint as

$$M \mapsto M[1/f]$$

does not preserve infinite products. Nonetheless, we will be able to show the following theorem.

Theorem. There exists a category $D_{\blacksquare}(X) \supset D_{\mathrm{qc}}(X)$ for every scheme which carries a six functor formalism which restricts to the usual one for the five out of six defined functors in that case.

In particular, this will yield a new proof of coherent duality (eg. Serre duality) using condensed maths.

[1] Remember from your algebraic topology I days that this is a really mild condition to impose. Any first countable space is compactly generated!