Proposal for Baby-Seminar: p-adic uniformization of Shimura curves

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1 Main objects of interest

The main objects of interest would be some Shimura curve representing some moduli problem involving abelian schemes. Some of them admit p-adic uniformization (i.e. can be viewed as a quotient of the p-adic upper half plane) and we want to see in which case this happens and to understand the proof.

Consider the following moduli problem (say over $\mathbb{Z}[1/N]$). We fix a quaternion algebra $B/\mathbb{Q}$ indefinite (i.e. $B \otimes \mathbb{R} \cong M_2(\mathbb{R})$) and ramified only at the primes dividing N^-, a divisor of $N = N^+N^-$. We want to classify, for every $\mathbb{Z}[1/N]$-scheme S, triples (A, i, C) with

1. A an abelian S-scheme of relative dimension 2;
2. $i : \mathcal{O}_B \to \mathrm{End}_S(A)$ a ring homomorphism, that is an action of a the maximal order $\mathcal{O}_B$ of B (they are all conjugate) on A;
3. C a finite flat subgroup scheme of $A[N^+]$ of order N^{+2} and stable and locally cyclic for the action of $\mathcal{O}_B$.

Shortly: abelian surfaces with quaternionic multiplication and some level N^+-structure.

This moduli problem is coarsely represented by a Shimura curve $X = X_{\hat{\mathcal{O}}^\times_{B,N^+}}$. The complex points can be viewed as a quotient of the complex upper half plane $\mathcal{H}$ (or, as below, of $\mathcal{H}^\pm := \mathbb{C} - \mathbb{R} = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$):

$$X(\mathbb{C}) = B^\times \setminus (\mathcal{H}^\pm \times (B \otimes \mathbb{A}_f)^\times)/(\mathcal{O}_{B,N^+} \otimes \hat{\mathbb{Z}})^\times.$$

When $N^- = 1$, this is the more familiar open modular curve $Y(\Gamma_0(N))$, parametrizing elliptic curves with a subgroup of order N.

In fact you could take even more general open compact subgroups U of $(B \otimes \mathbb{A}_f)^\times$ and you would get a range of interesting Shimura curves X_U that are analogous (in the sense that they represent a similar moduli problem) to the more familiar curves $Y(\Gamma_1(N))$ or $Y(N)$ representing elliptic curves with some structure. In fact X_U is a modular curve in the case when $B = M_2(\mathbb{Q})$ and $U = \widehat{\Gamma_1(N)}, \widehat{\Gamma(N)}$. Moreover, if one takes U small enough, these Shimura curves actually represent the moduli problem (not only coarsely).

One is often interested to define a model also at primes dividing N, which are of course more difficult to study. For example one can define a model $\mathscr{X}_U$ of X_U over $\mathbb{Z}_p$ for $p \mid N$.

The p-adic uniformization helps to understand this model. In particular it gives a way to study the special fiber of $\mathscr{X}$ when p is in the ramification of B, a task that it is not trivial at all (and that it very different form the case of the modular curves, that do not admit p-adic uniformization, because the associated quaternion algebra $B = M_2(\mathbb{Q})$ is nowhere ramified).

2 The Cherednik-Drinfeld p-adic uniformization

We denote with $\mathcal{H}_p$ the p-adic upper half plane, that set theoretically is just $\mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p)$, but in fact carries a structure of a rigid analytic space.

Theorem. *If p divides the discriminant of B (so now $B = M_2(\mathbb{Q})$ is out of the game ...), one can obtain a rigid analytic parametrization over $\mathbb{Q}_p$:*

$$X_U^{an} \cong GL_2(\mathbb{Q}_p) \setminus (\mathcal{H}_p \hat{\otimes} \widehat{\mathbb{Q}_p^{ur}} \times (U^{(p)} \setminus (B_* \otimes \mathbb{A}_f)^\times / B_*^\times))$$

where B_* is the quaternion algebra ramified at the same places of B except that one has to swap p and ∞ (so it is a definite quaternion algebra!). One can also describe X_U^{an} as a quotient of the p-adic upper half plane by a Schottky group, after base changing to some finite extension.

In fact even more is true, and one can obtain also an ***integral version*** *of the above uniformization:*

$$\widehat{\mathscr{X}_U} \cong GL_2(\mathbb{Q}_p) \setminus (\Omega \hat{\otimes} \widehat{\mathbb{Q}_p^{ur}} \times (U^{(p)} \setminus (B_* \otimes \mathbb{A}_f)^\times / B_*^\times))$$

where $\widehat{\mathscr{X}_U}$ denotes the formal completion of $\widehat{\mathscr{X}_U}$ along the special fiber and Ω is a formal scheme over $\mathbb{Z}_p$, whose generic fiber is the p-adic upper half plane $\mathcal{H}_p$ (and $U^{(p)}$ is the prime-to-p part of U).

Probably the most interesting part of the whole story is that one proves the isomorphism in a very meaningful way. Indeed, one shows that also Ω represents a moduli problem (namely of formal p-divisible groups of dimension 2 and height 4, with an action of the maximal order of the unique quaternion division algebra over $\mathbb{Q}_p$ and some rigidification) and one really compares the moduli problem represented by $\mathscr{X}_U$ (if U is small enough) with the one represented by Ω.

It is also interesting to see why the algebra B_* pops up: it turns out that there is only one isogeny class of supersingular two-dimensional abelian varieties over $\overline{\mathbb{F}}_p$ endowed with an action of $\mathcal{O}_B$ and for any of them, say A, it holds that $\mathrm{End}_{\mathcal{O}_B}(A) \otimes \mathbb{Q} \cong B_*$. In fact, one can also show that any such A is isogenous to a product of two supersingular elliptic curves.

3 What you will learn in the seminar

As the title hints, the focus of the seminar is p-adic. However it would be nice to start with explaining the complex picture, that is very satisfying to describe, and it is also nice to compare to the p-adic one. A possible reference that I'm aware of is the last chapter of [Voi21], but I'm sure there are better ones. If you don't like the p-adic part, you can go for the complex one.

In order to fully understand the statement of the theorem, we have to cover some introductory material, which involve several topics that are very interesting in their own sake and very useful to know even if you are not interested in the p-adic uniformization or the Shimura variety business, namely

- Formal schemes
- p-divisible groups
- Cartier/Dieudonné theory for finite group schemes/ p-divisible groups

I do not plan, instead, to cover in detail the rigid analytic side of the story (that is the description as a quotient of the p-adic upper half plane by a Schottky group). This would occupy 0 to 1 talks and at most at the level of a survey.

The central part of the seminar will be then devoted to the proof of the Cherednik-Drinfeld theorem. The main reference is [BC91] (don't fear, there is the English translation, that is also easier to find than the original). Also [DT07] can be useful.

3.1 Some possible topics for the last talks

The topic is indeed very rich and I've have decided to restrict the attention to dimension 1 (i.e. Shimura curves rather that varieties) and defined over $\mathbb{Q}$ (rather that a more general totally real field). Good reasons are that the material is already dense enough and that the main reference, that explains this with enough details, sticks to this case.

Depending on the taste of the audience, several options are possible for the last talks:

- Make a survey of what can be said for curves over totally real fields [BZ95];
- Discuss the generalizations to higher dimension. For this we need to talk of Rapoport-Zink spaces, of which the Ω mentioned above is the easiest instance. Morally, they are the integral version of Drinfeld p-adic symmetric domains, n-dimensional versions of the p-adic upper half plane that set theoretically are $\mathbb{P}^n(\mathbb{C}_p) - \mathbb{Q}_p$-rational hyperplanes. For this a possible reference is [RZ16];
- Turn a bit to the analytic side of the story. Here is a topic that someone could find interesting (I do, but you can also avoid to read further if it looks too long for you): one classical way of studying rational points of elliptic curves $E/\mathbb{Q}$ is to take a Shimura curve parametrization $X_{N^+,N^-} \to E$ (where $N = N^+N^-$ is the conductor of E) and look at Heegner points defined over a quadratic imaginary field K. However this points exist on X_{N^+,N^-} only if the primes dividing N that are inert in K are exactly those dividing N^-. On the other hand the Shimura curve $X = X_{N^+,N^-}$ only exists if N^- is even. If N^- is odd there is no hope of finding such points. In this case p-adic uniformization comes to rescue us because, using the interchanging trick of p with ∞, it gives a p-adic uniformization $\mathcal{H}_p \to E_{\mathbb{C}_p}$. Also, in this case, the invariant differential on E will correspond to some harmonic function on $\mathcal{H}_p$ and this correspond to p-adic measures (quoting [Tei95]: the De Rham cohomology $H^1_{dR}(\mathcal{H}_p, \mathbb{Q}_p)$ is isomorphic to the space of harmonic functions on the corresponding Bruhat-Tits building) and one can construct an L-function and formulate some "non-usual" p-adic BSD conjecture. . . (you can have a look at [Dar04] for a more mathematically precise overview). The analytic side is far more vast

than this. We can try to come up with something that meets your taste more than elliptic curves, if you are interested.

References

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