Pro-étale cohomology for schemes

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December 10, 2024

The goal of this course is to give an introduction to (pro-)étale cohomology for schemes and explain how this theory defines a good notion of ℓ-adic cohomology. The course will be in two parts: a first one about étale cohomology (with an introduction to sheaf theory, including sheaf cohomology and a few facts about derived functors, study of the étale site of a scheme, some properties of étale sheaves). In the second part, I will (partially) explain the paper "The pro-étale topology for schemes" of Bhatt and Scholze (notion of locally weakly contractible topoi, replete topoi, weakly étale morphisms, comparison between étale and pro-étale and if time permits, constructible sheaves and 6-functors formalism in this setting).

Main references: The main reference for this course is the paper of Bhatt and Scholze [BS13]. For the étale cohomology I will mostly be using the books of Tamme [Tam94] and Milne [Mil80]. The chapters about étale cohomogy and pro-étale cohomology of the Stack Project [StackProject] can also be useful.

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1 Introduction: the Weil conjectures

One of the motivations for the introduction of étale cohomology comes from the so-called Weil conjectures. Those conjectures were stated by André Weil in the 40s and concern the number of points on varieties defined over finite fields. In this section, we briefly review the statement of these conjectures and how the construction of a "good" cohomology theory can help to solve them.

In this section X will be a smooth projective variety over a finite field $\mathbf{F}_q$, with $q = p^r$ for some prime p and $r \in \mathbf{N}_{\geq 1}$. We would like to count the $\mathbf{F}_{q^n}$-points of X, for $n \in \mathrm{N}$. This set is given by

$$X(\mathbf{F}_{q^n}) := \mathrm{Hom}_{\mathrm{Spec}(\mathbf{F}_q)}(\mathrm{Spec}(\mathbf{F}_{q^n}), X).$$

The above question can be reformulated using polynomials: for $f_1, \ldots, f_m$ in $\mathbf{F}_q[t_0, \ldots, t_d]$ homogeneous polynomials[1], we want to determine how many solutions the equations

$$f_1 = \cdots = f_m = 0$$

have in $\mathbf{F}_{q^n}$, for each $n \in \mathrm{N}$. To solve this problem, let us introduce the zeta function associated to the variety X:

$$Z(X,t) := \exp\Big(\sum_{n=1}^{\infty} |X(\mathbf{F}_{q^n})| \frac{t^n}{n}\Big) \in \mathbf{Q}[|t|].$$

Note that if the function $Z(X,t)$ is known, then the numbers $|X(\mathbf{F}_{q^n})|$ can be recovered via the formula:

$$|X(\mathbf{F}_{q^n})| = \frac{1}{(n-1)!} \frac{d^n}{dt^n} \log(Z(X,t))\Big|_{t=0}.$$

So it is enough to compute the function $Z(X,t)$.

Before stating the conjectures, we give some examples where the zeta function is known.

Example 1.1 (The affine space)**.** Recall that the affine space $\mathbb{A}^d_{\mathbf{F}_q}$ of dimension d over $\mathbf{F}_q$ is the space $\mathrm{Spec}(k[x])$ endowed with its Zariski topology. So we have $|\mathbb{A}^d_{\mathbf{F}_q}(\mathbf{F}_{q^n})| = q^{nd}$ and this gives:

$$Z(\mathbb{A}^d_{\mathbf{F}_q}, t) = \frac{1}{1 - q^d t}.$$

Example 1.2 (The projective space)**.** An $\mathbf{F}_{q^n}$-point in $\mathbb{P}^d(\mathbf{F}_{q^n})$ can be described by its homogeneous coordinates $[x_0, x_1, \ldots, x_d]$, with $x_i \in \mathbf{F}_{q^n}$ and at least one of the x_i's is non-zero. Two sets of coordinates give the same point if and only if one is the multiplication of the other by an element of $\mathbf{F}_{q^n}^{\times}$. This gives $|\mathbb{P}^d(\mathbf{F}_{q^n})| = \frac{q^{n(d+1)}-1}{q^n-1}$ and

$$Z(\mathbb{P}^d, t) = \frac{1}{(1-t)(1-qt)\ldots(1-q^d t)}.$$

[1] They are the polynomials such that the variety X is given by $X := \mathrm{Proj}\big(\frac{\mathbf{F}_q[t_0,\ldots,t_d]}{\langle f_1,\ldots,f_m\rangle}\big)$.

Example 1.3. There are other cases where the zeta function for the varieties X is known. For an elliptic curve E, it can be shown that the zeta function can be computed via the formula

$$Z(E,t) = \frac{(1-\alpha t)(1-\beta t)}{(1-t)(1-qt)}$$

where α and β are conjugated in $\mathbf{C}$ and with absolute value $q^{\frac{1}{2}}$ (see for example [Sil09, Chapter 5]). More generally, if X is a curve of genus g then $Z(X,t)$ can be written

$$Z(X,t) = \frac{f(t)}{(1-t)(1-qt)}$$

with $f(t) \in \mathbf{Z}[t]$ of degree $2g$.

These computations lead to the following conjectures:

Conjecture 1.4 (Weil Conjectures)**.** Let X be a smooth connected projective variety of dimension d over $\mathbf{F}_q$. Then the zeta function $Z(X,t)$ satisfies the following property:

(i). **Rationality**: $Z(X,t)$ is a rational function in the variable t, with coefficients in $\mathbf{Q}$. More precisely,

$$Z(X,t) = \frac{P_1 ... P_{2d-1}}{P_0 ... P_{2d}},$$

with $P_i(t) \in \mathbf{Z}[t]$. Moreover we have $P_0(t) = 1-t$, $P_{2d}(t) = 1-q^d t$ and for $1 \leq i \leq 2d-1$, $P_i(t)$ is of the form $\prod_j (1-\alpha_{i,j} t)$.

(ii). **Functional equation**: there exists an integer $N \in \mathbf{N}$ such that $Z(X, q^{-d}t^{-1}) = \pm q^{\frac{Nd}{2}} t^N Z(X,t)$.

(iii). **Riemann hypothesis for finite fields**: the $\alpha_{i,j}$'s have absolute values $q^{\frac{-i}{2}}$.

(iv). **Relation to topology**: If X comes from a smooth projective variety over some $R \subset \mathbf{C}$, i.e. if X can be written $Y \otimes_R \mathbf{F}_q$ where R surjects onto $\mathbf{F}_q$ and Y is smooth and projective over $\mathbf{C}$, then

$$\deg P_i(t) = \dim_{\mathbf{Q}} H^i_{\text{sing}}(X(\mathbf{C}), \mathbf{Q}).$$

This conjecture was stated by Weil in 1949 and he proved it for curves and for abelian varieties. Dwork showed the rationality of the zeta function using methods from p-adic functional analysis. The introduction and study of ℓ-adic cohomology by Artin and Grothendieck, then allowed to prove the functional equation and later, in 1973, Deligne used it to prove the Riemann hypothesis for finite fields. This ℓ-adic cohomology will be the main object of study of this course.

Let us now explain how cohomology can be useful to prove these conjectures. Denote by $\mathrm{Var}_{\mathbf{F}_q}$ the category of algebraic varieties over $\mathbf{F}_q$. For the moment, let us assume that there exists a cohomology theory:

$$H^\bullet : \mathrm{Var}_{\mathbf{F}_q}^{\mathrm{op}} \to \{\text{graded } \mathbf{Q}\text{-vector spaces}\} \tag{1.0.0.1}$$

such that for a variety X smooth and projective of dimension d, the $\mathbf{Q}$-vector space $H^i(X)$ is finite dimensional for all i and $H^i(X) = 0$ for $i > 2d$. Write $X_{\overline{\mathbf{F}}_q}$ the base change of X to an algebraic closure $\overline{\mathbf{F}}_q$ of $\mathbf{F}_q$. The variety $X_{\overline{\mathbf{F}}_q}$ is equipped with a Frobenius morphism $\varphi : X_{\overline{\mathbf{F}}_q} \to X_{\overline{\mathbf{F}}_q}$. Assume that the cohomology $H^\bullet$ satisfies the following formula (called "Lefschetz trace formula"):

$$|X(\mathbf{F}_{q^n})| = \sum_{i=0}^{2d}(-1)^i \mathrm{tr}(H^i(\varphi^n)), \quad \text{for all } n \geq 1.$$

Note that $X(\mathbf{F}_{q^m})$ corresponds to the set of fixed points of the morphism $\varphi^m : X_{\overline{\mathbf{F}}_q} \to X_{\overline{\mathbf{F}}_q}$. Now we can plug in this formula into the definition of the zeta function. This yields

$$\begin{aligned} Z(X,t) &= \exp\Big(\sum_{n=1}^{\infty} |X(\mathbf{F}_{q^n})| \frac{t^n}{n}\Big) = \exp\Big(\sum_{n=1}^{\infty}\big(\sum_{i=0}^{2d}(-1)^i \mathrm{tr}(H^i(\varphi^n))\frac{t^n}{n}\big)\Big) \\ &= \prod_{i=0}^{2d}\Big(\exp\big(\sum_{n=1}^{\infty}\mathrm{tr}(H^i(\varphi^n))\frac{t^n}{n}\big)\Big)^{(-1)^i} = \prod_{i=0}^{2d}\det\big(\mathrm{Id} - t\cdot H^i(\varphi^n)\big)^{(-1)^{i+1}}. \end{aligned}$$

This would prove the rationality of $Z(X,t)$ and the proof of the Riemann Hypothesis for finite fields is reduced to the study of the eigenvalues of $H^i(\varphi)$. If moreover the cohomology theory satisfies Poincaré duality, i.e. there exists a trace isomorphism $H^{2d}(X) \xrightarrow{\sim} \mathbf{Q}$ that induces a natural perfect pairing of $\mathbf{Q}$-vector spaces:

$$H^i(X) \times H^{2d-i}(X) \to \mathbf{Q}$$

giving $H^i(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Q}}(H^{2d-i}(X), \mathbf{Q})$, then similar computations prove the functional equation for $Z(X,t)$. If in addition $H^\bullet$ can be compared with singular homology, we would get the point (iv) of the Weil conjectures. So we see that cohomology can be used as a tool to transform an algebraic geometry problem to a problem of linear algebra.

A cohomology theory satisfying this kind of "nice" properties (finiteness, vanishing in higher degrees, Poincaré duality, some kind of Lefschetz trace formula) is called a Weil cohomology theory. When working with varieties over $\mathbf{C}$, singular cohomology satisfies the axioms of Weil cohomology. More generally, in characteristic zero, the de Rham cohomology also defines a Weil cohomology theory. However, when working over finite fields, a Weil cohomology theory as written in (1.0.0.1) does not exist: this is due to the existence of supersingular elliptic curve (Serre). Indeed, assuming that for an elliptic curve E, such a cohomology exists this would give an anti-homomorphism

$$(\mathrm{End}E) \otimes \mathbf{Q} \to \mathrm{End}(H^1(E, \mathbf{Q})).$$

But if E is supersingular, this implies that the quaternion algebra $(\mathrm{End}\, E) \otimes \mathbf{Q}$ is non-split at p and ∞ and we obtain that $(\mathrm{End}\, E) \otimes \mathbf{R}$ is the Hamilton quaternions algebra $\mathbf{H}$. Extending the scalar to $\mathbf{R}$ in the formula above, we could get an anti-homomorphism $\mathbf{H} \to \mathcal{M}_2(\mathbf{R})$, but this does not exist. In fact, this argument shows that it is not possible to define a Weil cohomology with values in $\mathbf{R}$ and $\mathbf{Q}_p$, but it is still possible to work with $\mathbf{Q}_\ell$-coefficients, for $\ell \neq p$: this uses étale cohomology.

We will see that for a variety X over $\mathbf{F}_q$, the étale cohomology $H^\bullet_{\text{ét}}(X, \mathbf{Z}/\ell^n\mathbf{Z})$ of X with $\mathbf{Z}/\ell^n\mathbf{Z}$-coefficients define a cohomology theory with nice properties. This cohomology groups are $\mathbf{Z}/\ell^n$-modules and we obtain a $\mathbf{Q}_\ell$ vector space by taking the limit over n and inverting ℓ:

$$H^i_{\text{ét}}(X, \mathbf{Q}_\ell) := \varprojlim_n H^i_{\text{ét}}(X, \mathbf{Z}/\ell^n\mathbf{Z}) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell \quad \text{for } i \geq 0. \tag{1.0.0.2}$$

This definition of the ℓ-adic cohomology works relatively well and was used for many years. However, the fact that it is defined using inverse limit can cause problems and makes it difficult to handle. In [BS13], Bhatt and Scholze have introduced a new topology, the pro-étale topology, that gives a setting in which inverse limits behave well. We will see that the ℓ-adic pro-étale cohomology theory extends the étale one, giving a good definition of ℓ-adic cohomology in the cases where the definition (1.0.0.2) is defective and recovering it in the cases where it works well.

Remark 1.5 (Weil cohomology with p-adic coefficients). For E a supersingular elliptic curve, as mentioned before, $(\operatorname{End} E) \otimes \mathbf{Q}_p$ is not split leading to the impossibility to construct a Weil cohomology theory with coefficients in $\mathbf{Q}_p$. However, the algebra $(\operatorname{End} E) \otimes F$ where F is the fraction field of the Witt vector ring $W(\overline{\mathbf{F}}_q)$ is split. This suggests that it should be possible to define a Weil cohomology in the p-adic case, as long as we work with F-coefficients instead of $\mathbf{Q}_p$ ones, and it is indeed the case. The crystalline cohomology defines such a cohomology theory.

2 Sheaf theory

2.1 Grothendieck topologies and presheaves

Definition 2.1. Let $\mathscr{C}$ be an arbitrary category. A Grothendieck topology on $\mathscr{C}$ is the data, for any object U in $\mathscr{C}$, of a set $\mathrm{Cov}(U)$ of families $\{\varphi_i : U_i \to U\}_{i\in I}$ of morphisms in $\mathscr{C}$, called the coverings of U, satisfying the following axioms:

(i). **Isomorphism**: If $\varphi : U' \to U$ is an isomorphism in $\mathscr{C}$ then $\{\varphi : U' \to U\}$ is in $\mathrm{Cov}(U)$.

(ii). **Locality**: If $\{\varphi_i : U_i \to U\}_{i\in I}$ is in $\mathrm{Cov}(U)$ and for all i there are coverings $\{\psi_{i,j} : V_{i,j} \to U_i\}_{j\in J_i}$ in $\mathrm{Cov}(U_i)$ then $\{\varphi_i \circ \psi_{i,j} : U_{i,j} \to U\}_{(i,j)\in\prod_{i\in I}\{i\}\times J_i}$ is in $\mathrm{Cov}(U)$.

(iii). **Base change**: If $\{\varphi_i : U_i \to U\}_{i\in I}$ is in $\mathrm{Cov}(U)$ and $U' \to U$ is a morphism in $\mathscr{C}$ then,

a) for all $i \in I$, $U_i \times_U U'$ exists in $\mathscr{C}$,

b) the family $\{U_i \times_U U' \to U'\}_{i\in I}$ is in $\mathrm{Cov}(U')$.

A site is the data of a category $\mathscr{C}$ together with a Grothendieck topology on $\mathscr{C}$. The set of all coverings in $\mathscr{C}$ is denoted by $\mathrm{Cov}(\mathscr{C})$.

Example 2.2. (i). Let X be a topological space. The category of open subsets of X together with the usual coverings (i.e. the families $\{U_i \subset U\}_{i\in I}$ such that $U = \bigcup_{i\in I} U_i$) defines a Grothendieck topology. For two open subsets U_1 and U_2 inside an open subset V of X, the fibre product $U_1 \times_W U_2$ is the intersection $U_1 \cap U_2$.

(ii). Let X be a topological space. Consider $\mathrm{Top}|_X$ the category of topological spaces over X: the objects are pairs (Y, f) where Y is a topological space and $f : Y \to X$ is a continuous map, and the morphisms are the continuous maps $Y \to Z$ such that the following diagram commutes:

$$\begin{array}{ccc} Y & \longrightarrow & Z \;. \\ \downarrow & \swarrow & \\ X & & \end{array}$$

For Y in $\mathrm{Top}|_X$, we say that a family of continuous maps $\{\varphi_i : Y_i \to Y\}_{i\in I}$ is a covering if $Y = \bigcup_{i\in I} \varphi_i(Y_i)$. This defines a Grothendieck topology on $\mathrm{Top}|_X$. The same holds if we require moreover the $\varphi_i : Y_i \to Y$ to be open immersions.

From now on, Ab will denote the category of abelian groups.

Definition 2.3. Let $\mathscr{C}$ be a category. A presheaf of sets (respectively, an abelian presheaf) on $\mathscr{C}$ is a functor $\mathscr{F} : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ (respectively, Ab). If U is an object in $\mathscr{C}$, we write $\Gamma(U, \mathscr{F}) := \mathscr{F}(U)$ and the elements in $\Gamma(U, \mathscr{F})$ are called sections of $\mathscr{F}$ on U. For $\varphi : V \to U$ a morphism in $\mathscr{C}$, and s a section in $\mathscr{F}(V)$ we write

$$\mathscr{F}(\varphi)(s) = s|_V$$

and the map $\mathscr{F}(\varphi)$ is called the restriction map.

The category of presheaves (where the morphisms are the natural transformations of functors) of sets on $\mathscr{C}$ is denoted by $\mathrm{PreShv}(\mathscr{C})$. The category of abelian presheaves is denoted by $\mathrm{PreShvAb}(\mathscr{C})$.

Example 2.4. Let $\mathscr{C}$ be a category and X an object in $\mathscr{C}$. The following functor defines a presheaf on $\mathscr{C}$:

$$h_X : \begin{cases} \mathscr{C}^{\mathrm{op}} & \to \mathrm{Set} \\ U & \mapsto h_X(U) := \mathrm{Hom}_{\mathscr{C}}(U, X). \end{cases}.$$

The Yoneda lemma states that for two objects X, Y in $\mathscr{C}$, there is a natural bijection:

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{PreShv}(\mathscr{C})}(h_X, h_Y).$$

2.2 Sheaves

2.2.1 Definition

Definition 2.5. Let X, Y, Z be sets, and let $\alpha : X \to Y$ and $\beta, \gamma : Y \to Z$ be maps. We say that the diagram

$$X \xrightarrow{\alpha} Y \overset{\beta}{\underset{\gamma}{\rightrightarrows}} Z$$

is exact if α is injective and the image of α is equal to the equalizer of (β, γ), that is

$$\mathrm{Im}(\alpha) = \{y \in Y \mid \beta(y) = \gamma(y)\}.$$

Note that if X, Y, Z are abelian groups and α, β, γ are linear, then the diagram above is exact if and only if the sequence

$$0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta-\gamma} Z \to 0$$

is exact.

Definition 2.6. Let $\mathscr{C}$ be a site, and let $\mathscr{F}$ be a presheaf of sets or abelian groups on $\mathscr{C}$. We say that $\mathscr{F}$ is a sheaf if for every covering $\{U_i \to U\}_{i \in I}$ in $\mathrm{Cov}(\mathscr{C})$, the diagram

$$\mathscr{F}(U) \longrightarrow \prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{(i,j) \in I^2} \mathscr{F}(U_i \times_U U_j) \tag{2.2.1.1}$$

is exact, where the two arrows on the right are given by $(s_i)_i \mapsto (s_i|_{U_i \times_U U_j})_{i,j}$ and $(s_i)_i \mapsto (s_j|_{U_i \times_U U_j})_{i,j}$ respectively.

If $\mathscr{C}$ is a site and X an object in $\mathscr{C}$, then we define the site $\mathscr{C}_X$ in the following way. The objects of $\mathscr{C}_X$ are morphisms $Y \to X$ with Y an object of $\mathscr{C}$. Morphisms between objects $Y \to X$ and $Y' \to X$ are morphisms $Y \to Y'$ in $\mathscr{C}$ that make the obvious diagram commute and a family of morphisms $\{Y_i \to Y\}_i$ of objects over Y is a covering in $\mathscr{C}_X$ if and only if it is a covering in $\mathscr{C}$.

For the empty covering (i.e. when $I = \varnothing$), this implies that $\mathscr{F}(\varnothing)$ is an empty product, which is a final object in the corresponding category (so, a singleton for Set and Ab). We denote $\mathrm{Shv}(\mathscr{C})$ (respectively $\mathrm{ShvAb}(\mathscr{C})$) the full subcategory of $\mathrm{PreShv}(\mathcal{C})$ (respectively $\mathrm{PreShvAb}(\mathscr{C})$) which objects are sheaves.

It can be showed that for a site $\mathscr{C}$, the categories $\mathrm{PreShvAb}(\mathscr{C})$ and $\mathrm{ShvAb}(\mathscr{C})$ are abelian[2] (see for example [Tam94, §3]).

Example 2.7 (Sheaves vs presheaves)**.** Let X be a topological space. Then, (1) the presheaf $U \mapsto \{$ functions $U \to \mathbf{Z}\}$ is a sheaf.

(2) the presheaf $U \mapsto \{$ constant functions $U \to \mathbf{Z}\}$ is not a sheaf (the glueing does not always exist).

(3) the presheaf $U \mapsto \begin{cases} 0 & \text{if } U \neq X \\ \mathbf{Z} & \text{if } U \neq X \end{cases}$ is not a sheaf (the glueing in not necessarily unique).

Example 2.8 (Sheaves on $G - \mathrm{Set}$)**.** This example is important and will come back later in the course. Let G be a group. We denote by $G-\mathrm{Set}$ the category whose objects are sets endowed with a left G-action and morphisms are equivariant maps. We endow $G - \mathrm{Set}$ with the Grothendieck topology in which the coverings are the families $\{\varphi_i : U_i \to U\}_{i\in I}$ such that $U = \bigcup_{i\in I} \varphi_i(U_i)$. Note that G is itself an object in $G - \mathrm{Set}$ (the action is given by left translations). Let us denote by $\mathcal{T}_G$ this site.

Lemma 2.9. *The functor*

$$\begin{cases} \mathrm{Shv}(\mathcal{T}_G) & \to G - \mathrm{Set} \\ \mathscr{F} & \mapsto \mathscr{F}(G) \end{cases} \tag{2.2.1.2}$$

defines an equivalence of categories.

Proof. We first check that it is well-defined, i.e. $\mathscr{F}(G)$ is in $G - \mathrm{Set}$. Using the isomorphism

$$\begin{cases} G & \xrightarrow{\sim} \mathrm{Aut}_G(G) \\ g & \mapsto (h \mapsto hg) \end{cases}$$

we see that any $g \in G$ gives rise to an element of $\mathrm{Aut}_{G-\mathrm{Set}}(G)$ and so to a map $\mathscr{F}(G) \to \mathscr{F}(G)$. Hence, we get a left action of G on $\mathscr{F}(G)$.

[2]This means that the hom-sets are abelian groups, we can define kernels and cokernels and they behave nicely.

To prove it is an equivalence of categories, we will show that the functor

$$\begin{cases} G - \mathrm{Set} & \to \mathrm{Shv}(\mathcal{T}_G) \\ Z & \mapsto h_Z : U \mapsto \mathrm{Hom}_G(U, Z) \end{cases}$$

defines a quasi-inverse for (2.2.1.2). Let Z be a G-set. The isomorphism $h_Z(G) \xrightarrow{\sim} Z$ is given by the map $\varphi \mapsto \varphi(1_G)$. Conversely, let $\mathscr{F}$ be in $\mathrm{Shv}(\mathcal{T}_G)$, we want to prove that we have an isomorphism of sheaves

$$\mathscr{F} \xrightarrow{\sim} \mathrm{Hom}_G(-, \mathscr{F}(G)).$$

Let Z be a G-set. The set $\{G \xrightarrow{\varphi_z} Z\}_{z \in Z}$ where $\varphi_z(g) = g \cdot z$ for all $z \in Z$ and $g \in G$, is a covering in $\mathcal{T}_G$. So, by definition, the following diagram is exact

$$\mathscr{F}(Z) \longrightarrow \prod_{z \in Z} \mathscr{F}(G) \rightrightarrows \prod_{(z_1, z_2) \in Z \times Z} \mathscr{F}(G \times_Z G)$$

Note that the term is the middle $\prod_{z \in Z} \mathscr{F}(G)$ is equal to $\mathrm{Hom}(Z, \mathscr{F}(G))$ (with no G-structure in the hom-set). To finish the proof it suffices to prove that the kernel of the right map in the diagram above is equal to the subset $\mathrm{Hom}_G(Z, \mathscr{F}(G))$ of $\mathrm{Hom}(Z, \mathscr{F}(G))$. But this follows from the definition of the maps in the diagram, noting that for z_1, z_2 in Z, the product of the two corresponding copies of G is equal to G if there exists $g \in G$ such that $z_2 = g \cdot z_1$ and empty otherwise. □

Replacing the category of sheaves of set by the category of abelian sheaves, we obtain:

Corollary 2.10. *The category of left G-modules is equivalent to the category of abelian sheaves on the canonical topology $\mathcal{T}_G$. The equivalence is given by the quasi-inverse functors $\mathscr{F} \mapsto \mathscr{F}(G)$ and $M \mapsto \mathrm{Hom}_G(-, M)$.*

2.2.2 Sheafification

Let $\mathscr{T}$ be a site. The goal of this section is to define a sheafifacation functor $(-)^\sharp : \mathrm{PreShvAb}(\mathscr{T}) \to \mathrm{ShvAb}(\mathscr{T})$, which is left-adjoint to the inclusion functor $i : \mathrm{ShvAb}(\mathscr{T}) \to \mathrm{PreShvAb}(\mathscr{T})$. As a first approximation of the sheafification of the presheaf $\mathscr{F}$, we introduce the following definition:

Definition 2.11. Let $\mathscr{F}$ be a presheaf on the site $\mathscr{T}$ and $\mathscr{U} = \{U_i \to U\}$ in $\mathrm{Cov}(\mathscr{T})$. We define the 0-th Čech cohomology group of $(\mathscr{U}, \mathscr{F})$ by

$$\check{H}^0(\mathscr{U}, \mathscr{F}) = \big\{(s_i)_{i \in I} \in \prod_{i \in I} \mathscr{F}(U_i) \mid s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}\big\}.$$

Note that there is a natural map $\mathscr{F}(U) \to \check{H}^0(\mathscr{U},\mathscr{F})$. We would like to make the covering $\mathscr{U}$ in $\check{H}^0(\mathscr{U},\mathscr{F})$ vary. To do this, we need the notion of refinement of a covering: for $\mathscr{U} = \{U_i \to U\}_{i\in I}$ in $\mathrm{Cov}(\mathscr{T})$, a covering $\mathscr{V} = \{V_j \to U\}_{j\in J}$ is a refinement of $\mathscr{U}$ if there exists a map $\alpha : J \to I$ and for all $j \in J$, a commutative diagram

$$\begin{array}{ccc} V_j & \xrightarrow{f_j} & U_{\alpha(j)} \\ \downarrow & \swarrow & \\ U & & \end{array} .$$

Note that for every refinement $f : \mathscr{V} \to \mathscr{U}$ in $\mathrm{Cov}(U)$, we get a canonical map

$$\check{H}^0(\mathscr{U},\mathscr{F}) \to \check{H}^0(\mathscr{V},\mathscr{F}),$$

given by $(s_i)_{i\in I} \mapsto ((s_{\alpha(j)})|_{V_j})_{j\in J}$. We can show that this map is independent of the choices of α and the f_j's.

Definition 2.12. Let $\mathscr{F}$ be an abelian presheaf on $\mathscr{T}$. For every $U \in \mathscr{T}$, we define

$$\mathscr{F}^+(U) = \varinjlim_{\mathscr{U}\in \mathrm{Cov}(U)} \check{H}^0(\mathscr{U},\mathscr{F}).$$

Let $V \to U$ be a morphism in $\mathscr{T}$. If $\mathscr{U} = \{U_i \to U\}_{i\in I}$ is a covering of U, then $\mathscr{V} = \{U_i \times_U V \to V\}_{i\in I}$ is a covering of V and we get a morphism $\check{H}^0(\mathscr{U},\mathscr{F}) \to \check{H}^0(\mathscr{V},\mathscr{F})$. Taking the colimit, we get a morphism $\mathscr{F}^+(U) \to \mathscr{F}^+(V)$. This gives to $\mathscr{F}^+$ the structure of presheaf on $\mathscr{T}$.

Proposition 2.13. *Let $\mathscr{T}$ be a site and $\mathscr{F}$ an abelian presheaf on $\mathscr{T}$. Then $\mathscr{F}^\sharp := (\mathscr{F}^+)^+$ is a sheaf and the canonical map induces a functorial isomorphism*

$$\mathrm{Hom}_{\mathrm{PreShv}(\mathscr{C})}(\mathscr{F},\mathscr{G}) = \mathrm{Hom}_{\mathrm{Shv}(\mathscr{C})}(\mathscr{F}^\sharp,\mathscr{G})$$

for any $\mathscr{G} \in \mathrm{Shv}(\mathscr{T})$.

The proof of the proposition uses the notion of separated presheaf: a presheaf $\mathscr{F}$ is separated if for every U in $\mathscr{C}$ and $\mathscr{U}$ in $\mathrm{Cov}(U)$, the canonical map

$$\mathscr{F}(U) \to \prod_{i\in I} \mathscr{F}(U_i)$$

is injective.

Sketch of proof. The proof is in three steps:

(1) There is a canonical map of presheaves $\mathscr{F} \to \mathscr{F}^+$.

(2) If $\mathscr{F}$ is a separated presheaf then $\mathscr{F}^+$ is a sheaf and the map $\mathscr{F} \to \mathscr{F}^+$ is injective.

(3) The presheaf $\mathscr{F}^+$ is separated.

Details can be read in [Tam94, §3]. $\square$

Theorem 2.14. *Let $\mathscr{C}$ be a site. The category $\mathrm{ShvAb}(\mathscr{C})$ of abelian sheaves on $\mathscr{C}$ is an abelian category. The inclusion functor $i : \mathrm{ShvAb}(\mathscr{C}) \to \mathrm{PreShvAb}(\mathscr{C})$ is left exact and the sheafification functor $(-)^\sharp : \mathrm{PreShvAb}(\mathscr{C}) \to \mathrm{ShvAb}(\mathscr{C})$ is exact.*

Proposition 2.15 (Examples and properties). *(i). If $\mathscr{F}$ is a sheaf then $\mathscr{F} \simeq \mathscr{F}^\sharp$.*

(ii). If $f : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves then the presheaf

$$\mathrm{Ker}(f) := U \mapsto (f_U : \mathscr{F}(U) \to \mathscr{G}(U))$$

is a sheaf.

(iii). If $f : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, we define the image of f, denoted by $\mathrm{Im}(f)$ as the sheafification of the presheaf:

$$U \mapsto \mathrm{Im}(f_U : \mathscr{F}(U) \to \mathscr{G}(U)).$$

(iv). Let $\mathscr{C}$ be a site and $f : X \to Y$ in $\mathscr{C}$. The direct image functor is defined as

$$f_* : \begin{cases} \mathrm{Shv}(\mathscr{C}_X) \to \mathrm{Shv}(\mathscr{C}_Y) \\ \mathscr{F} \mapsto f_*\mathscr{F} = (U \mapsto \mathscr{F}(U \times_Y X)) \end{cases}.$$

(As an exercise: check that $f_*\mathscr{F}$ is indeed a sheaf.)

(v). Let $\mathscr{C}$ be a site and $f : X \to Y$ in $\mathscr{C}$. The inverse image $f^{-1}\mathscr{F}$ of a sheaf $\mathscr{F}$ over Y is defined as the sheafification of $U \mapsto \mathrm{colim}_V \mathscr{F}(V)$ where the colimit is over the schemes $V \to Y$ such that there is a map $U \to X \times_Y V$. The inverse image functor is the functor

$$f^{-1} : \begin{cases} \mathrm{Shv}(\mathscr{C}_Y) \to \mathrm{Shv}(\mathscr{C}_X) \\ \mathscr{F} \mapsto f^{-1}\mathscr{F} \end{cases}.$$

3 Crash-course on derived categories

Let $\mathscr{C}$ and $\mathscr{D}$ be abelian category and $F : \mathscr{C} \to \mathscr{D}$ a left exact functor. This means that if we have an exact sequence of object in $\mathscr{C}$:

$$0 \to A \to B \to C \to 0$$

after applying F, we get an exact sequence

$$0 \to F(A) \to F(B) \to F(C).$$

We would like to extend this exact sequence further. To do that we will define the higher derived functors of F: they are functors $\mathrm{R}^i F$ for all $i \geq 0$, such that we have a long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to \mathrm{R}^1 F(A) \to \mathrm{R}^1 F(B) \to \mathrm{R}^1 F(C) \to \mathrm{R}^2 F(A) \to \cdots$$

When we are in the case $\mathscr{C} = \mathrm{ShvAb}(\mathscr{T})$ for some site $\mathscr{T}$ with a base X, $\mathscr{D} = \mathrm{Ab}$ and $F = \Gamma(X, -)$, these derived functors will define the cohomology of a sheaf $\mathscr{F}$:

$$H^i(X, \mathscr{F}) := \mathrm{R}^i\Gamma(X, \mathscr{F}),$$

in other words, for a short exact sequence of sheaves $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$, we obtain a long exact sequence

$$0 \to H^0(X, \mathscr{F}) = \mathscr{F}(X) \to H^0(X, \mathscr{G}) \to H^0(X, \mathscr{H}) \to H^1(X, \mathscr{F}) \to H^1(X, \mathscr{G}) \to H^1(X, \mathscr{H}) \to \cdots$$

Many details in this section will be skipped. More precise statements and proofs can be read in [Wei94] or [StackProject].

3.1 Definition of derived category

3.1.1 The homotopy category

In this section $\mathscr{A}$ will always be an abelian category. A chain complex $K^\bullet$ is a sequence

$$\cdots \xrightarrow{d} K^{i-1} \xrightarrow{d} K^i \xrightarrow{d} K^{i+1} \xrightarrow{d} \cdots$$

such that the composition $d \circ d$ is zero. A morphism of chain complexes $f : K^\bullet \to L^\bullet$ is a sequence of morphisms $\{f_i : K^i \to L^i\}_{i \in \mathbf{Z}}$ such that $d \circ f_i = f_{i+1} \circ d$. We will denote by $\mathrm{Ch}(\mathscr{A})$ the category of chain complexes in $\mathscr{A}$. It can be showed that since $\mathscr{A}$ is abelian, then $\mathrm{Ch}(\mathscr{A})$ is also abelian. We will also write $\mathrm{Ch}^+(\mathscr{A})$ (respectively $\mathrm{Ch}^-(\mathscr{A})$) for the full subcategory of bounded below (respectively bounded above) chain complexes, i.e. those $K^\bullet$ with $K^i = 0$ for $i << 0$ (respectively, $i >> 0$). The full subcategory of bounded (below and above) chain complexes will be denoted by $\mathrm{Ch}^b(\mathscr{A})$. In the category $\mathrm{Ch}(\mathscr{A})$, let us define the following operation:

- For $K^\bullet$ a chain complex and i an integer, the shift $K^\bullet[i]$ of K by i, is the chain complex such that the n-th term is K^{i+n}. Alternatively, $K^\bullet$ can be viewed as the tensor product $K^\bullet \otimes S^i$ where S^i is the chain complex whose terms are all zero except in degree $-i$ where it is $\mathbf{Z}$ and the tensor product in $\mathrm{Ch}(\mathscr{A})$ is defined by the formula:

$$(K^\bullet \otimes L^\bullet)^n = \bigoplus_{i+j=n} A^i \otimes B^j \quad \text{for all } n \in \mathbf{Z}.$$

- For a morphism of chain complexes $f : K^\bullet \to L^\bullet$, the cone of f is the chain complex $\mathrm{Cone}(f)$ such that the n-th term is $K^{n+1} \oplus L^n$ and the differentials are given by $\begin{pmatrix} d_K & 0 \\ f & d_L \end{pmatrix}$. Note that we obtain a short exact sequence of chain complexes

$$0 \to L^\bullet \to \mathrm{Cone}(f) \to K^\bullet[1] \to 0.$$

  Alternatively, $\mathrm{Cone}(f)$ can be defined as the push-out

$$\begin{array}{ccc} L^\bullet \otimes S^0 & \longrightarrow & L^\bullet \otimes D^1 \\ \downarrow & & \downarrow \\ K^\bullet \otimes S^0 & \longrightarrow & \mathrm{Cone}(f) \end{array}$$

  where S^0 is the chain complex whose terms are all zero except in degree 0 where it is $\mathbf{Z}$ and D^1 is the chain complex whose terms are all zero except in degree -1 and 0 where they are $\mathbf{Z}$ (and the differential between the two copies of $\mathbf{Z}$ is the identity).

- For $K^\bullet$ a chain complex and i an integer, the i-th cohomology of $K^\bullet$ is defined by the formula

$$H^i(K^\bullet) = \mathrm{Ker}(K^i \to K^{i+1})/\mathrm{Im}(K^{i-1} \to K^i) \quad \text{for all } i \in \mathbf{Z}.$$

  The group $\mathrm{Ker}(K^i \to K^{i+1})$ is called the group of i-cocyles of $K^\bullet$ and we denote it by $Z^i(K^\bullet)$ while $\mathrm{Im}(K^{i-1} \to K^i)$ is the group of i-coboundaries of $K^\bullet$ and is denoted by $B^i(K^\bullet)$. If $0 \to A \to B \to C \to 0$ is a short exact sequence of chain complexes then taking cohomology yields a long exact sequence

$$\cdots \to H^{i-1}(C) \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots$$

Definition 3.1. Let $f : K^\bullet \to L^\bullet$ be a morphism of chain complexes. We say that f is a quasi-isomorphism if for all integer i, the morphisms $H^i(f) : H^i(K^\bullet) \to H^i(L^\bullet)$ are isomorphisms.

There is a notion of chain homotopy between two morphisms of chain complexes (see for example [Wei94, §1.4] for a definition). If f and g are morphisms of chain complexes $K^\bullet \to L^\bullet$ such that there exists a chain homotopy between f and g, we write $f \sim g$. This defines an equivalence relation on $\mathrm{Hom}_{\mathrm{Ch}(\mathscr{A})}(K^\bullet, L^\bullet)$. Note that if $f \sim g$ then $H^i(f) = H^i(g)$ for all $i \in \mathbf{Z}$. We say that two complexes $K^\bullet$ and $L^\bullet$ are homotopy equivalent if there exist $f : K^\bullet \to L^\bullet$ and

$g : L^\bullet \to K^\bullet$ such that $f \circ g \sim \mathrm{Id}$ and $g \circ f \sim \mathrm{Id}$. It can be proved that if $K^\bullet$ and $L^\bullet$ are homotopy equivalent then they are quasi-isomorphic (the converse is not true).

We define the homotopy category of $\mathscr{A}$ as the category $K(\mathscr{A})$ whose objects are chain complexes and sets of morphisms are the homotopy equivalence classes of maps of chain complexes, i.e. $\mathrm{Hom}_{K(\mathscr{A})}(K^\bullet, L^\bullet) = \mathrm{Hom}_{\mathrm{Ch}(\mathscr{A})}(K^\bullet, L^\bullet)/\sim$. Note that $K(\mathscr{A})$ satisfies the following universal property: for any functor $F : \mathrm{Ch}(\mathscr{A}) \to \mathscr{D}$ sending homotopy equivalence to isomorphism there exist a unique functor $\overline{F} : K(\mathscr{A}) \to \mathscr{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Ch}(\mathscr{A}) & \xrightarrow{F} & \mathscr{D} \\ \downarrow & \nearrow_{\overline{F}} & \\ K(\mathscr{A}) & & \end{array}.$$

Denote by $K(\mathscr{A})^+$, $K(\mathscr{A})^-$ and $K(\mathscr{A})^b$ the subcategories corresponding to $\mathrm{Ch}(\mathscr{A})^+$, $\mathrm{Ch}(\mathscr{A})^-$ and $\mathrm{Ch}(\mathscr{A})^b$.

Exact triangles in the homotopy category. Let $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ be a sequence of morphisms in $K(\mathscr{A})$. We say that the triangle (u, v, w) is exact if there exist f, g, h homotopy equivalences such that there is a commutative diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[-1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f[-1]} \\ A' & \xrightarrow{u'} & B' & \longrightarrow & \mathrm{Cone}(u') & \longrightarrow & A'[-1]. \end{array}$$

In particular, note that this implies that we have a long exact sequence:

$$\cdots \to H^{i-1}(C) \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots$$

Remark 3.2. The category $K(\mathscr{A})$ is called a triangulated category. More generally a triangulated category if an additive category $\mathscr{D}$ equipped with a functor $[1] : \mathscr{D} \to \mathscr{D}$ defining an auto-equivalence and a class of exact triangles $\mathscr{T}$ satisfying certain axioms. See for example [Wei94, §10.2] for the precise definition of triangulated category and exact triangles.

3.1.2 The derived category

Recall that we work with $\mathscr{A}$ an abelian category.

Definition 3.3. Let $\mathscr{C}$ be a category and S a class of morphisms in $\mathscr{C}$. The localisation of the category $\mathscr{C}$ with respect to S is the universal functor $Q : \mathscr{C} \to \mathscr{C}[S^{-1}]$ sending elements of S to isomorphisms: i.e. for any functor $F : \mathscr{C} \to \mathscr{D}$ sending elements of S to isomorphisms, there exists a unique functor $\overline{F} : \mathscr{C}[S^{-1}] \to \mathscr{D}$ making the following diagram commutative:

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & \mathscr{D} \\ \downarrow & \nearrow_{\overline{F}} & \\ \mathscr{C}[S^{-1}] & & \end{array} .$$

In particular, the homotopy category $K(\mathscr{A})$ is the localisation of $\mathrm{Ch}(\mathscr{A})$ with respect to homotopy equivalences.

Example 3.4. The name "localisation" comes from the following example: let R is a ring and $S \subset R$ a multiplicatively closed subset and consider the category $\mathscr{C}_R$ whose only object is a point $*$ and the set of morphisms $\mathrm{Hom}(*, *)$ is equal to R (the composition being the multiplication in R). Then $\mathscr{C}_R[S^{-1}] = \mathscr{C}_{R[S^{-1}]}$.

Definition 3.5. The derived category $\mathscr{D}(\mathscr{A})$ is defined as the localisation of $K(\mathscr{A})$ with respect to the class of quasi-isomorphisms:

$$Q : K(\mathscr{A}) \to \mathscr{D}(\mathscr{A}) := K(\mathscr{A})[\mathrm{qis}^{-1}].$$

It can be proved that the derived category $D(\mathscr{A})$ is a triangulated category. More generally, we have the following proposition:

Proposition 3.6. *Let $(\mathscr{C}, [1], \mathscr{T})$ be a triangulated category. Then there exists a unique structure of a triangulated category on $S^{-1}\mathscr{C}$ such that $[1] \circ Q = Q \circ [1]$ and the localization functor $Q : \mathscr{C} \to S^{-1}\mathscr{C}$ sends exact triangles to exact triangles.*

For a proof of the above proposition, see for example [StackProject, 05R6].

We denote by $D^+(\mathscr{A})$, $D^-(\mathscr{A})$ and $D^b(\mathscr{A})$ the subcategories corresponding to $K^+(\mathscr{A})$, $K^-(\mathscr{A})$ and $K^b(\mathscr{A})$.

Remark 3.7. Let $\mathscr{C}$ be a abelian category and S a saturated multiplicative system [3]. For Y an object of $\mathscr{C}$, we define Y/S as the category whose objects are morphisms $s : Y \to Y'$ in S and a morphism between two objects $s : Y \to Y'$ and $t : Y \to Y''$ is a morphism $Y' \to Y''$ in $\mathscr{C}$ (not necessarily in S) making the obvious diagram commute. Then, the sets of morphisms in the category $S^{-1}\mathscr{C}$ can be described as follows:

$$\mathrm{Hom}_{S^{-1}\mathscr{C}}(X, Y) = \mathrm{colim}_{(s:Y \to Y') \in Y/S} \mathrm{Hom}_{\mathscr{C}}(X, Y').$$

Dually, for an object X of $\mathscr{C}$, the category S/X is defined as the category whose objects are morphisms $s : X' \to X$ in S and the Hom-sets are defined in a similar way as above. As before, we have:

$$\mathrm{Hom}_{S^{-1}\mathscr{C}}(X, Y) = \mathrm{colim}_{(s:X' \to X) \in (S/X)^{\mathrm{op}}} \mathrm{Hom}_{\mathscr{C}}(X', Y).$$

[3] See [StackProject, 04VB] for a definition of "saturated multiplicative system". In the following we will apply this to S the set of quasi-isomorphisms in the homotopy category.

3.2 Derived functors

3.2.1 Derived functors in general

Consider $F : \mathscr{D} \to \mathscr{D}'$ a functor between two triangulated category and S a saturated multiplicative system in $\mathscr{D}$. We will first define the notion of right and left derived functor $\mathrm{R}F$ and $\mathrm{L}F$ for such a functor F.

Definition 3.8. Let X be an object in $\mathscr{D}$.

(1) We say that the right derived functor $\mathrm{R}F$ is defined at X if the diagram

$$\begin{cases} (X/S) & \to \mathscr{D}' \\ (s : X \to X') & \mapsto F(X') \end{cases}$$

is essentially constant[4]. If $\mathrm{R}F$ is defined at X, we denote by $\mathrm{R}F(X)$ its value.

(2) Dually, we say that the left derived functor $\mathrm{L}F$ is defined at X if the diagram

$$\begin{cases} (S/X) & \to \mathscr{D}' \\ (s : X' \to X) & \mapsto F(X') \end{cases}$$

is essentially constant. If $\mathrm{L}F$ is defined at X, we denote by $\mathrm{L}F(X)$ its value.

It can be shown that if $s : X \to Y$ is in S, then $\mathrm{R}F$ (respectively $\mathrm{L}F$) is defined at X if and only if it is defined at Y and $\mathrm{R}F(X) \xrightarrow{\sim} \mathrm{R}F(Y)$. Also, $\mathrm{R}F$ is defined at $X \in \mathscr{D}$ if and only if it is defined at $X[1]$ and in that case, $\mathrm{R}F(X)[1] = \mathrm{R}F(X[1])$. Moreover, if (X, Y, Z) is an exact triangle in $\mathscr{D}$ and $\mathrm{R}F$ is defined at two of the three of X, Y, Z then it is defined at the third one and $(\mathrm{R}F(X), \mathrm{R}F(Y), \mathrm{R}F(Z))$ is an exact triangle. We get:

Proposition 3.9. *The full subcategory $\mathscr{E}$ of $\mathscr{D}$ consisting of objects where $\mathrm{R}F$ is defined is a triangulated category and $\mathrm{R}F$ defines a functor $\mathscr{E} \to \mathscr{D}$ sending exact triangles to exact triangles. Elements of S with source or target in $\mathscr{E}$ are morphisms of $\mathscr{E}$, $\mathrm{R}F$ sends elements of $S_{\mathscr{E}} := \mathrm{Arrows}(\mathscr{E}) \cap S$ to isomorphisms and it induces a functor of triangulated categories $\mathrm{R}F : S_{\mathscr{E}}^{-1}\mathscr{E} \to \mathscr{D}$ (sending exact triangles to exact triangles).*

We have a similar result replacing $\mathrm{R}F$ by $\mathrm{L}F$.

We will say that an object X in $\mathscr{D}$ computes $\mathrm{R}F$ (respectively $\mathrm{L}F$) if $\mathrm{R}F$ (respectively $\mathrm{L}F$) is defined at X and $F(X) \xrightarrow{\sim} \mathrm{R}F(X)$ (respectively $\mathrm{L}F(X) \xrightarrow{\sim} F(X)$).

Lemma 3.10. *If for all X in $\mathscr{D}$, there exists $s : X \to X'$ (respectively $s : X' \to X$) in S such that X' computes $\mathrm{R}F$ (respectively $\mathrm{L}F$) then $\mathrm{R}F$ (respectively $\mathrm{L}F$) is defined everywhere.*

[4] This means that in the associated ind-category $\mathrm{Ind}(\mathscr{D}')$, it is isomorphic to a filtered diagram consisting of a single object Y and the morphisms are all equal to identity.

3.2.2 Derived functors on the derived category

The above results can be made more explicit in the case where we consider the homotopy categories of abelian categories.

Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories and $F : \mathscr{A} \to \mathscr{B}$ an additive functor. We write F (respectively F^+, F^-) for the induced functor $F : K(\mathscr{A}) \to D(\mathscr{B})$ (respectively $K^+(\mathscr{A}) \to D^+(\mathscr{B})$, $K^-(\mathscr{A}) \to D^-(\mathscr{B})$).

Lemma 3.11. *(i).* $\mathrm{R}F$ *is defined at* $X \in K^+(\mathscr{A})$ *if and only if it* $\mathrm{R}F^+$ *is defined at* X *and in that case, they have the same values.*

(ii). $\mathrm{L}F$ *is defined at* $X \in K^-(\mathscr{A})$ *if and only if it* $\mathrm{L}F^-$ *is defined at* X *and in that case, they have the same values.*

(iii). For $X \in K^+(\mathscr{A})$*,* X *computes* $\mathrm{R}F$ *if and only if it computes* $\mathrm{R}F^+$.

(iv). For $X \in K^-(\mathscr{A})$*,* X *computes* $\mathrm{L}F$ *if and only if it computes* $\mathrm{L}F^-$.

We defined the right (respectively left) derived functor as the functor $\mathrm{R}F$ (respectively $\mathrm{L}F$) going from a full subcategory of $D(\mathscr{A})$ to $D(\mathscr{B})$. We say that an object A in $\mathscr{A}$ is right (respectively, left) acyclic for F if $A[0]$[5] computes $\mathrm{R}F$ (respectively $\mathrm{L}F$).

Definition 3.12. Assume $\mathrm{R}F$ is defined everywhere on $D(\mathscr{A})^+$. Let $i \in \mathbf{Z}$. The i-th derived functor of F is the functor

$$\mathrm{R}^iF = H^i \circ \mathrm{R}F : D(\mathscr{A})^+ \to \mathscr{B}.$$

The following lemma explains why we will mostly be interested in left exact functor when computing right derived functor.

Lemma 3.13. *With the assumptions from Definition 3.12, then* $\mathrm{R}^iF = 0$ *for* $i < 0$*,* R^0F *is left exact and the map* $F \to \mathrm{R}^0F$ *is an isomorphism if and only if* F *is left exact. Moreover, if* A *is an object in* $\mathscr{A}$ *then* A *is right acyclic if and only if* $F(A) \xrightarrow{\sim} \mathrm{R}^0F(A)$ *and* $\mathrm{R}^iF(A) = 0$ *for* $i > 0$.

To compute right derived functors, our main tool will be the following result (and its corollary below):

Proposition 3.14 (Leray's acyclicity)**.** *Let* $F : \mathscr{A} \to \mathscr{B}$ *be an additive functor between abelian categories. Let* $K^\bullet$ *be a bounded below complex of right* F*-acyclic objects such that* $\mathrm{R}F$ *is defined at* $K^\bullet$*. Then, the canonical map* $F(K^\bullet) \to \mathrm{R}F(K^\bullet)$ *is an isomorphism in* $D^+(\mathscr{B})$*, i.e.,* $K^\bullet$ *computes* $\mathrm{R}F$.

[5] i.e. the complex whose all terms are zero except in degree 0 where it is A.

This result combined with Lemma 3.10 gives:

Corollary 3.15. *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor of abelian categories.*

(i). If every object of A injects into an object acyclic for $\mathrm{R}F$, then $\mathrm{R}F$ is defined everywhere on $K^+(\mathscr{A})$ and we obtain a functor $\mathrm{R}F : D^+(\mathscr{A}) \to D^+(\mathscr{B})$ sending exact triangles to exact triangles. Moreover, any bounded below complex $K^\bullet$ whose terms are acyclic for $\mathrm{R}F$ computes $\mathrm{R}F$.

(ii). If every object of A is a quotient of an object acyclic for $\mathrm{L}F$, then $\mathrm{L}F$ is defined everywhere on $K^-(\mathscr{A})$ and we obtain a functor $\mathrm{L}F : D^-(\mathscr{A}) \to D^-(\mathscr{B})$ sending exact triangles to exact triangles. Moreover, any bounded below complex $K^\bullet$ whose terms are acyclic for $\mathrm{L}F$ computes $\mathrm{L}F$.

3.3 Sheaf cohomology

We will now apply the previous construction to the category of abelian sheaves. The acyclic objects will be given by complexes of injective sheaves. The i-th cohomology group of a sheaf is then defined as the i-th right derived functor of the global section functor.

3.3.1 Injective objects and resolutions

Recall that for an object I of an abelian category $\mathscr{A}$, the contravariant functor $A \mapsto \mathrm{Hom}_{\mathscr{A}}(A, I)$ is left exact. We say that an object I in $\mathscr{A}$ is injective if the functor $A \mapsto \mathrm{Hom}_{\mathscr{A}}(A, I)$ is exact. Equivalently, I is injective if for any object A with a subobject $A' \subset A$ and a morphism $A' \to I$, then this morphism can be extended to a morphism $A \to I$. We will see later a criterion for an abelian group to be injective (see Proposition 3.20).

Definition 3.16. Let $\mathscr{A}$ be an abelian category.

(i). If A is an object of $\mathscr{A}$, an injective resolution of A is a chain complex $I^\bullet$ together with a map $A \to I^0$ such that $I^n = 0$ for $n < 0$, the objects I^n are injective for all n and the cohomology of the complex is computed by

$$A \xrightarrow{\sim} \ker(d_I^0) \text{ and } H^i(I^\bullet) = 0 \text{ for } i > 0.$$

In other words, $A[0] \to I^\bullet$ is a quasi-isomorphism.

(ii). If $K^\bullet$ is in $D(\mathscr{A})$, an injective resolution of $K^\bullet$ is a chain complex $I^\bullet$ together with a map $\alpha : K^\bullet \to I^\bullet$ such that $I^n = 0$ for $n << 0$, the objects I^n are injective for all n and α is a quasi-isomorphism.

Definition 3.17. We say that $\mathscr{A}$ has enough injectives if for all A in $\mathscr{A}$ there exists a monomorphism $A \to I$ with I injective.

Proposition 3.18. *Assume that $\mathscr{A}$ has enough injectives. Then,*

(i). any object A in $\mathscr{A}$ admits an injective resolution and,

(ii). if $K^\bullet$ is a chain complex such that $H^n(K^\bullet) = 0$ for $n << 0$ then $K^\bullet$ admits an injective resolution.

Note that if $H^n(K^\bullet) = 0$ for $n << 0$ then there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with $L^\bullet$ bounded below: it suffices to take $L := \tau_{\geq n} K^\bullet$ where the truncation $\tau_{\geq n}$ is defined by $\tau_{\geq n} K = (\cdots \to 0 \to 0 \to \operatorname{coker}(d_{n-1}) \to K_{n+1} \to K_{n+2} \to \cdots)$.

Sketch of proof of Proposition 3.18. For the first point, let A be an object in $\mathscr{A}$ and take a monomorphism into an injective object $A \hookrightarrow I^0$. Consider the object I^0/A and choose I^1 injective such that I^0/A injects into I^1. Write d^0 for the map $I^0 \to I^1$. Let us now consider the object $A/\mathrm{im}(d^0)$. As before we can take I^2 injective such that $I^1/\mathrm{im}(d^0)$ injects into I^2 and denote by d^1 the natural map $I^1 \to I^2$. Iterating the construction, we obtain the complex $I^\bullet$ as wanted.

For the second point, we proceed by induction on the degree. Let a be an integer such that $K^i = 0$ for $i < a$. Consider the following induction hypothesis, for $n \geq a$:

(IH_n)

For $i \leq n$ there is a complex $\big(I_a \to I_{a+1} \to \cdots \to I_{n-1} \to I_n\big)$ with a map $\alpha : K^\bullet \to I^\bullet$ such that $H^i I^\bullet \simeq H^i K^\bullet$ for $i < n$ and $K^{n+1} \to K^n \oplus I^{n-1} \to I^n$ is exact.

Define C as the cokernel of the map $K^n \oplus I^{n-1} \to K^{n+1} \oplus I^n$ sending (x, y) to $(d(x), d(y) - \alpha(x)$. Choose I^{n+1} injective such that C injects into I^{n+1}. Then, using I^{n+1} we obtain (IH_{n+1}). ☐

Proposition 3.19. *Let $\mathscr{A}$ be an abelian category and I in $\mathscr{A}$ an injective object. Then I is right acyclic for any additive functor $F : \mathscr{A} \to \mathscr{B}$ (with $\mathscr{B}$ an abelian category).*

Sketch of proof. We more generally prove that a bounded below complex of injectives $I^\bullet$ computes the derived functor $\mathrm{R}F$. By definition, it suffices to prove that

$$\begin{cases} I^\bullet/\mathrm{Qis}^+(\mathscr{A}) & \to D^+(\mathscr{B}) \\ (I^\bullet \xrightarrow{\sim} K^\bullet) & \mapsto F(K^\bullet) \end{cases}$$

is essentially constant with value $F(I^\bullet)$. This comes from the fact that since the I^n are injective objects, each $\alpha : I^\bullet \xrightarrow{\sim} K^\bullet$ has a left inverse (see [StackProject, 013P]). ☐

3.3.2 Application to the category of abelian sheaves

Proposition 3.20. *An abelian group M is injective if and only if M is divisible, that is, for every integer $n \in \mathrm{N}_{\geq 1}$, the multiplication by n from M to M is surjective.*

Proof. Assume first that M is an injective abelian group. Let m an element of M and $n \geq 1$ an integer. Consider the morphism from $f : n\mathbf{Z} \to M$ sending n to m. Since M is injective f can be extended in a morphism $\tilde{f}$ from $\mathbf{Z}$ to M. Then, since $\tilde{f}$ is linear,

$$m = f(n) = \tilde{f}(n) = n \cdot \tilde{f}(1)$$

and m is divisible by n.

Now, let M be a divisible abelian group. Let N be an abelian group and N' be a subgroup of N. Let $f' : N' \to M$ be a linear map, we need to extend f' to a linear map $f : N \to M$. Consider the set of all morphisms $\tilde{f} : \tilde{N} \to M$ extending f', where $\tilde{N}$ is an intermediate subgroup between N' and N. This is partially ordered and every chain has an upper bound so it admits at least one maximal element $\tilde{f} : \tilde{N}_0 \to M$. We will show that $\tilde{N}_0 = N$. Assume the inclusion is strict and choose an element x in $N \setminus \tilde{N}_0$. Consider its projection $\bar{x}$ to $N/\tilde{N}_0$. If $\bar{x}$ has infinite order then the group generated by $\tilde{N}_0$ and x is isomorphic to $\tilde{N}_0 \oplus \mathbf{Z}$ so $\tilde{f}$ can be extended to $\langle \tilde{N}_0, x \rangle$, which is a contradiction. So $\bar{x}$ must have finite order n with $n \geq 2$. Since M is divisible, there exists m in M such that $n \cdot m = \tilde{f}(nx)$ and again, we can extend $\tilde{f}$ to $\langle \tilde{N}_0, x \rangle$. This gives a contradiction, so M is injective. □

Example 3.21. The abelian groups $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$ are injective.

Theorem 3.22. *The abelian category* Ab *has enough injectives.*

Proof. Let N be an abelian group, we want to embed N into an injective abelian group M. Take $M := (\mathbf{Q}/\mathbf{Z})^{\mathrm{Hom}(N,\mathbf{Q}/\mathbf{Z})}$. Since $\mathbf{Q}/\mathbf{Z}$ is injective and arbitrary products of injective objects are injective, M is injective. Consider the map

$$\begin{cases} N & \to M \\ x & \mapsto (f(x))_{f \in \mathrm{Hom}(N,\mathbf{Q}/\mathbf{Z})}. \end{cases}$$

We will prove that this map is injective. Take $x \neq 0$ in N, it sufficed to find $f : N \to \mathbf{Q}/\mathbf{Z}$ such that $f(x) \neq 0$. Consider the subgroup $\mathbf{Z} \cdot x$ of N. If the order of x is finite, we can take $f(x) = \frac{1}{n}$. If the order of x is not finite, sending x to any non-zero element of $\mathbf{Q}/\mathbf{Z}$ gives such a map f. □

More generally, we have:

Theorem 3.23. *Let $\mathscr{A}$ be an abelian category.*

(i). If $\mathscr{A}$ has (arbitrary) direct sums, satisfies (Ab5)[6] *and has a generator then $\mathscr{A}$ has enough injectives.*

[6] i.e. filtered colimits are exact.

(ii). If $\mathscr{A}$ satisfies the conditions of the previous point and $\mathscr{B}$ is any category, then $\mathrm{Fun}(\mathscr{B}, \mathscr{A})$ also satisfies the above conditions and, in particular, $\mathrm{Fun}(\mathscr{B}, \mathscr{A})$ has enough injectives.

Corollary 3.24. *Let $\mathscr{T}$ be a category. The category $\mathrm{PreShvAb}(\mathscr{T})$ has enough injectives.*

In the above result, a generator of $\mathscr{A}$ is an object X of $\mathscr{A}$, such that for all Y in $\mathscr{A}$, there exists an epimorphism

$$\bigoplus_I X \to Y \to 0$$

with I arbitrary. Let $\mathscr{T}$ be a site. Let us give example of generators for $\mathrm{PreShvAb}(\mathscr{T})$ and $\mathrm{ShvAb}(\mathscr{T})$. For U an object in $\mathscr{T}$, we define the presheaf

$$\mathbf{Z}_U(V) = \bigoplus_{\mathrm{Hom}(V,U)} \mathbf{Z}.$$

In particular, for any abelian presheaf $\mathscr{F}$ there is a canonical isomorphism $\mathscr{F}(U) \simeq \mathrm{Hom}(\mathbf{Z}_U, \mathscr{F})$. Then the presheaf $\mathbf{Z} := \bigoplus_U \mathbf{Z}_U$ defines a generator of the category $\mathrm{PreShvAb}(\mathscr{T})$. Taking the sheafification $\mathbf{Z}^\sharp$, we get a generator for the category $\mathrm{ShvAb}(\mathscr{T})$.

We can then deduce from the first point of Theorem 3.23 the following result:

Theorem 3.25. *Let $\mathscr{T}$ be a site. The category $\mathrm{ShvAb}(\mathscr{T})$ has enough injectives.*

We can now define sheaf cohomology. Let $\mathscr{T}$ be a site. Note that the functor

$$\Gamma(U, -) : \mathrm{ShvAb}(\mathscr{T}) \to \mathrm{Ab}$$

is left exact as the composition of the left exact functor $\mathrm{ShvAb}(\mathscr{T}) \to \mathrm{PreShvAb}(\mathscr{T})$ and the exact functor $\Gamma(U, -) : \mathrm{PreShvAb}(\mathscr{T}) \to \mathrm{Ab}$. For $\mathscr{F}$ an abelian sheaf and i an integer, the i-the cohomology group of $\mathscr{F}$ is defined as the i-th derived functor of the global section functor:

$$H^i(U, \mathscr{F}) := \mathrm{R}^i\Gamma(U, \mathscr{F})$$

for any object U of $\mathscr{T}$.

4 Étale site

We would like to define a cohomology theory that is an algebraic geometry version of the singular cohomology for varieties over $\mathbf{C}$. A first guess would be to use the Zariski topology. However, this topology has not enough open sets: for example, for a complex variety, any two Zariski open sets meet. So, when computing the cohomology of a constant sheaf (i.e. the sheafification of a constant presheaf) we obtain that the restriction maps are surjective. This implies $H^i_{\mathrm{zar}}(X, \mathscr{F}) = 0$ for $i > 0$ and $\mathscr{F}$ constant, and the Zariski topology does not detect cohomology in higher degrees. Hence, we need to find a finer topology. To do that, we will first define morphisms that are algebraic analogues to local homeomorphisms. There are two obstructions for a morphism of complex varieties to be a local homeomorphism: firstly, there cannot be branch points and secondly, the dimensions of the fibers cannot vary. In the algebraic geometry world, a morphism with no branch points will be called unramified and a morphism with fibers of locally constant dimension will be called flat. An étale morphism will be a morphism that is flat and unramified.

4.1 Étale morphisms

We will assume that all rings are noetherian and all schemes are locally noetherian. Before defining unramified morphisms, let us recall a few facts about flat morphisms.

Definition 4.1. We say that a morphism of rings $f : A \to B$ is flat if the functor $- \otimes_A B : \mathrm{Mod}_A \to \mathrm{Mod}_B$ is exact. A morphism of schemes $f : X \to Y$ is flat if for all $y \in Y$ the map $\mathscr{O}_{X,f(y)} \to \mathscr{O}_{Y,y}$ is flat.

Note that equivalently, a morphism $f : X \to Y$ is flat if and only if for any open affines U of X and V of Y such that $f(V) \subset U$, the morphism $\Gamma(U, \mathscr{O}_X) \to \Gamma(V, \mathscr{O}_Y)$ is flat. Open immersions are flat. The property of being flat is stable by base change and by composition.

Example 4.2. (i). If K is a field, every K-module is flat.

(ii). If A is a ring and $S \subset A$ is a multiplicatively closed subset, then the localization $A \to A[S^{-1}]$ is flat. An A-module M is flat if and only if for every prime ideal $\mathfrak{p} \subset A$ (respectively every maximal ideal $\mathfrak{m} \subset A$), the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ (respectively the $A_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$) is flat.

(iii). Let A be a ring. Then $A[X_1, \dots, X_d]$ is flat over A (in other words: the affine space $\mathbb{A}^d_A$ is flat over $\mathrm{Spec}(A)$).

(iv). Let Z be an hypersurface in $\mathbb{A}^d_A$, i.e. a scheme of the form $\mathrm{Spec}(A[X_1, \dots, X_d])/(P)$ with $P \neq 0$. Then Z is flat over A if and only if for all maximal ideal $\mathfrak{m}$ in A, $Z \otimes_A k(\mathfrak{m})$ is not equal to $\mathbb{A}^d_{k(\mathfrak{m})}$. In other words, an hypersurface in $\mathbb{A}^d_A$ is flat if and only if its closed fibers over A all have the same dimension.

(v). Standard examples of non-flat morphisms are given by blowups. Consider for example the blowup $\widetilde{\mathbb{A}}^2_k$ of $\mathbb{A}^2_k = \mathrm{Spec}(k[x,y])$ at the origin. The points of $\widetilde{\mathbb{A}}^2_k$ can be described by the pairs $((x,y),[X:Y]))$ in $\mathbb{A}^2_k \times \mathbb{P}^1_k$ such that $xY = yX$. The fiber of $\widetilde{\mathbb{A}}^2_k \to \mathbb{A}^2_k$ over a point $(x,y) \neq (0,0)$ is given by a single point in $\mathbb{P}^1_k$ while the fiber over the origin is the entire projective line. The blowup $\widetilde{\mathbb{A}}^2_k$ can be covered by the two open affines $\mathrm{Spec}(k[x,\frac{y}{x}])$ and $\mathrm{Spec}(k[\frac{x}{y},y])$. The morphisms $k[x,y] \to k[x,\frac{y}{x}]$ and $k[x,y] \to k[\frac{x}{y},y]$ are not flat.

Definition 4.3. We say that a morphism of rings $f : A \to B$ of finite-type is unramified at a prime $\mathfrak{q} \in \mathrm{Spec}(B)$ if the ideal $\mathfrak{p} := f^{-1}(\mathfrak{q})$ generates the maximal ideal in $B_{\mathfrak{q}}$ (i.e. $\mathfrak{q}B_{\mathfrak{q}} = f(\mathfrak{p})B_{\mathfrak{q}}$) and $k(\mathfrak{q})$ is a finite separable field extension of $k(\mathfrak{p})$. We say that f is unramified if f is unramified at every prime. A morphism of schemes $f : Y \to X$ that is locally of finite-type is unramified at $y \in Y$ if $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}$ is a finite separable extension of $k(x)$. It is unramified if it is unramified at all $y \in Y$.

In particular, a morphism $f : Y \to X$ is unramified if and only if for all $x \in X$, its fibers $Y_x \to \mathrm{Spec}(k(x))$ is unramified and it can be proved that this is true if and only if all geometric fibers of f are unramified (see [Mil80, Proposition 3.2]). Open immersions are unramified. Moreover, the property of being unramified is stable by base change and composition.

Example 4.4. (i). Let k be a field. We denote by $\overline{k}$ an algebraic closure of k. Recall that a finite k-algebra A is separable over k if and only if it is isomorphic to a finite product of separable field extensions of k and this is true if and only if $A \otimes_K \overline{k}$ is isomorphic to a finite product of copies of $\overline{k}$. Using that, it can be proved that $f : Y \to X$ is unramified if and only if for all $x \in X$, the fiber Y_x is isomorphic to a co-product $\amalg_i \mathrm{Spec}(k_i)$, where the k_i are finite separable field extensions of $k(x)$.

(ii). Let k be a field. The morphisms $k[x] \to k[x,y]/(x+y)(x-y)$ is unramified everywhere except at the origin. The same goes for $k[x] \to k[x], x \mapsto x^2$ if $\mathrm{char}(k) \neq 2$.

(iii). Take $k := \mathbf{F}_p(t)$. The morphism $k[x] \to k[x,y]/(y^p - xy - t)$ is unramified everywhere except at $(x, y^p - t)$ where it becomes the inseparable extension $\mathbf{F}_p(t) \to \mathbf{F}_p(t^{\frac{1}{p}})$.

The following alternative definition of unramified morphism can also sometimes be useful:

Proposition 4.5. *Let $f : Y \to X$ be locally of finite type. We have the following equivalences:*

(i). f is unramified.

(ii). The sheaf $\Omega^1_{Y/X}$ is zero.

(iii). The diagonal morphism $\Delta_{Y/X} : Y \to Y \times_X Y$ is an open immersion.

Sketch of the proof. We just recall the main ideas for the proof, more details can be found in [Mil80, Chapter I, Proposition 3.5] or [StackProject, 02G3]. Assume assertion (i). Using the compatibility of $\Omega^1_{Y/X}$ with base change and localisation, the proof of (ii) can be reduced to prove that $\Omega_{B/A} = 0$ for $A \to B$ a local morphism between local rings. Using Nakayama's lemma, we see that it suffices then to check that $\Omega_{L/K}$ is zero for L/K a finite separable extension of fields. To prove that the second point implies the third one, first note that the diagonal morphism is always locally closed. So, we can find some open U such that $\Delta_{X/Y} : Y \to U$ is closed and we denote by $\mathscr{I}$ the associated ideal. Using that $\mathscr{I}/\mathscr{I}^2 \simeq \Omega^1_{Y/X}$, we can then find an open V in U such that $\mathscr{I}|_V = 0$ and $Y \simeq V \to U \to Y \times_X Y$ gives the open immersion we want. Assume now that $\Delta_{Y/X} : Y \to Y \times_X Y$ is an open immersion. Passing to geometric fibers, we can assume $X = \mathrm{Spec}(k)$ with k an algebraically closed field. If $y \to Y$ is a closed point of Y, we can use the hypothesis to prove that the diagonal morphism associated to $\mathrm{Spec}(\mathscr{O}_{Y,y}) \to \mathrm{Spec}(k)$ is an open immersion. Counting dimension, this yields $\mathrm{Spec}(\mathscr{O}_{Y,y}) \simeq \mathrm{Spec}(k)$ and it follows from the first point in Example 4.4 that f is unramified.

□

Note that, when working with affine schemes, the diagonal is always a closed immersion. But a closed immersion is open if and only if it is flat (see for example [StackProject, 0819]). So a morphism $A \to B$ of finite type is unramified if and only if $B \otimes_A B \to B$ is flat.

Definition 4.6. A morphism of schemes (or rings) is étale if it is flat and unramified.

Open immersions are étale. The property of being étale is stable by base change and composition. Moreover, it can be showed that if $f : X \to Y$ and $g : Y \to Z$ are morphisms of schemes with g unramified and $g \circ f$ étale then f is étale.

Example 4.7. (i). Let k be a field and $k \to A$ a finite k-algebra. Then A is étale if and only if $A \simeq L_1 \times \cdots \times L_n$ for some finite separable field extensions L_i/k.

(ii). **Jacobian criterion:** We say that a morphism of rings $A \to B$ is standard smooth if there exist integers $c \leq n$ and a presentation

$$B \simeq A[x_1, ..., x_n]/\langle f_1, \dots, f_c\rangle$$

such that $\det\left(\frac{\partial f_i}{\partial x_j}\right)_{1\leq i,j\leq c}$ is invertible in A. An étale morphism $A \to B$ is standard smooth. More precisely, a morphism $A \to B$ is étale if and only if there exists a presentation as above with $c = n$.

(iii). Suppose $Y \to X$ is a morphism of smooth affine $\mathbf{C}$-varieties. Then $Y \to X$ is étale if and only if $Y(\mathbf{C}) \to X(\mathbf{C})$ is a local homeomorphism of topological spaces.

Remark 4.8 (Relation between étale and smooth morphisms). There exists an equivalent definition of étale morphism, using the notion of relative dimension: for a morphism of schemes $f : X \to Y$ locally of finite type, we say that f has relative dimension $d \geq 0$ if every non-empty fiber X_y for $y \in Y$ has pure dimension d. For example, for every integer $d \geq 0$, the morphisms

$\mathbb{A}^d_S \to S$ and $\mathbb{P}^d_S \to S$ have relative dimension d. For every ring A and any integer $n \geq 1$, the morphism $f : \mathbb{A}^1_A \to \mathbb{A}^1_A$ given by $x \mapsto x^n$ has relative dimension 0. More generally, for any finite A-algebra B, the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ has relative dimension 0.

For $f : X \to Y$ is a morphism of affine schemes, we say that f is standard smooth if the induced ring map $\mathscr{O}(Y) \to \mathscr{O}(X)$ is standard smooth. If $f : X \to Y$ is a morphism of (arbitrary) schemes, we say that f is smooth at $x \in X$ if there exist affine open subsets $U \subset X$ and $V \subset Y$ with $x \in U$ and $f(U) \subset V$ such that the induced map $f|_U : U \to V$ is standard smooth. We say that f is smooth if it is smooth at every point of X. A morphism of schemes $f : X \to Y$ is smooth if and only if f is locally of finite presentation, flat and for every $y \in Y$, the geometric fibre $X_y = X \times_Y \mathrm{Spec}(k(y))$ is a non-singular variety (see [StackProject, 01VD, 01V7, 01V8]).

Using the Jacobian criterion, we obtain:

Proposition 4.9. *Let $f : X \to Y$ be a morphism of schemes. Then f is étale if and only if f is smooth of relative dimension* 0.

Moreover, it can be proved that smooth schemes are étale-locally like affine spaces: a morphism of schemes $f : X \to Y$ is smooth if and only if locally on the source and target, f can be written as follows:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathbb{A}^d_Y \\ & {\scriptstyle f}\searrow & \downarrow \\ & & Y \end{array}$$

where $d \geq 0$ is an integer and φ is étale.

4.2 The étale topology

Let X be a scheme. We denote by $\acute{\mathrm{E}}\mathrm{t}|_X$ the category of étale X-schemes[7]. Note that $\acute{\mathrm{E}}\mathrm{t}|_X$ has finite fiber products and any morphisms between étale X-schemes is étale. We say that a family of morphisms $\{\varphi_i : U_i \to U\}_{i \in I}$ in $\acute{\mathrm{E}}\mathrm{t}|_X$ is a covering if $U = \bigcup_{i \in I} \varphi_i(U_i)$. This defines a Grothendieck topology on $\acute{\mathrm{E}}\mathrm{t}|_X$ and we write $X_{\text{ét}}$ the site defined that way.

Remark 4.10. The site $X_{\text{ét}}$ is the small étale site. We can also define the big étale site $\mathrm{Sch}|_{X,\text{ét}}$: it is the category of all X-schemes endowed with the Grothendieck topology in which the coverings are the families of étale morphisms $\{\varphi : U_i \to U\}_{i \in I}$ such that $U = \bigcup_{i \in I} \varphi_i(U_i)$. Since a morphism between étale X-schemes is étale, there is a canonical morphism from $X_{\text{ét}}$ to $\mathrm{Sch}|_{X,\text{ét}}$. If $\mathscr{F}$ is an abelian sheaf on $\mathrm{Sch}|_{X,\text{ét}}$, then $\mathscr{F}|_{X_{\text{ét}}}$ is a sheaf on $X_{\text{ét}}$ and $H^i(X_{\text{ét}}, \mathscr{F}|_{X_{\text{ét}}}) = H^i(X, \mathscr{F})$ for all $i \geq 0$.

[7]In particular, X is a final object in $\acute{\mathrm{E}}\mathrm{t}|_X$.

4.2.1 The fpqc topology

The fpqc topology is coarser than the étale topology (see Lemma 4.14 below) but finer than the Zariski topology. In particular, we have than if a presheaf $\mathscr{F}$ is a sheaf for the fpqc topology, it will also be a sheaf for the étale topology. The fpqc topology has already been studied during the problem sessions (see Exercise Sheets 1 and 2). For clarity, we quickly summarize here what are the main results we have proved.

Definition 4.11. Let X be a scheme.

(i). Let U be a scheme over X. A Zariski covering of U is a family of morphisms $\{\varphi_i : U_i \to U\}_{i\in I}$ of schemes such that each φ_i is an open immersion and such that $U = \bigcup_{i\in I} \varphi_i(U_i)$. This defines a Grothendieck topology on $\mathrm{Sch}|_X$.

(ii). Let U be a scheme over X. An fpqc covering of U is a family $\{\varphi_i : U_i \to U\}_{i\in I}$ such that each φ_i is a flat morphism, $U = \bigcup_{i\in I} \varphi_i(U_i)$ and for each affine open $T \subset U$ there exists a finite set K, a map $\iota : K \to I$ and affine opens $T_{\iota(k)} \subset U_{\iota(k)}$ such that $T = \bigcup_{k\in K} \varphi_{\iota(k)}(T_{\iota(k)})$. This defines a Grothendieck topology on $\mathrm{Sch}|_X$.

Note that any Zariski covering is an fpqc-covering. If A is a ring, a A-module M is called faithfully flat if a sequence of A-modules $N_1 \to N_2 \to N_3$ is exact if and only if the sequence $M \otimes_A N_1 \to M \otimes_A N_2 \to M \otimes_A N_3$ is exact. We say that a morphism of schemes $f : X \to Y$ is faithfully flat if it is flat and surjective. A morphism of affine scheme $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ is an fpqc covering if and only if $A \to B$ is faithfully flat.

Lemma 4.12. *Let X be a scheme. For a presheaf $\mathscr{F}$ of sets (or abelian groups) on the fpqc site the following are equivalent:*

(i). $\mathscr{F}$ is an fpqc sheaf.

(ii). The gluing property is satisfied for fpqc coverings of the following types:

a) $\{U_i \to U\}_{i\in I}$ a surjective family of open immersions,

b) $\{V \to U\}$ a single surjective morphism of affine schemes.

The above lemma can be used to prove that any representable presheaf is a sheaf in the fpqc topology. We say that the fpqc topology is subcanonical. More precisely, we have:

Proposition 4.13. *(i). Let R' be a faithfully flat R-algebra, and let $R'' = R' \otimes_R R'$. Consider the two maps $R' \to R''$ given by $x \mapsto x \otimes 1$ and $x \mapsto 1 \otimes x$. The following diagram is exact:*

$$R \longrightarrow R' \rightrightarrows R''.$$

(ii). Let $f : S' \to S$ a faithfully flat and quasi-compact morphism of schemes and X and Y schemes over S. Denote by X', Y' (respectively X'', Y'') their base changes to S'

(respectively $S'' = S' \times_S S'$). Then the following diagram is exact:

$$\mathrm{Hom}_S(X,Y) \to \mathrm{Hom}_{S'}(X',Y') \rightrightarrows \mathrm{Hom}_{S''}(X'',Y'').$$

4.2.2 Étale sheaves

We will now give some example of étale sheaves. To be able to use the results from the preceding section, let us first prove the following lemma:

Lemma 4.14. *Any étale covering is an fpqc-covering.*

Proof. Let $\{U_i \xrightarrow{\varphi_i} U\}_{i\in I}$ be an étale covering. An étale morphism is flat and by construction, an étale covering is a family of jointly surjective morphisms, so we only have to check the quasi-compactness. Let $V \subset U$ be an affine open, and write $\varphi_i^{-1}(V) = \bigcup_{j\in J_i} V_{i,j}$ for some affine opens $V_{i,j} \subset U_i$. Since φ_i is open (étale morphisms are flat and locally of finite presentation so they are open), we obtain that $V = \bigcup_{i\in I}\bigcup_{j\in J_i} V_{i,j}$ is an open covering of V. But V is quasi-compact, so this covering admits a finite refinement. This concludes the proof. □

The étale topology being finer than the fpqc one, we obtain that any fpqc sheaf is an étale sheaf. In particular, we obtain:

Proposition 4.15. *The étale topology is subcanonical. More precisely, for X a scheme and Z an (arbitrary) X-scheme, the functor $U \mapsto \mathrm{Hom}_X(U,Z)$ is a sheaf of sets on $X_{\text{ét}}$.*

We also have the following analogue of Lemma 4.12.

Lemma 4.16. *Let X be a scheme. For a presheaf $\mathscr{F}$ of sets (or abelian groups) on $X_{\text{ét}}$ the following are equivalent:*

(i). $\mathscr{F}$ is a sheaf.

(ii). The gluing property is satisfied for coverings in $X_{\text{ét}}$ of the following types:

a) $\{U_i \to U\}_{i\in I}$ a surjective family of open immersions,

b) $\{V \to U\}$ a single surjective morphism of affine schemes.

Example: étale sheaf associated to a group scheme. Let X be a scheme and G be a group scheme over X[8]. We denote by G_X the sheaf on $X_{\text{ét}}$ represented by G. Then G_X is a sheaf of groups on $X_{\text{ét}}$: by definition, for each étale X-scheme U, the set $G_X(U) = \mathrm{Hom}_X(U,G)$ is equipped with a group structure. Moreover, if G is a commutative group scheme on X, then G_X is an abelian sheaf on $X_{\text{ét}}$.

[8]Recall that a group scheme over X is a pair (G,m) where G is a scheme over X and $m : G \times_X G \to G$ is a morphism of schemes over X such that for every scheme Y over X the pair $(G(X),m)$ is a group.

Example 4.17. (i). **The additive group $\mathbb{G}_a$.** It is defined by $\mathbb{G}_a := \operatorname{Spec} \mathbf{Z}[t]$, with multiplication law given by $\mathbf{Z}[t] \to \mathbf{Z}[t_1, t_2], t \mapsto t_1 + t_2$. For any scheme X, the base change $(\mathbb{G}_a)_X := \mathbb{G}_a \times_{\operatorname{Spec}(\mathbf{Z})} X$ is a group scheme over X and for every U in $X_{\text{ét}}$, we have

$$\begin{aligned}(\mathbb{G}_a)_X(U) &= \operatorname{Hom}_X(U, \operatorname{Spec}(\mathbf{Z}[t]) \times_{\operatorname{Spec}(\mathbf{Z})} X) \\ &= \operatorname{Hom}(U, \operatorname{Spec}(\mathbf{Z}[t])) \\ &= \operatorname{Hom}(\mathbf{Z}[t], \mathscr{O}(U)) \\ &= \mathscr{O}_U(U).\end{aligned}$$

We obtain the structure sheaf of the étale site $X_{\text{ét}}$.

(ii). **The multiplicative group $\mathbb{G}_m$.** It is defined by $\mathbb{G}_m := \operatorname{Spec}(\mathbf{Z}[t, t^{-1}])$, with multiplication law $t \mapsto t_1 t_2$. Note that for any ring R, we have $\mathbb{G}_m(R) = R^\times$ (and not $R \setminus \{0\}$). For a scheme X and U in $X_{\text{ét}}$, we have

$$\begin{aligned}(\mathbb{G}_m)_X(U) &= \operatorname{Hom}_X(U, \operatorname{Spec}(\mathbf{Z}[t, t^{-1}]) \times_{\operatorname{Spec}(\mathbf{Z})} X) \\ &= \operatorname{Hom}(U, \operatorname{Spec}(\mathbf{Z}[t, t^{-1}])) \\ &= \operatorname{Hom}(\mathbf{Z}[t, t^{-1}], \mathscr{O}(U)) \\ &= \mathscr{O}_U(U)^\times.\end{aligned}$$

(iii). **The group of roots of unity μ_n.** The group scheme of n-th roots of unity is defined by $\mu_n := \operatorname{Spec}(\mathbf{Z}[t]/(t^n - 1))$. Let X be a scheme and U in $X_{\text{ét}}$, similar computations as above give:

$$(\mu_n)_X(U) = \{s \in \mathscr{O}_U(U) \mid s^n = 1\}.$$

For each $n \in \mathrm{N}_{\geq 1}$ we have the following exact sequence of abelian sheaves on $X_{\text{ét}}$:

$$0 \to (\mu_n)_X \to (\mathbb{G}_m)_X \xrightarrow{n} (\mathbb{G}_m)_X$$

where $(\mathbb{G}_m)_X \xrightarrow{n} (\mathbb{G}_m)_X$ denotes the n-th power morphism $s \mapsto s^n$.

4.3 The étale fundamental group

For a scheme X, we denote by $\text{FÉt}|_X$ the category of finite étale X-schemes (the morphisms are the X-morphisms). We assume moreover that X is connected. Let $\overline{x} : \operatorname{Spec}(k) \to X$ be a geometric point. Consider the functor

$$F_{\overline{x}} : \begin{cases} \text{FÉt}|_X & \to \text{Set} \\ Y & \mapsto Y_{\overline{x}} = \operatorname{Hom}_X(\overline{x}, Y), \end{cases}$$

which associates to any finite étale cover of X its fibre over $\overline{x}$. This functor is not representable in the category of finite étale X schemes, however it is pro-représentable, that is:

Theorem 4.18. *Let X and $\overline{x}$ be as above. There exists a projective system $\widetilde{X} = (X_i)_{i\in I}$ of finite étale morphisms $X_i \to X$ (indexed by a directed set I), such that for every finite étale cover Y of X, we have*

$$F_{\overline{x}}(Y) = \mathrm{Hom}_X(\widetilde{X}, Y) =: \varinjlim_{i\in I} \mathrm{Hom}_X(X_i, Y).$$

Moreover, we can choose the X_i to be Galois coverings of X (see [StackProject, 0BN2]): this means that the cardinality of $\mathrm{Aut}_X(X_i)$ is equal to the degree[9] of $X_i \to X$.

Definition 4.19. The étale fundamental group of X at $\overline{x}$ is the group

$$\pi^1_{\text{ét}}(X, \overline{x}) := \mathrm{Aut}_X(\widetilde{X}) =: \varprojlim_{i\in I} \mathrm{Aut}_X(X_i).$$

Since each $\mathrm{Aut}_X(X_i)$ is a finite group, the étale fundamental group is a profinite group.

Note that in the above, we could take $\overline{x}$ to be of the form $\mathrm{Spec}(k) \to X$ with k separabely closed.

Example 4.20. (i). Let k be a field and let $s : k \to k^{\text{sep}}$ be a separable closure. Then we have $\pi^1_{\text{ét}}(\mathrm{Spec}(k), s) := \mathrm{Gal}(k^{\text{sep}}/k)$.

(ii). Let $X = \mathbb{A}^1_{\mathbf{C}} \setminus \{0\}$ and let $f_n : X \to X$ be the finite étale cover given by $x \mapsto x^n$. Then $\mathrm{Aut}(f_n) \simeq \mu_n(\mathbf{C})$, so $\pi^1_{\text{ét}}(X, \overline{x}) = \varprojlim_{n\geq 1} \mu_n(\mathbf{C}) \simeq \widehat{\mathbf{Z}}$.

(iii). For $X = \mathrm{Spec}(\mathbf{Z})$ and $\overline{x}$ as above, we can show that $F_{\overline{x}}$ is represented by $\mathrm{Spec}(\mathbf{Z})$, so $\pi^1_{\text{ét}}(X, \overline{x}) = \{1\}$ (see Exercise Sheet 4).

Remark 4.21 (Comparison with the usual fundamental group). Let X be a finite type scheme over $\mathbf{C}$. A finite étale map $f : X \to Y$ induces a covering (in the topological sense) $f : X(\mathbf{C}) \to Y(\mathbf{C})$ of finite degree. We get a functor $\mathrm{F\acute{E}t}|_X \to \mathrm{FCov}|_{X(\mathbf{C})}$. It can be proved (but this is hard) that this functor is an equivalence of categories. As a consequence, we obtain a natural map $\pi^1(X(\mathbf{C}), x) \to \pi^1_{\text{ét}}(X, \overline{x})$ with dense image, identifying the finite quotients of the usual fundamental group with the ones of the étale fundamental group. This implies that the étale fundamental group $\pi^1_{\text{ét}}(X)$ is isomorphic to the pro-finite completion of $\pi^1(X(\mathbf{C}))$.

Remark 4.22. Since every X_i is finite étale over X, the transition maps $X_i \to X_j$ are also finite étale so they are affine. It follows that the inverse limit $\widehat{X} = \varprojlim_i X_i$ exists as a scheme (see [StackProject, 01YV]). But $\widehat{X}$ is not locally of finite presentation over X and in particular, it is not étale over X. We will see that $\widehat{X} \to X$ defines in fact a pro-étale covering of X.

[9]The degree of a finite étale morphism is the cardinality of any geometric fiber.

4.4 Stalks of étale sheaves

4.4.1 Henselian rings

Definition 4.23. Let $(A, \mathfrak{m})$ be a local ring with residue field $k := A/\mathfrak{m}$. We say that A is henselian if Hensel's lemma holds in A that is, for every monic $f \in R[t]$ and every $a \in k$ which is a simple root of $\overline{f} \in k[t]$, there exists a unique lift $\widetilde{a} \in A$ of a such that $f(\widetilde{a}) = 0$. We say that A is strictly henselian if moreover k is separably closed.

Example 4.24. (i). Any complete discrete valuation ring is henselian ($\mathbf{Z}_p$, etc).

(ii). If A is henselian with residue field k then $\pi^1_{\text{ét}}(\mathrm{Spec}(A)) \simeq \pi^1_{\text{ét}}(\mathrm{Spec}(k)) \simeq \mathrm{Gal}(k^{\mathrm{sep}}/k)$.

Definition 4.25. Let A be a local ring. An henselization of A is an henselian extension[10] $A \to A^{\mathrm{h}}$ such that every henselian extension $A \to B$ factors through A^{h}. A strict henselization of A is a strictly henselian extension $A \to A^{\mathrm{sh}}$ such that every strictly henselian extension $A \to B$ factors through A^{sh}.

One can show that the henselization (respectively strict henselization) of a local ring A exists and is unique up to isomorphism. In general, one constructs A^{sh} as the filtered inductive limit of all étale A-algebras.

Example 4.26. The henselization (respectively strict henselization) of $\mathbf{Z}_{(p)}$ is given by the integral closure of $\mathbf{Z}_{(p)}$ in $\mathbf{Z}_p$ (respectively $\mathbf{Z}_p^{\mathrm{unr}}$).

4.4.2 Geometric point and stalks

Definition 4.27. Let X be a scheme.

(i). Let $x \in X$. An étale neighborhood of x in X is an étale morphism of schemes $U \to X$ together with a point $u \in U$ mapping to x.

(ii). Let $\overline{x} : \mathrm{Spec}(k) \to X$ be a geometric point. An étale neighborhood of $\overline{x}$ in X is an étale morphism of schemes $U \to X$ together with a geometric point $\overline{u} : \mathrm{Spec}(k) \to U$ mapping to $\overline{x}$. In other words, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(k) & \xrightarrow{\overline{u}} & U \\ & \searrow^{\overline{x}} & \downarrow \\ & & X \end{array}$$

[10]An extension of A is a local ring B together with a local morphism $A \to B$ (i.e. the inverse image of the maximal ideal of B is the maximal ideal of A).

Example 4.28. (i). Let X be a scheme and $x \in X$. A Zariski open neighborhood of x defines in particular an étale neighborhood of x.

(ii). The morphism $\mathrm{Spec}(k') \to \mathrm{Spec}(k)$ for any finite separable extension k'/k defines an étale neigborhood.

Note that the category of étale neighborhoods of $\overline{x}$ in X is filtered.

Definition 4.29. Let X be a scheme, and let $\overline{x}$be a geometric point of X. Let $\mathcal{F}$ be an abelian sheaf on $X_{\text{ét}}$. The stalk of $\mathcal{F}$ at $\overline{x}$ is the abelian group $\mathcal{F}_{\overline{x}} := \varinjlim_{(U,\overline{u})} \mathcal{F}(U)$ where the limit is taken over the étale neighborhood of $\overline{x}$.

Note that a sequence $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ of abelian sheaves on $X_{\text{ét}}$ is exact in $\mathrm{Shv}(X_{\text{ét}})$ if and only if for every geometric point $\overline{x}$ of X, the sequence of abelian groups $\mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}} \to \mathcal{H}_{\overline{x}}$ is exact.

Example 4.30 (Strict localisation)**.** Let X be a scheme and $\overline{x}$ a geometric point. The strict localisation of X at $\overline{x}$ is the ring $\mathscr{O}_{X,\overline{x}} := \varinjlim_{(U,\overline{u})} \mathscr{O}(U)$. It is a local ring with residue field $k(\overline{x})$. Since every Zariski neighborhood is an étale neighborhood, there is a canonical map $\mathscr{O}_{X,\overline{x}} \to \mathscr{O}_{X,x}$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{Spec}(\mathscr{O}_{X,\overline{x}}) & \longrightarrow & \mathrm{Spec}(\mathscr{O}_{X,x}) & \longrightarrow & X \\ \uparrow & & \uparrow & \nearrow & \\ \mathrm{Spec}(k(\overline{x})) & \longrightarrow & \mathrm{Spec}(k(x)) & & \end{array}$$

Proposition 4.31. *Let X be a scheme, and let $x \in X$. Let $k(x)^s$ be a separable closure of $k(x)$ and let $\overline{x}$ be the associated geometric point of X. Then $\mathscr{O}_{X,\overline{x}}$ is the strict henselization of the local ring $\mathscr{O}_{X,x}$.*

In particular, note that $\mathscr{O}_{X,\overline{x}}$ depends only on $\mathscr{O}_{X,x}$.

4.5 Cohomology of a point

If G is a topological group, a G-set Z is said to be continuous if the map $G \times Z \to Z$ is continuous, where Z is given the discrete topology. This is equivalent to say that every element of Z has open stabilizer in G (in particular, every G-orbit in Z is finite).

Proposition 4.32. *Let k be a field and k^{sep} a separable closure of k. Consider*

$$G := \mathrm{Aut}_{\mathrm{Spec}(k)}(\mathrm{Spec}(k^{\mathrm{sep}})) = \mathrm{Gal}(k^{\mathrm{sep}}/k)$$

as a topological group and denote $G - \mathrm{Set}$ *(respectively* $\mathscr{C}^0(G - \mathrm{Set})$*) the category of (respectively continuous) left* G*-sets. Then the functor*

$$F : \begin{cases} \mathrm{\acute{E}t_k} & \to G - \mathrm{Set} \\ X & \mapsto \mathrm{Hom}_k(\mathrm{Spec}(k^{\mathrm{sep}}), X) = X(k^{\mathrm{sep}}) \end{cases}$$

induces an equivalence of category between $\mathrm{\acute{E}t}_k$ *and* $C^0(G - \mathrm{Set})$.

Sketch of the proof. Let us first check that the functor F is well-defined. For X a k-scheme, an element of $X(k^{\mathrm{sep}})$ is a point x of X together with a k-embedding $k(x) \to k^{\mathrm{sep}}$. If moreover X is étale over k, then $k(x)$ is a finite separable extension of k. So $X(k^{\mathrm{sep}})$ is indeed a continuous G-set and the functor F is well-defined.

To prove that F defines an equivalence of categories is to construct a left adjoint $\widetilde{F}$ to F then prove that the unit and counit $1 \to F \circ \widetilde{F}$ and $\widetilde{F} \circ F \to 1$ are isomorphisms. To prove the existence of a left adjoint, it suffices to show that for all Z continuous G-set, the functor

$$X \mapsto \mathrm{Hom}_G(Z, X(k^s)) \tag{4.5.0.1}$$

is representable (and we will define $\widetilde{F}(Z)$ as the étale k-scheme representing the above functor).

Decomposing Z into a disjoint union of its orbits, we can reduce the proof to $Z := G/H$ for H an open subgroup of G. Then H corresponds to a finite separable extension k' of k, with $k' \subset k^{\mathrm{sep}}$. Then for X an étale k-scheme,

$$\mathrm{Hom}_G(G/H, X(k^s)) \simeq X(k^{\mathrm{sep}})^H \simeq X(k')$$

and we obtain that k' represents the functor (4.5.0.1) when $Z = G/H$ (in other words we have $\widetilde{F}(G/H) := \mathrm{Spec}(k')$).

To finish the proof we still have to check that for Z a G-set, $Z \to F \circ \widetilde{F}(Z)$ is an isomorphism and for X an étale k-scheme, $X \to \widetilde{F} \circ F(Z)$. The proof of the first part can be reduced to the case where $Z = G/H$ and the one of the second part to the case $X = k'$ with k' a finite separable extension of k. The result then follows from the definition of $\widetilde{F}(G/H)$. □

Let $G - \mathrm{Mod}$ be the category of continuous G-modules, i.e. the (discrete) abelian groups endowed with a continuous and linear action of G.

Theorem 4.33. *The stalk functor defines an equivalence of categories:*

$$\begin{cases} \mathrm{ShvAb}((\mathrm{Spec}(k))_{\acute{e}t}) & \to \mathrm{G - Mod} \\ \mathscr{F} & \mapsto \varinjlim_{k'/k \text{ finite ext. inside } k^{\mathrm{sep}}} \mathscr{F}(\mathrm{Spec}(k')) \\ \mathscr{F}_M : (k'/k \mapsto M^{\mathrm{Gal}(k^{\mathrm{sep}}/k')}) & \leftarrow M \end{cases}$$

Let us say a few words about the proof. It suffices to prove that both sides are equivalent to the category $\mathrm{ShvAb}(\mathcal{T}_G)$ of abelian sheaves on the site $\mathcal{T}_G$ of continuous $G - \mathrm{Set}$. For the right-hand side it follows from Lemma 2.9. As for the other side, consider the morphism of sites $\mathrm{Spec}(k)_{\text{ét}} \to \mathcal{T}_G$ from the above proposition. We need to prove that it defines an equivalence of categories $\mathrm{ShvAb}(\mathcal{T}_G) \to \mathrm{ShvAb}((\mathrm{Spec}(k))_{\text{ét}})$. This follows from the following observation:

Lemma 4.34. *Let $f : X \to Y$ be a morphism of étale schemes over k. The map f is surjective if and only if $f(k^{\mathrm{sep}}) : X(k^{\mathrm{sep}}) \to Y(k^{\mathrm{sep}})$ is surjective.*

The two results below follow immediately from the theorem:

Corollary 4.35. *Let k be a field, k^{sep} a separable closure of k and let denote G the Galois group of k^{sep}/k. Then for any $\mathscr{F}$ abelian sheaf on* $\mathrm{Spec}(k)_{\text{ét}}$*,*

$$H^i_{\text{ét}}(\mathrm{Spec}(k), \mathscr{F}) = H^i(G, M)$$

where $M := \varinjlim_{k'/k \text{ finite ext. inside } k^{\mathrm{sep}}} \mathscr{F}(\mathrm{Spec}(k'))$.

Corollary 4.36. *If k is separably closed then the functor $\mathscr{F} \mapsto \mathscr{F}(k)$ induces an equivalence of categories* $\mathrm{ShvAb}(X_{\text{ét}}) \xrightarrow{\sim} \mathrm{Ab}$ *and* $H^i_{\text{ét}}(\mathrm{Spec}(k), \mathscr{F}) = 0$ *for* $i > 0$.

In other words, a geometric point has no cohomology in strictly positive degree.

5 Overview of ℓ-adic étale cohomology

Let ℓ be a prime number. In this section, we will briefly explain how the above construction of étale cohomology can be used to define a Weil cohomology in the ℓ-adic case. More details about ℓ-adic sheaves can be found in [FR88, Chapter 1, § 12].

5.1 Local systems and constructible sheaves

Let A be a discrete abelian group and X a scheme. We denote by A_X the sheaf associated to the presheaf $U \mapsto A$ on $X_{\text{ét}}$. The sheaf A_X is called the constant sheaf with values in A. For $U \in X_{\text{ét}}$, we can show that

$$A_X(U) = \{s : U \to A \text{ such that } s \text{ is locally constant for the Zariski topology }\}.$$

In particular, if U is connected, then $A_X(U) = A$. Moreover, the stalk of A_X at a geometric point $\overline{x}$ of X is equal to A.

We have

$$\begin{aligned} A_X(U) &= \prod_{\text{connected components of } U} A \\ &= \operatorname{Hom}_X(U, \coprod_A X). \end{aligned}$$

In other words, the constant sheaf A_X is represented by the étale group scheme $\coprod_A X$ with the group structure induced by A.

Example 5.1. Take $A = \mathbf{Z}/n\mathbf{Z}$. Then the sheaf $(\mu_n)_X$ is isomorphic to the constant sheaf $(\mathbf{Z}/n\mathbf{Z})_X$ if and only if there exists at least one primitive root n-th of 1 on X. If we assume that n is relatively prime to the characteristics of all local residue fields of X, then we obtain that the sheaf $(\mu_n)_X$ is locally isomorphic to the sheaf $(\mathbf{Z}/n\mathbf{Z})_X$, i.e. there is a covering $\{X_i \to X\}$ in $X_{\text{ét}}$, such that the restrictions $(\mu_n)_X|_{X_i}$ are isomorphic to $(\mathbf{Z}/n\mathbf{Z})_{X_i}$.

Definition 5.2. Let X be a scheme. A locally constant sheaf on $X_{\text{ét}}$ (or local system on $X_{\text{ét}}$) is a sheaf $\mathscr{F}$ on $X_{\text{ét}}$ which is locally constant for the étale topology, i.e. there exists an étale covering $\{U_i \to X\}_{i\in I}$ such that for all i, the restriction of $\mathscr{F}$ to U_i is a constant sheaf. We say that a locally constant sheaf $\mathscr{F}$ on $X_{\text{ét}}$ is finite if its stalks are finite abelian groups.

Proposition 5.3. *Let X be a connected scheme and let $\mathscr{F}$ be a locally constant sheaf on $X_{\text{ét}}$. Then the stalks of $\mathscr{F}$ are all non canonically isomorphic: there exists an abelian group A and an étale covering $\{U_i \to X\}_{i\in I}$ of X such that $\mathscr{F}|_{U_i} \simeq A_{U_i}$ for all $i \in I$.*

Proof. By definition there exists a covering $\{U_i \to X\}_{i\in I}$ such that $\mathscr{F}|_{U_i} \simeq A_i$ for some A_i abelian groups. If $U_i \times_X U_j \neq$ then A_i and A_j are isomorphic. For A an abelian group define $I_A := \{i \in I \mid A_i \simeq A\}$ and $U_A := \bigcup_{I_A} \mathrm{Im}(U_i \to X)$. Then U_A is open (since an étale map is open) and if A and A' are two non isomorphic abelian groups then U_A and $U_{A'}$ are disjoint. Since the set of the U_A's cover X and X is connected, we obtain that there exists some A such that $X = U_A$. This concludes the proof. □

Let us consider the case of a point $X := \mathrm{Spec}(k)$ where k is a field. Let k^{sep} be a separable closure of k and G the Galois group of k^{sep}/k. We have seen that the category of abelian sheaves on $(\mathrm{Spec}(k))_{\text{ét}}$ is equivalent to the category of continuous G-modules. Let $\mathscr{F} \in \mathrm{ShvAb}((\mathrm{Spec}(k))_{\text{ét}})$ and let M be the corresponding Galois module. Then $\mathscr{F}$ is constant if and only if G acts trivially on M and F is locally constant if and only if there exists a finite separable extension k' of k inside k^{sep} such that $\mathrm{Gal}(k^{\mathrm{sep}}/k')$ acts trivially on M (in other words the action of G on M factors through a finite quotient).

We have the following theorem (see [StackProject, 0DV5]).

Theorem 5.4. *Let X be a connected scheme and let $\overline{x}$ be a geometric point of X. There is an equivalence of categories between the finite locally constant abelian sheaves on $X_{\text{ét}}$ and the $\pi^1_{\text{ét}}(X, \overline{x})$-modules.*

The proof of this theorem is in two steps.

- One first proves a general version of the Galois correspondence (see [StackProject,]): taking the $\overline{x}$-points induces an equivalence of categories

$$\begin{cases} \mathrm{F\acute{E}t}|_X & \xrightarrow{\sim} \{\text{Finite } \pi^1_{\text{ét}}(X, \overline{x})\text{-sets}\} \\ Y & \mapsto Y_{\overline{x}} = \mathrm{Hom}_X(\overline{x}, Y). \end{cases}$$

  The strategy to prove this result is similar of the one used in the proof of Proposition 4.32 (see for example [StackProject, 0BND]).

- We then use étale descent to prove that a sheaf $\mathscr{F}$ is finite locally constant if and only if $\mathscr{F}$ is the representable sheaf h_U associated to some finite étale cover $U \to X$ (see [StackProject, 03RV]).

Definition 5.5. Let X be a quasi-compact and quasi-separated scheme. A sheaf $\mathscr{F}$ on $X_{\text{ét}}$ is said to be constructible if there exists a finite decomposition of X into locally closed subsets

$$X = \coprod_i X_i$$

such that $\mathscr{F}|_{X_i}$ is finite locally constant for all i.

It can be showed that if X is a noetherian scheme then the full subcategory of constructible sheaves is abelian.

5.2 ℓ-adic sheaves

Definition 5.6. Let X be a noetherian scheme. A $\mathbf{Z}_\ell$-sheaf on X is an inverse system $\{\mathscr{F}_n\}_{n\geq 1}$ where for all n, $\mathscr{F}_n$ is a constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-module on $X_{\text{ét}}$ and the transition maps $\mathscr{F}_{n+1} \to \mathscr{F}_n$ induce isomorphisms $\mathscr{F}_{n+1} \otimes_{\mathbf{Z}/\ell^{n+1}\mathbf{Z}} \mathbf{Z}/\ell^n\mathbf{Z} \xrightarrow{\sim} \mathscr{F}_n$.

The category of $\mathbf{Z}_\ell$-sheaves on X noetherian is abelian. We say that a $\mathbf{Z}_\ell$ sheaf $\mathscr{F}$ is torsion if there exists n such that the map $\ell^n : \mathscr{F} \to \mathscr{F}$ is zero. We define the category of $\mathbf{Q}_\ell$-sheaves on X as the (Serre) quotient of the category of $\mathbf{Z}_\ell$-sheaves by the subcategory of torsion sheaves. Concretely, the objects of the resulting category are $\mathbf{Z}_\ell$-sheaves and the morphisms are given by

$$\mathrm{Hom}_{\mathbf{Q}_\ell}(\mathscr{F}, \mathcal{G}) = \mathrm{Hom}_{\mathbf{Z}_\ell}(\mathscr{F}, \mathcal{G}) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

There is a natural functor $\mathscr{F} \mapsto \mathscr{F} \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell$ going from the category of $\mathbf{Z}_\ell$-sheaves to the one of $\mathbf{Q}_\ell$-sheaves, right adjoint to the inclusion functor.

If X is a separated scheme of finite type over an algebraically closed field k and $\mathscr{F} = \{\mathscr{F}_n\}_{n\geq 1}$ is a $\mathbf{Z}_\ell$-sheaf on X, then we define, for all $i \geq 0$

$$H^i_{\text{ét}}(X, \mathscr{F}) := \varprojlim_{n\geq 1} H^i_{\text{ét}}(X, \mathscr{F}_n) \quad \text{and} \quad H^i_{\text{ét}}(X, \mathscr{F} \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell) := H^i_{\text{ét}}(X, \mathscr{F}) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

The i-th ℓ-adic cohomology group of X is $H^i_{\text{ét}}(X, \mathbf{Q}_\ell)$.

Example 5.7. If we want to obtain a cohomology compatible with singular cohomology when working with complex schemes, it is necessary to see $\mathbf{Q}_\ell$ as a $\mathbf{Q}_\ell$-sheaf and not as the constant étale sheaf associated to $\mathbf{Q}_\ell$. For example if X is a smooth projective connected curve of genus g over an algebraically closed field k, then we have

$$H^i_{\text{ét}}(X, \mathbf{Q}_{\ell,X}) = \begin{cases} \mathbf{Q}_\ell & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases} \quad \text{whereas} \quad H^i_{\text{ét}}(X, \mathbf{Q}_\ell) = \begin{cases} \mathbf{Q}_\ell & \text{if } i = 0 \\ \mathbf{Q}_\ell^{2g} & \text{if } i = 1 \\ \mathbf{Q}_\ell & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

where in the first case $\mathbf{Q}_{\ell,X}$ is the constant étale sheaf associated to $\mathbf{Q}_\ell$ and in the second case $\mathbf{Q}_\ell$ is seen as a $\mathbf{Q}_\ell$-sheaf.

When working over an algebraically closed field, the groups $H^i(X, \mathbf{Q}_\ell)$'s are finite $\mathbf{Q}_\ell$-vector spaces and it can be proved that we obtain that way a nice Weil cohomology. However because of the fact that the cohomology groups do not arise as derived functors, in other cases (when the cohomology groups are not finite) there can be some functoriality problems, especially when trying to get the usual long exact sequence associated to a short exact sequence of $\mathbf{Q}_\ell$-sheaves. One possible alternative definition was proposed by Jannsen in [Jann88], this construction is often called continuous ℓ-adic étale cohomology. Jannsen defines $H^i_{\text{ét,cont}}(X, \{\mathscr{F}_n\}_n)$ as the derived functor of

$$\begin{cases} \{ \text{ Inverse system } \{\mathscr{F}_n\}_n \text{ of étale sheaves on } X \ \} & \to \mathrm{Ab} \\ \{\mathscr{F}_n\}_n & \mapsto \varprojlim_n \mathscr{F}_n(X). \end{cases}$$

In particular, this yields a short exact sequence:

$$0 \to \mathrm{R}^1\mathrm{lim}_n H^{i-1}_{\text{ét}}(X, \mathbf{Z}/\ell^n\mathbf{Z}) \to H^i_{\text{ét,cont}}(X, \mathbf{Z}_\ell) \to \mathrm{lim}_n H^i_{\text{ét}}(X, \mathbf{Z}/\ell^n\mathbf{Z}) \to 0$$

and we see that in the cases where the groups $H^{i-1}_{\text{ét}}(X, \mathbf{Z}/\ell^n\mathbf{Z})$'s are finite, the $\mathrm{R}^1\mathrm{lim}_n H^{i-1}$ will be zero and we recover the previous definition. One advantage of this definition is that for X a scheme of finite type over k, it satisfies a Hochschild-Serre spectral sequence:

$$E_2^{p,q} := H^p(\mathrm{Gal}(\overline{k}/k), H^q_{\text{ét,cont}}(X_{\overline{k}}, \mathbf{Z}_\ell(r))) \Rightarrow H^{p+q}_{\text{ét,cont}}(X, \mathbf{Z}_\ell(r)).$$

In order to do more sophisticated applications however, this is still not sufficient: it is often useful to work directly at the level of the derived categories so we need to understand what are the derived categories of $\mathbf{Z}_\ell$- and $\mathbf{Q}_\ell$-sheaves. Deligne ([Del80]) and Ekedahl ([Eke90]) define the derived category of constructible $\mathbf{Z}_\ell$-sheaves as the 2-limit of the derived categories of constructible $\mathbf{Z}/\ell^n\mathbf{Z}$-sheaves. However the objects in this category are complicated to work with and the idea of Bhatt and Scholze was to define a category that recovers this previous definition but can be described as an honest derived category (and not as a limit). To understand where their definition comes from, we need to understand why working with étale $\mathbf{Z}/\ell^n\mathbf{Z}$-coefficients works well and what is the problem of $\mathbf{Z}_\ell$-coefficients: this is in fact due to representability. We have seen that étale descent implies that finite locally constant étale sheaves are representable by finite étale morphisms. Now we would like to be able to take limit of locally constant sheaves: this means that we need to enlarge the category of finite étale morphisms by adding limits. We will then define pro-étale morphisms as limit of étale morphisms.

5.3 Ind-étale algebras

Definition 5.8. Let $A \to B$ be a map of rings. We say that f is ind-étale if it is a filtered colimit of étale A-algebras.

Note that the property of being ind-étale is stable by base change and composition and a filtered colimit of ind-étale maps is ind-étale. If $A \to B$ is an ind-étale map with $B = \varinjlim_i B_i$ then as topological spaces

$$|\operatorname{Spec}(\varinjlim_i B_i)| \simeq \varinjlim_i |\operatorname{Spec}(B_i)|.$$

Example 5.9. (i). If A is a ring and $\mathfrak{p} \in \operatorname{Spec}(A)$ then $A \to A_{\mathfrak{p}}$ is ind-étale.

(ii). Let k be a field and k^{sep} a separable closure of k. Then $k \to k^{\mathrm{sep}}$ is ind-étale.

(iii). Let k be an algebraically closed field. Then, for any profinite set S, there exists an ind-étale map $k \to A$ such that $S \simeq |\operatorname{Spec}(A)|$ (as topological spaces).

We would like to define pro-étale morphisms as the dual of ind-étale morphisms, i.e. as limit of étale maps, and using this notion to construct the pro-étale site of a scheme X (see for example [Sch13]). However, this is not exactly the definition used by Bhatt and Scholze in [BS13]. Rather, they use the notion of weakly étale map:

Definition 5.10. A morphism of schemes $f : X \to Y$ is weakly étale if it is flat and the diagonal morphism $Y \to Y \times_X Y$ is flat.

We will see later that in fact weakly étale maps and ind-étale maps generate the same topology. The reason why they prefer to use weakly étale morphisms instead of pro-étale ones is because the property of being proétale is not local on the target: an example of a morphism that is locally proétale but not globally proétale is explained in [BS13, 4.1.12]. We briefly explain here the main ideas to construct this example: take S the one-point compactification of $\mathbf{Z}$. Note that it can be realized as the image of the map

$$\begin{cases} \mathbf{Z} \amalg \{\infty\} & \to \mathbf{C} \\ \infty & \mapsto -1 \\ n & \mapsto \exp(\pi i(1 - \frac{1}{2^n})) \text{ if } n \geq 0 \\ n & \mapsto \exp(\pi i(2^n - 1)) \text{ if } n \leq 0. \end{cases}$$

The set S is equipped with a translation operator $T : n \mapsto n+1$ fixing the point at infinity. Choose X_1 and X_2 two irreducible smooth curves inside $\mathbb{A}^1_{\mathbf{C}}$ that meet transversally in two points p and q. Let $X := X_1 \cup X_2 \subset \mathbb{A}^1_{\mathbf{C}}$. Define $Y \to X$ as the scheme obtained by glueing $S \otimes X_1$ and $S \otimes X_2$ using the identity at p and the translation T at q.

[A drawing of $Y \to X$ may be added later.]

Then $Y \to X$ is locally proétale: away from p (respectively q) this becomes the proétale map $S \otimes (X \setminus \{p\}) \to X \setminus \{p\}$ (respectively $S \otimes (X \setminus \{q\}) \to X \setminus \{q\}$). However the map is not globally proétale. Indeed, assume that Y can be written as a limit $\varprojlim_i Y_i$ with $Y_i \to X$ étale and denote by π_i the projections $Y \to Y_i$. Consider the section $s : X \to Y$ given by the point at infinity $\infty \in S$. Then $s_i := \pi_i \circ s : X \to Y_i$ defines a section to the étale map $Y_i \to X$. This means that Y_i can be decomposed as $X \coprod X_i$ where the inclusion $X \to Y_i$ is given by s_i. We obtain that $s(X)$ can be written as the intersection over i of clopen $U_i := \pi^{-1}(X)$ of Y_i. Looking at the fiber over p, we see that each $U_{i,p}$ will be a clopen of S stable under the action of T, so for any i, $U_{i,p} = S$. This gives $s(X)_p = S$, but by construction $s(X) = \{\infty\}$, contradiction.

6 Weakly contractible objects

We will define define precisely the pro-étale site $X_{\text{proét}}$ in a later section and we will see that this site as the property of being "locally contractible": this means that there exists a pro-étale covering $\{U_i \to X\}_{i\in I}$ such that for all covering $\{V_{i,j} \to U_i\}_{j\in J_i}$ the natural map $\coprod_{j\in J_i} V_{i,j} \to U_i$ admits a section (in particular, this implies that $H^n(U_i, \mathscr{F}) = 0$ for $\mathscr{F}$ an abelian sheaf and $n > 0$, so we will say that U_i is weakly contractible). The goal of this section if to prove the ring version if this property. More precisely, we define:

Definition 6.1. Let A be a ring. We say that A is w-contractible if every faithfully flat ind-étale map $A \to B$ has a section.

The main result is the following:

Theorem 6.2. *For any ring A, there is an ind-étale faithfully flat A-algebra A' with A' w-contractible.*

The proof will be in four steps:

(i). We first prove a Zariski version of the result: for A a ring, there exists an A-algebra A^Z such that $A \to A^Z$ is a faithfully flat ind-(Zariski localization) and A^Z is w-local.

(ii). We prove that there is a surjection from a pr-finite set $T \to \operatorname{Spec}(A^Z/J_{A^Z})$ where J_{A^Z} is the Jacobson radical of A^Z.

(iii). We give T a structure of affine scheme $\operatorname{Spec}(A_0)$.

(iv). The ring A' will be obtained by taking the henselianization of A_0 along A^Z, *i.e.* the colimit of the étale A^Z-algebras B with a map to A_0.

6.1 Construction of the localization functor

We first construct the localization functor for spectral spaces X and then transfer the result to the ring category. Morally X^Z will be the "smallest" Zariski cover of X.

6.1.1 Localization of spectral spaces

A spectral space is a topological space that is homeomorphic to the spectrum of a commutative ring. In other words, spectral spaces are defined as the image of the functor sending a ring to its set of primes equipped with the Zariski topology:

$$|\operatorname{Spec}(-)| : \text{Ring}^{\text{op}} \to \text{Top}.$$

A continuous map $f : X \to Y$ of spectral spaces is called spectral if the inverse image of a quasi-compact open is quasi-compact. We will denote by $\mathcal{S}$ the category of spectral spaces. Spectral spaces can be written as limits of finite spectral spaces (or equivalently, finite T_0-spaces). We will denote by $\mathcal{S}_f$ the category of finite spectral spaces and we have $\mathcal{S} = \mathrm{Pro}(\mathcal{S}_f)$.

Definition 6.3. We say that a spectral space X is w-local if it satisfies:

(i). All open covers split, i.e. for every open cover $\{U_i \hookrightarrow X\}_i$, the map $\coprod_i U_i \to X$ has a section.

(ii). The subspace $X^c \subset X$ of closed points is closed.

We say that a map $f : X \to Y$ of w-local spaces is w-local if it is spectral and $f(X^c) \subset Y^c$.

We denote by $\mathcal{S}^{\mathrm{wl}}$ the subcategory of w-local spaces with w-local maps. Then $\mathcal{S}^{\mathrm{wl}}$ admits all small limits and the inclusion $i : \mathcal{S}^{\mathrm{wl}} \to \mathcal{S}$ preserves limits (see [BS13, 2.1.9]). A finite disjoint unions of w-local spectral spaces is w-local. If $X \in \mathcal{S}^{\mathrm{wl}}$ and $Z \subset X$ is closed, then $Z \in \mathcal{S}^{\mathrm{wl}}$. This is because any open cover of Z extends to an open cover of X (adding $X \setminus Z$) so they split and $Z^c = X^c \cap Z$. We give below (Example 6.7) some other examples of w-local spaces.

Lemma 6.4. *A spectral space X is w-local then every connected component of X has a unique closed point and the composition $X^c \to X \to \pi_0(X)$ is a homeomorphism.*

Here, $\pi_0(X)$ is the set of connected components of X, equipped with the quotient topology induced by the canonical projection $\pi : X \to \pi_0(X)$. For X a spectral space, it can be showed that $\pi_0(X)$ is profinite (see [StackProject, 0906]).

Proof. Let $Y \subset X$ be a connected component of X. Since Y is closed and X w-local, Y is w-local. Take y_1 and y_2 two closed points in Y and assume $y_1 \neq y_2$. Consider the open cover $\{Y \setminus \{y_1\}, Y \setminus \{y_2\}\}$ of Y. Since Y is w-local, the map $(Y \setminus \{y_1\}) \coprod (Y \setminus \{y_2\}) \to Y$ admits a section. This means that Y can be decomposed into two clopen, which contradicts the fact that Y is connected. This proves that Y has at most one closed point. Since a closed subspace of a spectral space is spectral and any spectral space has at least one closed point, we obtain that Y has a unique closed point.

This proves that the map $X^c \to \pi_0(X)$ is bijective. To prove that it is an homeomorphism, we apply the following lemma:

Lemma 6.5. *Let $f : Z_1 \to Z_2$ be a continuous map of topological spaces. If f is bijective, Z_1 quasi-compact and Z_2 Hausdorff then f is an homeomorphism.*

□

Remark 6.6. (i). The above lemma shows that if X is w-local then there exists a specialization map $s : X \to \pi_0(X) \simeq X^c$ (in other words: any point x of X specializes into a unique closed point $s(x)$).

(ii). In fact, the above result is an equivalence: X is w-local if and only if $X^c \subset X$ is closed and every connected component has a unique closed point (equivalently: every point x of X specializes into a unique closed point).

Example 6.7. (i). Any profinite space S equipped with the pro-finite topology is a w-local spectral space.

(ii). If X is a scheme then the underlying topological spaces $\mathrm{Spec}(\mathcal{O}_{X,x})$ are w-local spectral spaces (they have a unique closed point).

Let $X = |\mathrm{Spec}(A)|$ be a spectral space. Then we can equip X with the constructible topology: the family of constructible sets of $|\mathrm{Spec}(A)|$ is the smallest family closed under finite intersection, finite union, complement and containing $V(I)$ for I finitely generated ideal (if A is noetherian, any ideal is finitely generated) and the constructible topology on X is the topology generated by the constructible sets. Equivalently, it is the topology which has as a subbase of opens the sets U and U^c where U is a quasi-compact open of X. For X a spectral space, the constructible topology is Hausdorff, totally disconnected and quasi-compact (see [StackProject, 0901]).

Theorem 6.8. *The inclusion $i : \mathcal{S}^{\mathrm{wl}} \to \mathcal{S}$ admits a right adjoint $X \mapsto X^Z$. The counit $X^Z \to X$ is a pro-(open cover) for all X, and the composite $(X^Z)^c \to X$ is a homeomorphism for the constructible topology on X.*

Sketch of proof. Using that $\mathcal{S} = \mathrm{Pro}(\mathcal{S}_f)$ and that the inclusion i of w-local spaces to spectral spaces preserves limits, we can reduce the proof to the case where X is a finite spectral space. In that case, the constructible topology is the same as the discrete topology. Then X^Z is defined as $X^Z := \coprod_{x\in X} X_x$ where $X_x := \{y \in X \mid y \text{ specializes to } x\} = \bigcap_{x\in U} U$. □

Remark 6.9. If X is a topological space, a stratification of X is a decomposition $X = \coprod_{i\in I} X_i$ together with a partial ordering on I such that the topological closure $\overline{X}_j \subset \bigcup_{i\leq j} X_i$. Let X be a spectral space, then the localization X^Z of X can be described via the following formula:

$$X^Z = \lim_{\{X_i\subset X\}_{i\in I}} \coprod_{i\in I} \widetilde{X}_i$$

where $\widetilde{X}_i = \{y \in X \mid y \text{ specializes to a point of } X_i\}$ and the limit is taken over all the constructible stratifications of X. We can in fact restrict ourselves to taking the limit over finite stratifications (see [StackProject, 096U]). We also have

$$(X^Z)^c = \lim_{\{X_i\subset X\}_{i\in I}} \coprod_{i\in I} X_i.$$

6.1.2 Localization of rings

Definition 6.10. Let A be a ring.

(i). A is said to be w-local if $\mathrm{Spec}(A)$ is w-local.

(ii). A is said to be w-strictly local if A is w-local, and every faithfully flat étale map $A \to B$ has a retraction.

(iii). A map $f : A \to B$ of w-local rings is w-local if $\mathrm{Spec}(f)$ is w-local.

(iv). A map $f : A \to B$ is called a Zariski localization if $B = \prod_{i=1}^{n} A[\frac{1}{f_i}]$ for some $f_1, \ldots, f_n \in A$.

(v). An ind-(Zariski localization) is a filtered colimit of Zariski localizations.

Note that any cofiltered limit of w-strictly local rings along w-local maps is w-strictly local.

Example 6.11. Let A be a strictly henselian local ring. We claim that A is w-strictly local. This follows from the two following facts:

- A flat map $A \to B$ is faithfully flat if and only if the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is surjective.
- A ring A is henselian if and only if for all $A \to B$ étale and all $\mathfrak{p} \in \mathrm{Spec}(B)$ mapping to the closed point $\mathfrak{m}_A$ of A with $k(\mathfrak{p}) = k$ then $A \to B_{\mathfrak{p}}$ is an isomorphism.

In fact this can be generalized:

Lemma 6.12. *Let A be a w-local ring. Then A is w-strictly local if and only if for any $\mathfrak{m}$ in $\mathrm{Spec}(A)^c$, the local ring $A_{\mathfrak{m}}$ is henselian.*

Theorem 6.13 (Localization of rings)**.** *The inclusion of the category of w-local rings and maps inside all rings admits a left adjoint $A \mapsto A^Z$. The unit $A \to A^Z$ is a faithfully flat ind-(Zariski localization) and $\mathrm{Spec}(A)^Z = \mathrm{Spec}(A^Z)$ over $\mathrm{Spec}(A)$.*

Sketch of proof. Let $X = \mathrm{Spec}(A)$. Theorem 6.8 above gives a w-local spectral space X^Z. We can equip X^Z with a structure of ringed space by taking the pullback of the structure sheaf $\mathcal{O}_X$ along $\pi : X^Z \to X$. It remains to show that $(X^Z, \pi^{-1}\mathcal{O}_X)$ defines an affine scheme. This follows from the description of X^Z given in Remark 6.9. We have seen that X^Z can be written as a limit of affine schemes so it is affine. $\square$

Example 6.14 (Localization of $\mathrm{Spec}(\mathbf{Z})$)**.** Consider the spectral space $X = \mathrm{Spec}(\mathbf{Z})$. The points of X are give by the generic point $\eta := (0)$ and the closed point $x_p := (p)$ for p prime number. Then, as a set, we have

$$\mathrm{Spec}(\mathbf{Z})^Z = \{\eta\} \amalg \coprod_{p \text{ prime}} \{x_p, \eta_p\}.$$

The closed points of X^Z are the points x_p's and the point η (as the complement of the union of the $\{x_p, \eta_p\}$'s which are open). We have

$$\mathbf{Z}^Z = \big(\varinjlim_{p \text{ prime}} \mathbf{Z}_{(2)} \times \mathbf{Z}_{(3)} \times \cdots \times \mathbf{Z}_{(p)} \times \mathbf{Z}[\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{p}]\big).$$

6.1.3 Absolutely flat algebras

From now on, if A is a ring, we will write J_A for its Jacobson radical (i.e. the intersection of all its maximal ideals). A ring A is called absolutely flat if A is reduced with Krull dimension 0 (equivalently, B is reduced and $\mathrm{Spec}(B)$ Hausdorff). For example, the product $\overline{k} \otimes_k \overline{k}$ with k perfect is absolutely flat, we have $|\mathrm{Spec}(\overline{k} \otimes_k \overline{k})| \simeq \mathrm{Gal}(\overline{k}/k)$.

Lemma 6.15. *Let A be w-local. Then its Jacobson radical J_A cuts out* $(\mathrm{Spec}(A))^c \hookrightarrow \mathrm{Spec}(A)$ *with its reduced structure. In particular, the quotient A/J_A is absolutely flat.*

Proof. Let J be the ideal of A defining the reduced scheme of $(\mathrm{Spec}(A))^c$. Then J is contained inside $\mathfrak{m}$ for all maximal ideal $\mathfrak{m}$ of A and we see that $J \subset J_A$. Conversely, suppose that there exists $\mathfrak{m}$ a closed point such that $\mathfrak{m}$ is not in $\mathrm{Spec}(A/J_A)$. This means that J_A is not included in $\mathfrak{m}$, which contradicts the definition of J_A. □

In fact, for A a ring, the map $A \to A^Z/J_{A^Z}$ is the universal map from A to an absolutely flat ring, i.e. for any absolutely flat A-algebra B, there exists a map $A^Z/J_{A^Z} \to B$ making the obvious diagram commutes.

Proposition 6.16 (Strictly local cover)**.** *For any absolutely flat ring A, there is an ind-étale faithfully flat map $A \to \overline{A}$ with $\overline{A}$ w-strictly local and absolutely flat. For a map $A \to B$ of absolutely flat rings, we can choose such maps $A \to \overline{A}$ and $B \to \overline{B}$ together with a map $\overline{A} \to \overline{B}$ of A-algebras.*

Sketch of the proof. Let I be the set of isomorphism classes of faithfully flat étale A-algebra. For $J \subset I$ a finite subset, define $A_J := \bigotimes_{j \in J} A_J$. Set $T^1(A) := \mathrm{colim}_{j \subset I} A_J$ and for all n, $T^{n+1}(A) = T^1(T^n(A))$. Then $\overline{A} := \mathrm{colim}_n T^n(A)$ is an ind-étale faithfully flat A-algebra and it is absolutely flat as colimit of étale algebras over an absolutely flat algebra.

Take $\overline{A} \to B$ a faithfully flat étale map, we want to construct a retraction. It can be showed that there exists n such that B can be written $B = \overline{A} \otimes_{T^n(A)} \widetilde{B}$ with $T^n(A) \to \widetilde{B}$ faithfully flat étale. By definition of $T^{n+1}(A)$, this means that there exists a map of $T^n(A)$-algebras $\widetilde{B} \to T^{n+1}(A)$. Composing with the natural morphism $T^{n+1}(A) \to \overline{A}$ we obtain the desired map $B \to \overline{A}$. □

6.2 Local contractibility

We now study the étale version of "w-local".

6.2.1 Definition

Definition 6.17. A ring A is w-contractible if every faithfully flat ind-étale map $A \to B$ has a retraction.

Proposition 6.18. *A w-contractible ring A is w-local.*

Proof. Consider the map $\pi : \mathrm{Spec}(A^Z) \to \mathrm{Spec}(A)$. Since A is w-contractible, it admits a section $s : \mathrm{Spec}(A) \to \mathrm{Spec}(A^Z)$. By [StackProject, 01KT], the section of a separated morphism is a closed immersion[11], so s realized $\mathrm{Spec}(A)$ as a closed subset of $\mathrm{Spec}(A^Z)$. By definition $\mathrm{Spec}(A^Z)$ is w-local, so $\mathrm{Spec}(A)$ is w-local. □

6.2.2 Henselian pairs and henselianization functor

A henselian pair is a pair (A, I) where A is a ring and I an ideal of A such that $I \subset J_A$ and for all $f \in A[t]$ monic polynomial such that $\overline{f} = g_0 h_0$ in $A/I[t]$ with g_0 and h_0 monic polynomials generating the unit ideal in $A/I[t]$, there exists a factorization $f = gh$ in $A[t]$ with g, h monic and $g_0 = \overline{g}$ and $h_0 = \overline{h}$. If A is a local ring and $I = \mathfrak{m}$, we recover the previous definition of henselian ring. If A is a ring and I is locally nilpotent then (A, I) defines an henselien pair and the functor $B \mapsto B/I$ induces an equivalence of categories between étale A-algebras and étale A/I-algebras. Arbitrary pairs (A, I) can be henselianized: the inclusion functor from the category of henselian pairs to the category of pairs admits a left adjoint $(A, I) \mapsto (A^h, I^h)$ (see [StackProject, 0A02]). Concretely, A^h is constructed as the colimit of the étale ring maps $A \to B$ such that $A/I \to B/IB$ is an isomorphism and $I^h := IA^h$.

Definition 6.19. Let $A \to B$ be a map of rings. We denote by $\mathrm{Hens}_A(-)$ the functor from $\mathrm{Ind}(B_{\text{ét}})$ to $\mathrm{Ind}(A_{\text{ét}})$, which is the right adjoint to the base change functor $\mathrm{Ind}(A_{\text{ét}}) \to \mathrm{Ind}(B_{\text{ét}})$. Explicitly, for B_0 in $\mathrm{Ind}(B_{\text{ét}})$, we have $\mathrm{Hens}_A(B_0) = \mathrm{colim} A'$, where the colimit is taken over diagrams $A \to A' \to B_0$ with A' étale A-algebra.

Note that for any map $A \to B$ and C in $\mathrm{Ind}(B_{\text{ét}})$, the ring $\mathrm{Hens}_A(C)$ depends only on the A-algebra C and not on B.

Lemma 6.20. *For (A, I) a pair, the functor $\mathrm{Hens}_A(-) : \mathrm{Ind}((A/I)_{\text{ét}}) \to \mathrm{Ind}(A_{\text{ét}})$ is fully faithful. In particular, for any B ind-étale A/I-algebra,*

$$\mathrm{Hens}_A(B) \otimes_A A/I \simeq B.$$

[11] and π is separated as a morphism between affine schemes.

If (A, I) is a pair (any) then $\mathrm{Hens}_A(A/I)$ is A^h the henselianization of A from before.

Proposition 6.21. *Let A be a ring henselian along an ideal I. Then,*

(i). A is w-strictly local if and only if A/I is w-strictly local,

(ii). A is w-contractible if and only if A/I is w-contractible.

6.2.3 Profinite spaces and extremally disconnected spaces

Lemma 6.22. *Let A be a ring and $T \to \pi_0(\mathrm{Spec}(A))$ a continuous map of profinite sets. There exists an ind-(Zariski localization) $A \to B$ such that applying π_0 to the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ gives rise to the given map $T \to \pi_0(\mathrm{Spec}(A))$.*

Sketch of proof. Assume first that T is a closed subset of $\pi_0(\mathrm{Spec}(A))$. Let Z be the inverse image of T in $\mathrm{Spec}(A)$, then we can show that Z is the intersection $\bigcap_i Z_i$ of the open and closed subsets of $\mathrm{Spec}(A)$ containing Z. Each of the Z_i is the spectrum of some A_i for $A \to A_i$ a local isomorphism[12]. The ring B from the lemma is given by $\mathrm{colim}_i A_i$.

In general, let $T \to \pi_0(\mathrm{Spec}(A))$ be a continuous map. Write $T = \lim_i T_i$ with T_i finite sets. Let Z_i be the image of T in $\pi_0(\mathrm{Spec}(A)) \times T_i$. Since each $\mathrm{Spec}(A) \times T_i$ is the spectrum of $A_i = \prod_{t \in T_i} A$, we can apply the previous result to the closed subset $Z_i \subset \pi_0(\mathrm{Spec}(A) \times T_i)$. This gives some map $A_i \to B_i$. Then B is defined as the colimit of the B_i's. $\square$

We would like to compare the notion of w-strictly local ring and w-constructible. In order to do that we need the notion of extremally disconnected spaces:

Definition 6.23. A compact Hausdorff space is extremally disconnected if the closure of every open is open.

By a theorem of Gleason, we know that extremally disconnected spaces are exactly the objects X in the category of all compact Hausdorff spaces for which every continuous surjection $Y \to X$ splits.

Example 6.24 (The Stone-Čech compactification)**.** The fully faithful embedding $\mathrm{CHaus} \hookrightarrow \mathrm{Top}$ from the category of compact Hausdorff topological spaces into the category of topological spaces admits a left adjoint $\beta : \mathrm{Top} \to \mathrm{CHaus}$. The image βX of a topological X is called the Stone-Čech compactification of X. When X is given the discrete topology, it can be showed that the image βX of X is an extremally disconnected space. If X is a compact Hausdorff

[12] i.e. for all $\mathfrak{p}$ in $\mathrm{Spec}(A)$, there is g not in $\mathfrak{p}$ such that $\mathrm{Spec}(A_i)_g \to \mathrm{Spec}(A)$ is an open immersion.

space, then there exists a continuous surjection $\beta(\delta(X)) \to X$ where $\delta(X)$ is the set X endowed with the discrete topology. This shows that all compact Hausdorff spaces can be covered by extremally disconnected spaces.

Lemma 6.25. *A w-strictly local ring A is w-contractible if and only if $\pi_0(\mathrm{Spec}(A))$ is extremally disconnected.*

Sketch of proof. Assume that A is w-contractible. Let $T \to \pi_0(\mathrm{Spec}(A))$ be a continuous surjection of profinite sets. We want to show that this map has a section. Use Lemma 6.22 to get an ind-(Zariski-localisation) $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ such that $\pi_0(B)$ is homeomorphic to T. It has a section by w-contractibility. Composing with $\mathrm{Spec}(A)^c \to \mathrm{Spec}(A)$, we get a map $\mathrm{Spec}(A)^c \to \mathrm{Spec}(B) \to \pi_0(\mathrm{Spec}(B))$ and by w-locality of A, the map $\mathrm{Spec}(A)^c \to \pi_0(\mathrm{Spec}(A))$ is an isomorphism. This gives a section $\pi_0(\mathrm{Spec}(A)) \to T$.

Assume now that $\pi_0(\mathrm{Spec}(A))$ is extremally disconnected. The key point is that since A is w-strictly local, its residue fields are separably closed (by Lemma 6.12). Using that, we can deduce that any faithfully flat ind-étale map $A \to B$ induces isomorphisms on the associated local rings. We need to define a section $B \to A$. Using the previous observation, we can assume that the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ induces a continuous surjection of profinite sets $\pi_0(\mathrm{Spec}(B)) \to \pi_0(\mathrm{Spec}(A))$. By hypothesis, this admits a section s. The section $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is then defined as precomposing s with the natural projection $\mathrm{Spec}(A) \to \pi_0(\mathrm{Spec}(A))$ and then taking the composition with $\pi_0(\mathrm{Spec}(B)) \simeq \mathrm{Spec}(B)^c \to \mathrm{Spec}(B)$. □

6.2.4 End of the proof of Theorem 6.2

Let us recall first the statement of the theorem:

Theorem 6.26. *For any ring A, there is an ind-étale faithfully flat A-algebra A' with A' w-contractible.*

Proof. In short, the ring A' is defined by the following diagram (more details below):

$$\begin{array}{ccccccccc}
 & & & & & & & & \mathrm{Spec}(A') \\
 & & & & & & & & \| \\
T & \xleftarrow{6.22} & \mathrm{Spec}(A_0) & & \xrightarrow{6.20} & & & & \mathrm{Spec}(\mathrm{Hens}_{A^Z}(A_0)) \\
{\scriptstyle 6.24}\downarrow & & \downarrow & & & & & \swarrow & \downarrow \\
\pi_0(\mathrm{Spec}(\overline{(A^Z/J_{A^Z})})) & \longleftarrow & \mathrm{Spec}(\overline{(A^Z/J_{A^Z})}) & \xrightarrow{6.16} & \mathrm{Spec}(A^Z/J_{A^Z}) & \longrightarrow & \mathrm{Spec}(A^Z) & \xrightarrow{6.13} & \mathrm{Spec}(A)
\end{array}$$

By Theorem 6.13 there exists a faithfully flat ind-(Zariski localization) $A \to A^Z$ to A^Z a w-local A-algebra. Then A^Z/J_{A^Z} is absolutely flat by Lemma 6.15. Using Proposition 6.16, we obtain an ind-étale faithfully flat map $A^Z/J_{A^Z} \to \overline{A^Z/J_{A^Z}}$ with $\overline{A^Z/J_{A^Z}}$ a w-strictly local absolutely flat ring. Since $\overline{A^Z/J_{A^Z}}$ is w-local, the profinite set $\pi_0(\mathrm{Spec}(\overline{A^Z/J_{A^Z}}))$ is compact and we can consider the Stone-Čech compactification of the associated discrete space from Example 6.24: we obtain a continuous surjection from an extremally disconnected set T. By Lemma 6.22, there exists an ind-(Zariski localization) $\overline{A^Z/J_{A^Z}} \to A_0$ realizing the map $T \to \pi_0(\mathrm{Spec}(\overline{A^Z/J_{A^Z}}))$ after applying $\pi_0(-)$. We can check that $\mathrm{Spec}(A_0)$ is w-local (see for example [StackProject, 096C]) and using Lemma 6.12 we can see it is strictly w-local. Since T is extremally disconnected, using Lemma 6.25 we obtain that $\mathrm{Spec}(A_0)$ is w-contractible.

Now, define $A' := \mathrm{Hens}_{A^Z}(A_0)$. By Lemma 6.20 we see that $\mathrm{Hens}_{A^Z}(A_0) \otimes_{A^Z} A^Z/J_{A^Z} \simeq A_0$ is w-contractible and by Proposition 6.21, we deduce that A' is w-contractible. Moreover $A \to A'$ is ind-étale faithfully flat since $A \to A^Z$ and $A^Z \to A'$ are so. □

7 Replete topoï

Eventually our goal is to define a category $D(X_{\text{proét}}, \mathbf{Z}_\ell)$ where derived limits behave well. This will be the case for the derived category of sheaves on so-called "replete topoï". The goal of this section is to define and study some properties of replete topoï. In particular, we will see that for a sequence of surjective morphisms $\cdots \to \mathscr{F}_n \to \mathscr{F}_{n-1} \to \cdots \to \mathscr{F}_1 \to \mathscr{F}_0$ in a replete topos, the limit $\mathrm{R}\varprojlim_n \mathscr{F}_n$ is exact, i.e. $\mathrm{R}\varprojlim_n \mathscr{F}_n \simeq \varprojlim_n \mathscr{F}_n$. We will also see that locally weakly contractible topoï are in fact replete, which we will use later to prove that the pro-étale site defines a replete topos.

7.1 Definition and properties

Let us first recall some definition coming from topos theory.

Definition 7.1. (i). A topos is a category equivalent to a category of the form $\mathrm{Shv}_\tau(\mathscr{C})$ for a category $\mathscr{C}$ and τ a Grothendieck topology on $\mathscr{C}$. Recall that for X an object of $\mathscr{C}$, we can associate a sheaf h_X defined as the sheafification of the presheaf $\mathrm{Hom}_{\mathscr{C}}(-, X)$.

(ii). Let $\mathscr{X} := \mathrm{Shv}_\tau(\mathscr{C})$ be a topos. A morphism $\mathscr{F} \to \mathscr{G}$ of objects of $\mathscr{X}$ is surjective if for any object $X \in \mathscr{C}$ and section $s \in \mathscr{G}(X)$, there exists a covering $\{U_i \to X\}_{i \in I}$ such that $s|_{U_i}$ is in the image of $\mathscr{F}(U_i) \to \mathscr{G}(U_i)$ for each $i \in I$.

If $\mathscr{X} := \mathrm{Shv}_\tau(\mathscr{C})$ is a topos we denote by $D(\mathscr{X}) := D(\mathrm{ShvAb}_\tau(\mathscr{C}))$ its derived category. Note that if $\mathscr{C}$ is a site and $\{V_i \to Y\}_{i \in I}$ a covering family, then $\coprod_{i \in I} h_{V_i} \to h_Y$ is a surjective morphism of sheaves.

Definition 7.2. A topos $\mathscr{X}$ is replete if for every sequence of surjective morphisms $\cdots \to \mathscr{F}_2 \to \mathscr{F}_1 \to \mathscr{F}_0$, the induced morphisms $\varprojlim_{n \in \mathrm{N}} \mathscr{F}_n \to \mathscr{F}_m$ is surjective for all m.

Recall that for $(\mathscr{F}_n)_{n \in \mathrm{N}}$ a projective system of sheaves, the limit can be computed termwise, i.e. $(\varprojlim_n \mathscr{F}_n)(U) = \varprojlim_n (\mathscr{F}_n(U))$.

Example 7.3. (i). The category of sets defines a replete topos: it is the category of sheaves on the category $\{*\}$ with only one object (and one morphism: the identity) equipped with the trivial Grothendieck topology, i.e. the only covering family is $\{* \xrightarrow{\mathrm{Id}} *\}$.

(ii). It follows from the previous example that if $\mathscr{C}$ is a category equipped with the trivial Grothendieck topology (i.e. the covering families are of the form $\{U \xrightarrow{\mathrm{Id}} U\}$), then $\mathrm{PreShv}(\mathscr{C})$ is a replete topos.

(iii). If G be a discrete group, the the category of $G-\mathrm{sets}$ from Lemma 2.9 is a replete topos.

(iv). Let k be a field and let k^{sep} be a separable closure of k. Then $\mathscr{X} = \text{Shv}(\text{Spec}(k)_{\text{ét}})$ is replete if and only if k^{sep} is a finite extension of k. Indeed, let us first assume that k^{sep}/k is finite. The topos of sheaves on $\text{Spec}(k^{\text{sep}})_{\text{ét}}$ is just the category of sets so it is replete. We then use that $\{\text{Spec}(k^{\text{sep}}) \to \text{Spec}(k)\}$ is an étale covering to deduce that the topos of sheaves on $\text{Spec}(k)_{\text{ét}}$ is also replete. Conversely, if k^{sep}/k is not finite, then we can find a tower of finite separable extensions $k := k_0 \subset k_1 \subset k_2 \subset \cdots \subset k^{\text{sep}}$. For all $n \geq 1$, the map $\text{Spec}(k_n) \to \text{Spec}(k_{n-1})$ is an étale covering, in particular, $h_{\text{Spec}(k_n)} \to h_{\text{Spec}(k_{n-1})}$ is surjective. However, the limit

$$\varprojlim_n h_{\text{Spec}(k_n)} \to h_{\text{Spec}(k)}$$

is not surjective: consider the element $\text{Id}_k \in h_{\text{Spec}(k)}(\text{Spec}(k))$. For any étale covering $\text{Spec}(L) \to \text{Spec}(k)$, for $n >> 0$ we have $h_{\text{Spec}(k_n)}(\text{Spec}(L)) = 0$, so there is no element in the limit mapping to Id_k.

(v). Let $\mathscr{X}$ be the topos of fpqc sheaves on the category of affine schemes. Then $\mathscr{X}$ is replete. To prove this, consider a tower of surjective morphisms $\cdots \to \mathscr{F}_2 \to \mathscr{F}_1 \to \mathscr{F}_0$. Let $X = \text{Spec}(A)$ be an affine scheme and $s \in \mathscr{F}_0(X)$. Since $\mathscr{F}_1 \to \mathscr{F}_0$ is surjective, there exists $\text{Spec}(B_0) \to \text{Spec}(A)$ faithfully flat such that $s|_{\text{Spec}(B_0)}$ is in the image of $\mathscr{F}_1(B_0) \to \mathscr{F}_0(B_0)$. Let s_1 be a preimage of $s|_{\text{Spec}(B_0)}$. Repeating the argument, we obtain a sequence of faithfully flat morphisms

$$A \to B_0 \to B_1 \to \cdots \to B_{n-1} \to B_n \to \cdots$$

and elements $s_i \in \mathscr{F}_i(B_{i-1})$ such that s_i maps to $s_{i-1}|_{B_{i-1}}$. Define $B := \varinjlim_i B_i$. Then $A \to B$ is faithfully flat and the $s_i \in \mathscr{F}_i(B_{i-1})$ define an element t in $(\varprojlim_i \mathscr{F}_i)(B)$ mapping to s. We obtain that $\varprojlim_i \mathscr{F}_i \to \mathscr{F}_0$ is surjective. A similar argument proves that the other projections are surjective as well.

Our goal now is to prove that on a replete topos, a limit as in Lemma 7.2 does not have higher derived functors. To do that, we need to understand the behavior of projective limits with respect to surjective maps. In general taking inverse limit does not preserve surjections: consider for example the maps $\mathbf{Z} \to \mathbf{Z}/\ell^n$ for any n, we see that $\mathbf{Z} = \varprojlim_n \mathbf{Z} \to \varprojlim_n \mathbf{Z}/\ell^n = \mathbf{Z}_\ell$ is not surjective. However, on a replete topos, we have the following result:

Lemma 7.4. *Let $\mathscr{X} = \text{Shv}_\tau(\mathscr{C})$ be a replete topos. Consider the map*

$$\begin{array}{ccccccccccc}
\cdots \to & \mathscr{F}_n & \to & \mathscr{F}_{n-1} & \to \cdots \to & \mathscr{F}_2 & \to & \mathscr{F}_1 & \to & \mathscr{F}_0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \to & \mathscr{G}_n & \to & \mathscr{G}_{n-1} & \to \cdots \to & \mathscr{G}_2 & \to & \mathscr{G}_1 & \to & \mathscr{G}_0
\end{array}$$

between two sequences of surjective morphisms and assume that the induced maps $\mathscr{F}_n \to \mathscr{G}_n$ and $\mathscr{F}_{n+1} \to \mathscr{F}_n \times_{\mathscr{G}_n} \mathscr{G}_{n+1}$ are surjective for all i. Then $\varprojlim_n \mathscr{F}_n \to \varprojlim_n \mathscr{G}_n$ is surjective.

Proof. Set $\mathscr{G} := \varprojlim_n \mathscr{G}_n$ and let X be in $\mathscr{C}$, s in $\mathscr{G}(X)$ and write $s = (s_n)_n$ with $s_n \in \mathscr{G}_n(X)$. Since $\mathscr{F}_0 \to \mathscr{G}_0$ is surjective, there exists a surjective map $X_0 \to X$ and t_0 a section in $\mathscr{F}_0(X_0)$ such that t_0 maps to $s_0|_{X_0} \in \mathscr{G}_0(X_0)$. By induction, we construct a sequence of surjective morphisms

$$\cdots \to X_{n+1} \to X_n \to \cdots \to X_1 \to X_0 \to X$$

and elements $t_n \in \mathscr{F}_n(X_n)$ such that the map $\mathscr{F}_n(X_n) \to (\mathscr{F}_{n-1} \times_{\mathscr{G}_{n-1}} \mathscr{G}_n)(X_n)$ sends t_n to $(t_{n-1}|_{X_n}, s_n|_{X_n})$. Now, since $\mathscr{X}$ is replete, the map $\varprojlim_n h_{X_n} \to h_X$ is surjective, so we can lift the sections $t_n \in \mathscr{F}_n(X_n)$ to elements $\tilde{t}_n$ of $\mathscr{F}_n(X)$ and we obtain that way of pre-image $\tilde{t} := (\tilde{t}_n)_n \in \varprojlim_n \mathscr{F}_n(X)$ of s. □

We want to use this lemma to prove that on a replete topos, countable products are exact. Let us briefly recall how those are defined. Let $\mathscr{X} := \mathrm{Shv}_\tau(\mathscr{C})$ be a topos.

- The category $\coprod_{n\in \mathrm{N}} \mathscr{C}$ is defined as the category whose objects are pairs (n, U) with $n \in \mathrm{N}$ and U an object of $\mathscr{C}$ and the set of morphisms between two objects (n, U) and (m, V) is the empty set if $n \neq m$ and to $\mathrm{Hom}_{\mathscr{C}}(U, V)$ if $n = m$. We equip $\coprod_{n\in \mathrm{N}} \mathscr{C}$ with the coarsest topology such that the inclusions $\mathscr{C} \to \coprod_{n\in \mathrm{N}} \mathscr{C}$ send covers to covers. This defines a site. The topos $\prod_{n\in \mathrm{N}} \mathscr{X}$ is the category of sheaves on $\coprod_{n\in \mathrm{N}} \mathscr{C}$.
- The category $\mathrm{N} \times \mathscr{C}$ is the category whose objects are pairs (n, U) with with $n \in \mathrm{N}$ and U an object of $\mathscr{C}$ and morphisms are given by

$$\mathrm{Hom}((n, U), (m, V)) = \begin{cases} \varnothing & \text{if } n > m \\ \mathrm{Hom}_{\mathscr{C}}(U, V) & \text{otherwise.} \end{cases}$$

  We equip this category with the coarsest topology such that the inclusions $\mathscr{C} \to \mathrm{N} \times \mathscr{C}$ send covers to covers. The topos $\mathscr{X}^{\mathrm{N}}$ is the category of sheaves on $\mathrm{N} \times \mathscr{C}$.

We have functors

$$\prod_n : \prod_{n\in \mathrm{N}} \mathscr{X} \to \mathscr{X} \quad \text{and} \quad \varprojlim_n : \mathscr{X}^{\mathrm{N}} \to \mathscr{X}$$

and passing to the derived categories we obtain

$$\mathrm{R}\prod_n : D(\prod_{n\in \mathrm{N}} \mathscr{X}) \to D(\mathscr{X}) \quad \text{and} \quad \mathrm{R}\varprojlim_n : D(\mathscr{X}^{\mathrm{N}}) \to D(\mathscr{X}).$$

The relation between the two is given by the following: for any $(\mathscr{F}_n)_n \in D(\prod_{n\in \mathrm{N}} \mathscr{X})$ there is a quasi-isomorphism in $D(\mathscr{X})$:

$$\mathrm{R}\varprojlim_n \mathscr{F}_n \simeq \mathrm{Cone}\big(\mathrm{R}\prod_n \mathscr{F}_n \xrightarrow{(\pi_{n+1}-1_n)_n} \mathrm{R}\prod_n \mathscr{F}_n\big)[-1] \qquad (7.1.0.1)$$

where $\pi_{n+1} : \mathscr{F}_{n+1} \to \mathscr{F}_n$ are the transition maps.

Proposition 7.5. *Countable products are exact in a replete topos.*

Sketck of proof. Let $\mathscr{X}$ be a replete topos. For each $n \in \mathrm{N}$, let $f_n : \mathscr{F}_n \to \mathscr{G}_n$ be a surjective map in $\mathscr{X}$, we want to show that

$$f = (f_n)_n : \prod_n \mathscr{F}_n \to \prod_n \mathscr{G}_n$$

is surjective. To do that, first note that f can be written as $\varprojlim_n \prod_{i<n} f_i$ and then use Lemma 7.4. □

Proposition 7.6. *Let* $\mathscr{X} = \mathrm{Shv}_\tau(\mathscr{C})$ *be a replete topos and let*

$$\cdots \to \mathscr{F}_2 \to \mathscr{F}_1 \to \mathscr{F}_0$$

be a sequence of surjective morphisms in $\mathrm{ShvAb}_\tau(\mathscr{C})$*. Then,*

$$\mathrm{R}\varprojlim_n \mathscr{F}_n \simeq \varprojlim_n \mathscr{F}_n.$$

Sketck of proof. Combining formula (7.1.0.1) and Proposition 7.5, we see that

$$\mathrm{R}\varprojlim_n \mathscr{F}_n \simeq \mathrm{Cone}(\prod_n \mathscr{F}_n \xrightarrow{(\pi_{n+1}-1_n)_n} \prod_n \mathscr{F}_n)[-1],$$

and it suffices to show that $(\pi_{n+1} - 1_n)_n$ is surjective. To do that, we apply Lemma 7.4 to the following map of projective systems:

$$\mathscr{H}_n := \prod_{i \le n+1} \mathscr{F}_i, \quad \mathscr{G}_n := \prod_{i \le n} \mathscr{F}_i, \quad f_n := (\pi_{i+1} - 1_i)_{i \le n} : \mathscr{H}_n \to \mathscr{G}_n.$$

□

7.2 Weakly contractible topos

We say that a topos $\mathscr{X}$ is coherent if it can be written $\mathrm{Shv}_\tau(\mathscr{C})$ where $\mathscr{C}$ has finite limits and $(\mathscr{C}, \tau)$ is a site such that for every covering $\{U_i \to X\}_{i \in I}$ there is a finite set $\{i_1, \dots, i_n\} \subset I$ such that $\{U_{i_j} \to X\}_{1 \le j \le n}$ is a covering. A site that satisfies this condition will be called coherent site.

An important example of coherent topos is the Zariski topos for affine schemes $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{AffineSchemes})$. Note that since this topos is equivalent to $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Schemes})$, the Zariski topos $\mathrm{Shv}_{\mathrm{Zar}}(\mathrm{Schemes})$ is coherent even if the site $(\mathrm{Schemes}, \mathrm{Zar})$ is not coherent.

Definition 7.7. (i). An object $\mathscr{F}$ of a topos $\mathscr{X}$ is called weakly contractible if every surjection $\mathscr{G} \to \mathscr{F}$ has a section.

(ii). Suppose $\mathscr{X}$ is a coherent topos with coherent site of definition $(\mathscr{C}, \tau)$. We say that $\mathscr{X}$ is locally weakly contractible if every object $\mathscr{F}$ in $\mathscr{X}$ admits a surjection $\coprod_{i\in I} \mathscr{G}_i \to \mathscr{F}$ with $\mathscr{G}_i$ a weakly contractible object which is representable by objects of $\mathscr{C}$.

The topos Sets is locally weakly contractible. We will see later that the pro-étale topos is locally weakly contractible.

Remark 7.8. (i). If $\mathscr{C}$ is a site, we say that U an object of $\mathscr{C}$ is weakly contractible if the sheaf h_U is. It can be proved that the following statements are equivalent:

a) U is weakly contractible.

b) For all covering $\{U_i \to U\}_{i\in I}$, the surjective map $\coprod_{i\in I} h_{U_i} \to h_u$ splits.

c) For any surjective map of sheaves $\mathscr{F} \to \mathscr{G}$ the induced map $\mathscr{F}(U) \to \mathscr{G}(U)$ is surjective.

(ii). If U is a weakly contractible object, then $H^i(U, \mathscr{F})$ is zero for all $i \geq 1$ and $\mathscr{F}$ abelian sheaf.

(iii). We have the following characterization:

Proposition 7.9. *Let $\mathscr{X} = \mathrm{Shv}_\tau(\mathscr{C})$ be a topos and suppose that there exists a full subcategory $\mathscr{C}' \subset \mathscr{C}$ such that $\mathscr{C}'$ is a coherent site of definition for $\mathscr{X}$. Then $\mathscr{X}$ is locally weakly contractible if and only if for each $X \in \mathscr{C}$, there is a covering family $\{U_i \to X\}_{i\in I}$ such that each U_i is in $\mathscr{C}'$ and U_i is weakly contractible.*

Recall that for n an integer and a chain complex $K_\bullet$, we define the truncations $\tau_{\leq n}$ and $\tau_{\geq n}$ as:

$$\tau_{\geq n}K_\bullet = (\cdots \to 0 \to 0 \to \operatorname{coker}(d_{n+1}) \to K_{n+1} \to K_{n+2} \to \cdots)$$
$$\tau_{\leq n}K_\bullet = (\cdots \to K_{n-2} \to K_{n-1} \to \ker(d_n) \to 0 \to 0 \to \cdots).$$

Note that $H^i(\tau_{\leq n}K_\bullet) = H^i(K_\bullet)$ for $i \leq n$ and $H^i(\tau_{\leq n}K) = 0$ for $i > n$. Similarly, $H^i(\tau_{\geq n}K) = H^i(K)$ for $i \geq n$ and $H^i(\tau_{\leq n}K) = 0$ for $i < n$.

Proposition 7.10. *Let $\mathscr{X}$ be a locally weakly contractible topos. Then $\mathscr{X}$ is replete and for any object $\mathscr{F} \in D(\mathscr{X})$ we have*

$$\mathrm{R}\varprojlim_n \tau_{\geq -n}\mathscr{F} \simeq \mathscr{F}.$$

Sketch of proof. The key point is that in a locally weakly contractible topos, to prove that a map $\mathscr{F} \to \mathscr{G}$ is surjective (respectively an isomorphism) it suffices to check that $\mathscr{F}(U) \to \mathscr{G}(U)$ is surjective (respectively an isomorphism) for U a weakly contractible object. To show that $\mathscr{X}$ is replete, we need to prove that for $(\mathscr{F}_n)_n$ an inverse system with surjective transition maps, the map $\varprojlim_n \mathscr{F}_n \to \mathscr{F}_0$ is surjective. But for U weakly contractible, $\varprojlim_n \mathscr{F}_n(U) \to \mathscr{F}_0(U)$ is surjective (the topos Sets is replete and tha maps $\mathscr{F}_{n+1}(U) \to \mathscr{F}_n(U)$ are surjective). This proves the result.

For the second part of the statement, since $\mathscr{X}$ is replete, we have that $\mathrm{R}\varprojlim_n \tau_{\geq -n}\mathscr{F} \simeq \varprojlim_n \tau_{\geq -n}\mathscr{F}$ for any $\mathscr{F}$ in $D(\mathscr{X})$. So we need to prove that in all degree i, the map $\underline{H}^i(\mathscr{F}) \to \underline{H}^i(\varprojlim_n \tau_{\geq -n}\mathscr{F})$ is an isomorphism (here the notation $\underline{H}^*$ means that we consider the cohomology sheaf associated to the complex of sheaves $\mathscr{F}$). But for U a weakly contractible object, we have

$$\underline{H}^i(\mathscr{F})(U) = H^i(\mathscr{F}(U)) \text{ and } \underline{H}^i(\varprojlim_n \tau_{\geq -n}\mathscr{F})(U) = H^i(\varprojlim_n \tau_{\geq -n}\mathscr{F}(U)).$$

□

7.3 Left-completion of a derived category

Definition 7.11. Let $\mathscr{X} = \mathrm{Shv}_\tau(\mathscr{C})$ be a topos. We define the left-completion $\widehat{D}(\mathscr{X})$ of $D(\mathscr{X})$ as the full subcategory of $D(\mathscr{X}^{\mathrm{N}})$ spanned by the projective systems $(\mathscr{F}_n)_n$ in $\mathrm{Ch}(\mathrm{ShvAb}_\tau(\mathscr{C})^{\mathrm{N}})$ such that

(i). $\mathscr{F}_n \in D_{\geq -n}(\mathscr{X})$ (i.e. $\underline{H}^i(\mathscr{F}_n) = 0$ for $i < -n$).

(ii). The canonical map $\tau_{\geq -n}\mathscr{F}_{n+1} \to \mathscr{F}_n$ is an equivalence (i.e. the map $\underline{H}^i(\mathscr{F}_{n+1}) \to \underline{H}^i(\mathscr{F}_n)$ is an isomorphism for all $i \geq -n$).

There exists a natural map $\tau : D(\mathscr{X}) \to \widehat{D}(\mathscr{X})$ given by $\mathscr{F} \mapsto (\tau_{\geq -n}\mathscr{F})_n$. We say $D(\mathscr{X})$ is left-complete if τ is an equivalence. The functor

$$\mathrm{R}\varprojlim_n : \widehat{D}(\mathscr{X}) \to D(\mathscr{X}^{\mathrm{N}}) \to D(\mathscr{X})$$

is the right adjoint of τ. In particular, if $D(\mathscr{X})$ is left-complete, then $\mathscr{F} \simeq \mathrm{R}\varprojlim_n(\tau_{\geq -n}\mathscr{F})$ for any $\mathscr{F} \in \mathrm{Ch}(\mathrm{ShvAb}_\tau(\mathscr{C}))$.

Proposition 7.12. *Let $\mathscr{X} = \mathrm{Shv}_\tau(\mathscr{C})$ be a topos. If $\mathscr{X}$ is replete then $D(\mathscr{X})$ is left-complete.*

Sketch of proof. We need to prove that $\tau : D(\mathscr{X}) \to \widehat{D}(\mathscr{X})$ is an equivalence of categories. To prove that it is fully faithful, by adjunction, it suffices to prove $\mathscr{F} \xrightarrow{\sim} \mathrm{R}\varprojlim_n \tau_{\geq -n}\mathscr{F}$ for all $\mathscr{F}$ in $D(\mathscr{X})$. We can then proceed as in the proof of Proposition 7.6: since $\mathscr{X}$ is replete, it is enough to check that the map $\prod_n \tau_{\geq -n}\mathscr{F} \xrightarrow{(\pi_{n+1}-1_n)_n} \prod_n \tau_{\geq -n}\mathscr{F}$ is surjective.

Similarly, to show the essential surjectivity it suffices to check that $\mathscr{F}_n \xrightarrow{\sim} \tau_{\geq -n}\mathrm{R}\varprojlim_i \mathscr{F}_i$ for all $(\mathscr{F}_i)_i$ in $\widehat{D}(\mathscr{X})$. For a K-injective resolution $(\mathscr{I}_n)_n$ in $\mathrm{Ch}(\mathrm{ShvAb}_\tau(\mathscr{C})^{\mathrm{N}})$ of $(\mathscr{F}_n)_n$, since $\mathscr{I}_{n+1} \to \mathscr{I}_n$ is surjective (see [StackProject, 070L]), we have that $\mathrm{R}\varprojlim_n \mathscr{F}_n$ is computed by the kernel of $\prod_n \mathscr{I}_n \xrightarrow{(\pi_{n+1}-1_n)_n} \prod_n \mathscr{I}_n$. This kernel can be computed using the fact that $\underline{H}^i(\prod_n \mathscr{I}_n) \simeq \prod_{n\geq i} \underline{H}^i(\mathscr{I}_n)$. □

Example 7.13. (i). We will see later that the pro-étale topos is replete, hence $D(X_{\text{proét}})$ is left-complete. We will show that $\widehat{D}(X_{\text{ét}})$ defines a full-subcategory of $D(X_{\text{proét}})$.

(ii). If $\mathscr{X} = \text{Shv}(\mathscr{C}, \tau)$ is a topos such that for each $\mathscr{F}$ in $D(\mathscr{X})$ and $U \in \mathscr{C}$ there exists $d \geq 0$ such that $H^p(U, \underline{H}^q(\mathscr{F})) = 0$ for $p > d$ and $q \geq 0$ then $D(\mathscr{X})$ is left-complete. This is the case for the étale topos of $\text{Spec}(\mathbf{F}_q)$ or the one of $X_{\text{ét}}$ for X a smooth affine variety over an algebraically closed field.

(iii). When $\mathscr{X}$ is not replete, we can find example of derived categories that are not left complete: see Exercise Sheet 8.

7.4 ℓ-adic sheaves

In this section, we fix $\mathscr{X} = \text{Shv}(\mathscr{C}, \tau)$ a replete topos. We denote by $\text{Mod}_{\mathbf{Z}_\ell}$ the category of modules over $\mathbf{Z}_\ell$. Recall that a $\mathbf{Z}_\ell$ module is said to be (classically) complete if $M \simeq \varprojlim_n M/\ell^n M$. We write $\text{Mod}_{\mathbf{Z}_\ell}^{\text{comp}} \subset \text{Mod}_{\mathbf{Z}_\ell}$ for the full subcategory of (classically) complete modules.

We make the following observation:

Lemma 7.14. *Let M be a $\mathbf{Z}_\ell$-module and assume that M is ℓ-torsion free. Then M is in $\text{Mod}_{\mathbf{Z}_\ell}^{\text{comp}}$ if and only if both $\varprojlim_{x \mapsto \ell \cdot x} M$ and $\mathrm{R}^1 \varprojlim_{x \mapsto \ell \cdot x} M$ are zero.*

Proof. Let us denote by M_0 and M_1 the limits $\varprojlim(\cdots \xrightarrow{\ell} M \xrightarrow{\ell} M)$ and $\mathrm{R}^1 \varprojlim(\cdots \xrightarrow{\ell} M \xrightarrow{\ell} M)$. Consider the following inverse system of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\ell^{n+1}} & M & \longrightarrow & M/\ell^{n+1}M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \ell} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{\ell^{n}} & M & \longrightarrow & M/\ell^{n}M & \longrightarrow & 0 \end{array}$$

Taking the limit, we obtain an exact sequence:

$$0 \to M_0 \to M \to \varprojlim_n M/\ell^n M \to M_1 \to 0$$

and the result of the lemma follows. $\square$

This result leads to the following definition:

Definition 7.15. Let $\mathscr{F}$ be a complex in $D(\mathscr{X}, \mathbf{Z}_\ell)$. We set

$$T(\mathscr{F}) := \mathrm{R}\varprojlim(\ldots \xrightarrow{\ell} \mathscr{F} \xrightarrow{\ell} \mathscr{F} \xrightarrow{\ell} \mathscr{F}).$$

We say $\mathscr{F}$ is derived complete if $T(\mathscr{F}) \simeq 0$ in $D(\mathscr{X}, \mathbf{Z}_\ell)$. We write $D_{\text{comp}}(\mathscr{X}, \mathbf{Z}_\ell) \subset D(\mathscr{X}, \mathbf{Z}_\ell)$ for the full subcategory of derived complete objects.

Note that since $\mathscr{X}$ is replete, $T(\mathscr{F})$ can be computed as the cone of the map:

$$\mathrm{Cone}\big(\prod_{\mathbf{N}} \mathscr{F} \xrightarrow{\ell-\mathrm{Id}} \prod_{\mathbf{N}} \mathscr{F}\big)[-1].$$

In particular, we see that T preserves exact triangles.

If $\mathscr{X}$ is locally weakly contractible and $U \in \mathscr{C}$ a weakly contractible object, then for any $\mathscr{F}$ in $D(\mathscr{X}, \mathbf{Z}_\ell)$, we have $\mathrm{R}\Gamma(U, T(\mathscr{F})) = T(\mathrm{R}\Gamma(U, \mathscr{F}))$. We see that $\mathscr{F}$ is derived complete if and only if for all U weakly contractible, $T(\mathrm{R}\Gamma(U, \mathscr{F}))$ is zero.

Proposition 7.16. *A $\mathbf{Z}_\ell$-module $M \in \mathrm{Mod}_{\mathbf{Z}_\ell}$ is classically complete if and only if M is ℓ-adically separated (i.e. $\bigcap \ell^n M = 0$) and $\underline{M}$ is derived complete.*

For a proof see [BS13, Section 3.4].

Example 7.17. There exist derived complete modules that are not classically complete. Consider for example $M := \mathrm{coker}(\lambda)$ where λ is the map

$$\begin{cases} \mathbf{Z}_\ell\langle t\rangle & \to \mathbf{Z}_\ell\langle t\rangle \\ t & \mapsto \ell \cdot t \end{cases}.$$

Here $\mathbf{Z}_\ell\langle t\rangle$ denoted the ℓ-adic completion of $\mathbf{Z}_\ell[t]$.

The $\mathbf{Z}_\ell$-module M is derived complete: taking the cokernel preserves the property of being derived complete and $\mathbf{Z}_\ell\langle t\rangle$ is classically complete, hence derived complete. We claim that M is not separated. Indeed, consider the element $f = \sum_{n\geq 0} \ell^n t^n$ of $\mathbf{Z}_\ell\langle t\rangle$. Then the projection $\overline{f}$ of f to M is non-zero. But for any integer m, we can write $f = (1+\ell t+\cdots+\ell^{m-1}t^{m-1})+\ell^m\cdot(t^m f)$ so $\overline{f} = \ell^m \cdot (t^m \overline{f})$. We obtain that $\overline{f}$ is a non-zero element of $\bigcap_{m\geq 1} \ell^m \cdot M$.

Proposition 7.18. *A $\mathbf{Z}_\ell$-complex $\mathscr{F}$ in $D(\mathscr{X}, \mathbf{Z}_\ell)$ is derived complete if and only if for all $i \in \mathbf{Z}$, the sheaf $\underline{H}^i(\mathscr{F}) \in \mathrm{Shv}_\tau(\mathscr{C}, \mathbf{Z}_\ell)$ is derived complete.*

Proof. Let us assume that the cohomology sheaves are derived complete. We will first prove that for all m, the truncated complex $\tau_{\leq m}\mathscr{F}$ is derived complete. Since $\mathscr{X}$ is replete, we have a quasi-isomorphism $\tau_{\leq m}\mathscr{F} \simeq \mathrm{R}\varprojlim_n \tau_{\geq -n}(\tau_{\leq m}\mathscr{F})$. Note that since limits commute with limits, a (derived) limit of derived complete complexes is derived complete. So it suffices to prove that for all n, the complex $\tau_{\geq -n}(\tau_{\leq m}\mathscr{F})$ is derived complete. To do that, consider the following exact triangle:

$$\tau_{\geq m-i}\tau_{\leq m}\mathscr{F} \to \tau_{\geq m-i-1}\tau_{\leq m}\mathscr{F} \to \underline{H}^{m-(i+1)}\mathscr{F}$$

Since the sheaves $\underline{H}^*(\mathscr{F})$'s are derived complete, an induction on the degree i shows that for all $i \geq 0$, $\tau_{\geq m-i}\tau_{\leq m}\mathscr{F}$ is derived complete. This proves that $\tau_{\leq m}\mathscr{F}$ is derived complete. To prove that $\mathscr{F}$ is derived complete, apply $T(-)$ to the exact triangle:

$$\mathscr{F} \to \tau_{\geq m+1}\mathscr{F} \to (\tau_{\leq m}\mathscr{F})[1]$$

and use that $T(\tau_{\leq m}\mathscr{F})$ is zero. This yields an isomorphism $T(\mathscr{F}) \xrightarrow{\sim} T(\tau_{\geq m+1}\mathscr{F})$ for all $m \in \mathbf{Z}$, which implies that $T(\mathscr{F})$ is zero in degree $i < m$ for all $m \in \mathbf{Z}$, i.e. $T(\mathscr{F}) \simeq 0$.

Conversely, assume that $\mathscr{F}$ is derived complete. It is enough to prove that $\underline{H}^0(\mathscr{F})$ is derived complete. Let us first suppose that $\mathscr{F}$ is in $D_{\leq 0}(\mathscr{X}, \mathbf{Z}_\ell)$. Then there is an exact triangle

$$\tau_{\leq -1}\mathscr{F} \to \mathscr{F} \to \underline{H}^0(\mathscr{F}).$$

Using that $T(\mathscr{F}) = 0$, we obtain that $T(\tau_{\leq -1}\mathscr{F}) \simeq T(\underline{H}^0(\mathscr{F}))$. But the first term is in $D_{\leq -1}(\mathscr{X}, \mathbf{Z}_\ell)$ while the second is in$D_{\geq 0}(\mathscr{X}, \mathbf{Z}_\ell)$, so both are zero. For a general $\mathscr{F}$ in $D(\mathscr{X}, \mathbf{Z}_\ell)$, consider the exact triangle

$$\tau_{\leq 0}\mathscr{F} \to \mathscr{F} \to \tau_{\geq 1}\mathscr{F}$$

and use that $T(\mathscr{F}) = 0$. By a similar argument as before, we obtain that both $T(\tau_{\leq 0}\mathscr{F})$ and $T(\tau_{\geq 1}\mathscr{F})$ are zero. Since $\tau_{\leq 0}\mathscr{F}$ is in $D_{\leq 0}(\mathscr{X}, \mathbf{Z}_\ell)$, the previous step shows that $\underline{H}^0(\mathscr{F})$ is zero. □

For $\mathscr{F}$ in $D(\mathscr{X}, \mathbf{Z}_\ell)$, we define the derived completion of $\mathscr{F}$ as

$$\widehat{\mathscr{F}} := \mathrm{R}\varprojlim_n (\mathrm{Cone}(\mathscr{F} \xrightarrow{\ell^n} \mathscr{F})).$$

Note that for all n, we have a quasi-isomorphism $\mathscr{F} \otimes^{\mathrm{L}}_{\mathbf{Z}_\ell} \mathbf{Z}_\ell/\ell^n \simeq \mathrm{Cone}(\mathscr{F} \xrightarrow{\ell^n} \mathscr{F})$.

Proposition 7.19. *The functor sending $\mathscr{F}$ to $\widehat{\mathscr{F}}$ defines a left adjoint to the inclusion $D_{\mathrm{comp}}(\mathscr{X}, \mathbf{Z}_\ell) \subset D(\mathscr{X}, \mathbf{Z}_\ell)$.*

Sketch of proof. First note that for any $\mathscr{F}$ in $D(\mathscr{X}, \mathbf{Z}_\ell)$, we have an exact triangle:

(7.4.0.1) $$\mathscr{F} \to \widehat{\mathscr{F}} \to T(\mathscr{F})$$

that comes from the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathscr{F} & \xrightarrow{\ell} & \mathscr{F} & \xrightarrow{\ell} & \mathscr{F} & \xrightarrow{\ell} & \mathscr{F} \\
 & & \downarrow{\scriptstyle \ell^4} & & \downarrow{\scriptstyle \ell^3} & & \downarrow{\scriptstyle \ell^2} & & \downarrow{\scriptstyle \ell} \\
\cdots & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \mathscr{F} \otimes^{\mathrm{L}}_{\mathbf{Z}_\ell} \mathbf{Z}_\ell/\ell^4 & \longrightarrow & \mathscr{F} \otimes^{\mathrm{L}}_{\mathbf{Z}_\ell} \mathbf{Z}_\ell/\ell^3 & \longrightarrow & \mathscr{F} \otimes^{\mathrm{L}}_{\mathbf{Z}_\ell} \mathbf{Z}_\ell/\ell^2 & \longrightarrow & \mathscr{F} \otimes^{\mathrm{L}}_{\mathbf{Z}_\ell} \mathbf{Z}_\ell/\ell
\end{array}$$

We want to show that for $\mathscr{F}$ in $D(\mathscr{X}, \mathbf{Z}_\ell)$ and $\mathscr{G}$ in $D_{\mathrm{comp}}(\mathscr{X}, \mathbf{Z}_\ell)$ then, $\mathrm{Hom}_{D(\mathscr{X}, \mathbf{Z}_\ell)}(\widehat{\mathscr{F}}, \mathscr{G}) \to \mathrm{Hom}_{D(\mathscr{X}, \mathbf{Z}_\ell)}(\mathscr{F}, \mathscr{G})$ is an isomorphism. Using the previous exact sequence, it suffices to check that

$$\mathrm{Hom}_{D(\mathscr{X}, \mathbf{Z}_\ell)}(T(\mathscr{F}), \mathscr{G}) = 0.$$

This follows from the two following lemmas:

Lemma 7.20. *Let $\mathscr{F}$ be in $D(\mathscr{X}, \mathbf{Z}_\ell)$. Then $T(\mathscr{F})$ is in the essential image of the canonical functor $D(\mathscr{X}, \mathbf{Q}_\ell) \to D(\mathscr{X}, \mathbf{Z}_\ell)$.*

Proof. We use the following isomorphism: $\mathbf{Q}_\ell \simeq \mathrm{colim}_{x \mapsto \ell \cdot x}(\mathbf{Z}_\ell)$. For any $\mathscr{F}$ in $D(\mathscr{X}, \mathbf{Z}_\ell)$

$$
\begin{aligned}
\mathrm{RHom}_{\mathbf{Z}_\ell}(\mathbf{Q}_\ell, \mathscr{F}) &\simeq \mathrm{RHom}_{\mathbf{Z}_\ell}\big(\mathrm{colim}(\mathbf{Z}_\ell \xrightarrow{\ell} \mathbf{Z}_\ell \xrightarrow{\ell} \mathbf{Z}_\ell \xrightarrow{\ell} \mathbf{Z}_\ell \to \cdots), \mathscr{F}\big) \\
&\simeq \mathrm{Rlim}\big(\cdots \xrightarrow{\ell} \mathrm{RHom}_{\mathbf{Z}_\ell}(\mathbf{Z}_\ell, \mathscr{F}) \xrightarrow{\ell} \mathrm{RHom}_{\mathbf{Z}_\ell}(\mathbf{Z}_\ell, \mathscr{F})\big) \\
&= T(\mathscr{F}).
\end{aligned} \tag{7.4.0.2}
$$

□

Lemma 7.21. *Let $\mathscr{F}$ be in $D(\mathscr{X}, \mathbf{Q}_\ell)$ and $\mathscr{G}$ in $D(\mathscr{X}, \mathbf{Z}_\ell)$ with $\mathscr{G}$ derived complete. Then* $\mathrm{Hom}(\mathscr{F}, \mathscr{G}) = 0$.

Proof. From the above computation, we see that the lemma is true when $\mathscr{F} = \mathbf{Q}_\ell$. We then use that elements of $D(\mathscr{X}, \mathbf{Q}_\ell)$ can be written as the cokernel of morphisms between direct sums of sheaves of the form $j_!\mathbf{Q}_\ell$ where $j : U \to 1_{\mathscr{X}}$. □

To prove that if $\mathscr{F}$ in $D(\mathscr{X}, \mathbf{Z}_\ell)$ then $\widehat{\mathscr{F}}$ is in $D_{\mathrm{comp}}(\mathscr{X}, \mathbf{Z}_\ell)$, we use the exact triangle (7.4.0.1) for the sheaf $\widehat{\mathscr{F}}$ and the fact that $\widehat{\widehat{\mathscr{F}}} \simeq \widehat{\mathscr{F}}$.

□

8 The pro-étale topology

In this section, we define the pro-étale site $X_{\text{proét}}$ and prove that the associated topos is locally weakly contractible, hence replete. We then study in more details the pro-étale site of a point.

8.1 The pro-étale site and topos

Recall that a morphism of rings $A \to B$ is said to be weakly étale if it is flat and the diagonal morphism $B \otimes_A B \to B$ is flat. Similarly, a morphism $X \to Y$ of schemes is weakly étale if it is flat and the diagonal morphism $X \to X \times_Y X$ is flat. An ind-étale morphism is weakly étale. Let $f : A \to B$ and $g : B \to C$ be morphisms of rings. If f and g are weakly étale, then $g \circ f$ is weakly étale. If $g \circ f$ and f are weakly étale then g is weakly étale. If k is a field then $\mathrm{Spec}(A) \to \mathrm{Spec}(k)$ is weakly étale if and only if $k \to A$ is ind-étale. If X is a scheme and $\overline{x}$ a geometric point then $\mathrm{Spec}(\mathscr{O}^{\mathrm{sh}}_{X,\overline{x}}) \to X$ is weakly étale. Moreover, we have the following theorem (see Exercise Sheet 7 for more details about the proof):

Theorem 8.1. *Let $f : A \to B$ be weakly étale. Then there exists a faithfully flat ind-étale morphism $g : B \to C$ such that $g \circ f : A \to C$ is ind-étale.*

Definition 8.2. Let X be a scheme. We define the pro-étale site $X_{\text{proét}}$ of X as the category of weakly étale X-schemes, to which we give the structure of a site by defining a cover as a family $\{\varphi_i : U_i \to U\}_{i \in I}$ of maps in $X_{\text{proét}}$ such that for any affine open $V \subset U$ there exist a map $\alpha : \{1, \ldots, n\} \to I$ and affine open $V_j \subset U_{\alpha(j)}$ such that $V = \bigcup_{j=1}^{n} \varphi_{\alpha(j)}(V_j)$.

Note that by Theorem 8.1, any $f : X \to Y$ weakly étale is Zariski locally on the target and pro-étale locally on the source of the form $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ with $B \to A$ ind-étale.

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