Sheet 1: Reminders on sheaves and schemes, Grothendieck topologies

Exercise 1. Let X be a topological space.

(1) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of $\mathcal{A}$-modules. Show that the presheaf $\mathrm{Ker}\varphi : U \mapsto \mathrm{Ker}(\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U))$ is a sheaf, and that we have a canonical isomorphism $(\mathrm{Ker}\varphi)_x = \mathrm{Ker}(\varphi_x : \mathcal{F}_x \to \mathcal{G}_x)$ for every $x \in X$.

(2) Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{A}$-modules. Show that we have a canonical morphism $(\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G})_x \simeq \mathcal{F}_x \otimes_{\mathcal{A}_x} \mathcal{G}_x$, for every $x \in X$.

Exercise 2. Let X be a topological space.

(1) Show that a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is injective if and only if $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every open subset U of X.

(2) Show that a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective if and only if, for every open subset U of X and every $s \in \mathcal{G}(U)$, there exists an open cover $(U_i)_{i\in I}$ and sections $t_i \in \mathcal{F}(U_i)$ such that $\varphi_{U_i}(t_i) = s_i$ for every $i \in I$.

(3) Let $0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$ be a sequence of sheaves of abelian groups. Show that it is exact if and only if the following two conditions hold:

(a) for every open subset U of X, the sequence $0 \to \mathcal{F}(U) \xrightarrow{\alpha_U} \mathcal{G}(U) \xrightarrow{\beta_U} \mathcal{H}(U)$ is exact;

(b) the morphism of sheaves $\beta : \mathcal{G} \to \mathcal{H}$ is surjective.

(4) Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves, and suppose that $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism for every $x \in X$. Show that φ is an isomorphism.

Let A be a ring. The spectrum of A, denoted $\mathrm{Spec}(A)$, is the set of prime ideals of A. We endow $\mathrm{Spec}(A)$ with its Zariski topology, i.e. the topology generated by the open sets $D(f) := \{\mathfrak{p} \in \mathrm{Spec}(A) \mid f \notin \mathfrak{p}\}$, for $f \in A$. The closed subsets are given by $V(I) := \{\mathfrak{p} \in \mathrm{Spec}(A), I \subset \mathfrak{p}\}$, where I is an ideal of A. Note that a singleton $\{\mathfrak{p}\}$ in $\mathrm{Spec}(A)$ is closed if and only if $\mathfrak{p}$ is a maximal ideal of A. A locally ringed space $(X, \mathcal{O}_X)$ is a pair consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ such that the stalks of $\mathcal{O}_X$ are local rings. For $X := \mathrm{Spec}(A)$, we can define a structural sheaf $\mathcal{O}_X$ on X such that for any $f \in A$, $\mathcal{O}_X(D(f)) = A_f$. The locally ringed space $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ is called an affine scheme. A scheme is a locally ringed space with the property that every point has an open neighbourhood which is an affine scheme.

Exercise 3. Let A be a ring and set $X := \mathrm{Spec}(A)$. Let f be an element of A.

(1) Show that the locally ringed space $(D(f), \mathcal{O}_X|_{D(f)})$ is isomorphic to $\mathrm{Spec}(A_f)$.

(2) For another element $g \in A$ describe the restriction map $\mathcal{O}_X(D(f)) \to \mathcal{O}_X(D(fg))$ in terms of a ring homomorphism $A_f \to A_{fg}$.

Exercise 4. Consider $X_1 := \mathrm{Spec}(\mathbf{Q}[t_1, t_2]/(t_1t_2))$ and $X_2 := \mathrm{Spec}(\mathbf{Q}[t_1, t_2]/(t_1^2 + t_2^2))$.

(1) Compute $\mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(\mathbf{Q}), X_i)$ for $i = 1, 2$.

(2) Deduce that X_1 and X_2 are not isomorphic schemes.

(3) Let $X_1' := \mathrm{Spec}(\mathbf{Q}(i)[t_1, t_2]/(t_1t_2))$ and $X_2' := \mathrm{Spec}(\mathbf{Q}(i)[t_1, t_2]/(t_1^2 + t_2^2))$. Prove that X_1' and X_2' are isomorphic as schemes.

Exercise 5. Let $X := \mathbb{A}^2_{\mathbf{C}} \setminus \{0\} \subset \mathbb{A}^2_{\mathbf{C}}$.

(1) Prove that the restriction map $\mathcal{O}_{\mathbb{A}^2_{\mathbf{C}}}(\mathbb{A}^2_{\mathbf{C}}) \to \mathcal{O}_X(X)$ is an isomorphism.

(2) Show that the scheme X is not an affine scheme.

Let X be a locally ringed space and $U \subset X$ be an open subset. Let $\mathcal{O}_U = \mathcal{O}_X|_U$ be the restriction of $\mathcal{O}_X$ to U. For $u \in U$ the stalk $\mathcal{O}_{U,u}$ is equal to the stalk $\mathcal{O}_{X,u}$ and is a local ring. We obtain a locally ringed space $(U, \mathcal{O}_U)$ and the morphism $j : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ is called an open immersion. If $(X, \mathcal{O}_X)$ is a scheme then so is $(U, \mathcal{O}_U)$.

Recall that a module M over a ring A is flat if the functor $- \otimes_A M : \mathrm{Mod}_A \to \mathrm{Mod}_A$ is exact. If $f : X \to Y$ is a map of schemes, we say that f is flat at a point $x \in X$ if the local ring $\mathcal{O}_{X,x}$ is flat over the local ring $\mathcal{O}_{Y,f(x)}$. We say f is flat if f is flat at every point of X.

Exercise 6. Let X be a scheme.

(1) A Zariski covering of X is a family of morphisms $\{\varphi_i : X_i \to X\}_{i \in I}$ of schemes such that each φ_i is an open immersion and such that $X = \bigcup_{i \in I} \varphi_i(X_i)$. Prove that the Zariski coverings define a Grothendieck topology on X.

(2) An fpqc covering of X is a family $\{\varphi_i : X_i \to X\}_{i \in I}$ such that each φ_i is a flat morphism, $X = \bigcup_{i \in I} \varphi_i(X_i)$ and for each affine open $U \subset X$ there exists a finite set K, a map $\iota : K \to I$ and affine opens $U_{\iota(k)} \subset X_{\iota(k)}$ such that $U = \bigcup_{k \in K} \varphi_{\iota(k)}(U_{\iota(k)})$. Prove that the fpqc coverings define a Grothendieck topology on X.

Exercise 7. An A-module M is called faithfully flat if a sequence of A-modules $N_1 \to N_2 \to N_3$ is exact if and only if the sequence $M \otimes_A N_1 \to M \otimes_A N_2 \to M \otimes_A N_3$ is exact. We say that a morphism of schemes $f : X \to Y$ is faithfully flat if it is flat and surjective.

(1) Prove that $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ is an fpqc covering if and only if $A \to B$ is faithfully flat.

(2) Let R' be a faithfully flat R-algebra, and let $R'' = R' \otimes_R R'$. Consider the two maps $R' \to R''$ given by $x \mapsto x \otimes 1$ and $x \mapsto 1 \otimes x$. Prove that the following diagram is exact:

$$R \longrightarrow R' \rightrightarrows R''.$$