Sheet 2: Canonical topology, Zariski and fpqc sites

Exercise 1. Let X be the topological space $\mathbf{C} \setminus \{0\}$ with its usual topology and let $\mathcal{F}$ be the sheaf of nowhere-zero continuous complex-valued functions. Consider the morphism $\varphi : \mathcal{F} \to \mathcal{F}$ sending a function f to f^2. Prove that $U \mapsto \mathrm{Im}(\varphi_U)$ is not a sheaf.

Exercise 2. Let $\mathcal{C}$ be a category. A family $\{U_i \to U\}_{i \in I}$ is a family of effective epimorphisms if the diagram

$$\mathrm{Hom}(U, Z) \longrightarrow \prod_{i \in I} \mathrm{Hom}(U_i, Z) \rightrightarrows \prod_{(i,j) \in I^2} \mathrm{Hom}(U_i \times_U U_j, Z)$$

is exact, for all $Z \in \mathcal{C}$. We say that it is a family of universal effective epimorphisms if $\{U_i \times_U V\}_{i \in I}$ is a family of effective epimorphisms for each morphism $V \to U$ in $\mathcal{C}$. The canonical topology $\mathcal{T}$ on $\mathcal{C}$ is defined by taking for the set of coverings the collection of all families of universal effective epimorphisms in $\mathcal{C}$.

(1) Prove that it defines a Grothendieck topology.

(2) Prove that for this topology, representable presheaves are sheaves and that the canonical topology is the finest topology on $\mathcal{C}$ in which all representable presheaves are sheaves. A topology on $\mathcal{C}$ in which representable presheaves are sheaves is called subcanonical.

(3) Show that when $\mathcal{C}$ is the category $G-$Set from the lecture, then the topology $\mathcal{T}_G$ we have defined is in fact the canonical topology of $\mathcal{C}$ (in particular, for any G-set Z, the presheaf $\mathrm{Hom}_G(-, Z)$ is a sheaf).

Exercise 3. Let X be a scheme. The goal is to prove that if a presheaf $\mathcal{F}$ satisfies the two following properties:

(a) the sheaf property for every Zariski covering, and

(b) the sheaf property for families of the form $\{\varphi : V \to U\}$ with U, V affines and φ flat and surjective,

then $\mathcal{F}$ is a sheaf on X_{fpqc}.

(1) Prove that if a Zariski sheaf satisfies the following condition:

(c) the sheaf property for families of the form $\{\varphi : V \to U\}$ with φ flat, surjective and quasi-compact,

then it is a fpqc sheaf (hint: for $\{U_i \to U\}_{i \in I}$ a fpqc-covering, consider the covering given by the disjoint union $V := \amalg_{i \in I} U_i \to U$).

(2) Prove that if a sheaf satisfies condition (b) then it satisfies the following condition:

(d) the sheaf property for fpqc coverings of the form $\{\varphi_i : U_i \to U\}_{i \in I}$ with I finite and U_i affine.

(3) Prove that if $\mathcal{F}$ satisfies conditions (a) and (d) then it satisfies (c) and conclude.

Exercise 4. Let X be a scheme.

(1) Assume that Z is an affine scheme over X. Prove that the representable presheaf h_Z defines a sheaf for the fpqc topology (hint: use Exercise 7 of Sheet 1).

(2) Prove that any representable presheaf on the category of schemes over X defines a sheaf for the fpqc topology.

For the second question, we can assume the following fact: if $f : X \to Y$ is a faithfully flat morphism, a subset of Y is open if and only its inverse image in X is open in X.

Exercise 5. If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a map of ringed spaces, we define the pullback of a sheaf $\mathcal{F}$ of $\mathcal{O}_Y$-modules as $f^*\mathcal{F} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. This gives a right exact functor:

$$f^* : \mathcal{O}_Y\text{−Mod} \to \mathcal{O}_X\text{−Mod}.$$

If $\mathcal{F}$ is a module of $\mathcal{O}_X$-modules, we say that $\mathcal{F}$ is quasi-coherent if for every point $x \in X$ there exists an open neighborhood $x \in U \subset X$ such that $\mathcal{F}|_U$ is isomorphic to the cokernel of a map of the form $\bigoplus_{j\in J} \mathcal{O}_U \to \bigoplus_{i\in I} \mathcal{O}_U$.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\alpha : R \to \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism from a ring R into the ring of global sections on X. Let M be an R-module. Let $\pi : (X, \mathcal{O}_X) \to (\{*\}, R)$ be the morphism of ringed spaces giving α on the associated rings. Prove that the following three constructions give canonically isomorphic sheaves of $\mathcal{O}_X$-modules:

(a) $\mathcal{F}_1 := \pi^*M$,

(b) $\mathcal{F}_2 := \operatorname{coker}(\bigoplus_{j\in J} \mathcal{O}_X \to \bigoplus_{i\in I} \mathcal{O}_X)$ for $\bigoplus_{j\in J} R \to \bigoplus_{i\in I} R \to M \to 0$ a presentation of M. Note that the map on the component $\mathcal{O}_X$ corresponding to $j \in J$ is given by the section $\sum_{i\in I} \alpha(r_{i,j})$ where the $r_{i,j}$ are the matrix coefficients of the map in the presentation of M,

(c) $\mathcal{F}_3$ is equal to the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U) \otimes_R M$, where the map $R \to \mathcal{O}_X(U)$ is the composition of α and the restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$.

The sheaf obtained is usually denoted by $\mathcal{F}_M$.

Exercise 6. Let X be a scheme. Prove that for any quasi-coherent sheaf $\mathcal{F}$ on X_{Zar} the presheaf

$$\mathcal{F}_{\text{fpqc}} : \begin{cases} \text{Sch}(X_{\text{fpqc}}) & \to \text{Ab} \\ (f : U \to X) & \mapsto \Gamma(U, f^*\mathcal{F}) \end{cases}$$

is a sheaf on the fpqc site.