For X a scheme, we will denote by X_{zar} its Zariski site and X_{fpqc} its fpqc site.

Exercise 1. Let $\mathcal{T}$ be a site. For $\mathcal{U} := \{U_i \to U\}_{i\in I}$ a covering in $\mathcal{T}$ and $\mathcal{F}$ an abelian presheaf, the Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})$ is defined as

$$\prod_{i_0\in I}\mathcal{F}(U_{i_0}) \to \prod_{i_0,i_1\in I}\mathcal{F}(U_{i_0}\times_U U_{i_1}) \to \prod_{i_0,i_1,i_2}\mathcal{F}(U_{i_0}\times_U U_{i_1}\times_U U_{i_2}) \to \cdots$$

where the first term is in degree 0 and the differential in degree p is defined by the formula

$$d(s)_{i_0\dots i_{p+1}} = \sum\nolimits_{j=0}^{p+1}(-1)^j s_{i_0\dots\hat{i}_j\dots i_{p+1}}|_{U_{i_0\cdots i_{p+1}}}.$$

The Čech cohomology $\check{H}^\bullet(\mathcal{U},\mathcal{F})$ of the presheaf $\mathcal{F}$ associated to the covering $\mathcal{U}$ is defined as the cohomology of this complex $\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F})$.

(1) For $\mathcal{U}$ a covering, prove that the functor $\check{\mathcal{C}}^\bullet(\mathcal{U},-)$ is exact on the category $\mathrm{PreShvAb}(\mathcal{T})$.

(2) For a presheaf of sets $\mathcal{G}$ define $\mathbf{Z}_{\mathcal{G}}$ by $\mathbf{Z}_{\mathcal{G}}(U) := \bigoplus_{g\in\mathcal{G}(U)}\mathbf{Z} := \mathbf{Z}[\mathcal{G}(U)]$. The functor $\mathcal{G}\mapsto\mathbf{Z}_{\mathcal{G}}$ is left adjoint to the forgetful functor $\mathrm{PreShvAb}(\mathcal{T})\to\mathrm{PreShv}(\mathcal{T})$. When $\mathcal{G}=h_U$ we write $\mathbf{Z}_U := \mathbf{Z}_{h_U}$. Note that for any presheaf of sets $\mathcal{F}$,

$$\mathrm{Hom}_{\mathrm{PreShvAb}(\mathcal{T})}(\mathbf{Z}_U,\mathcal{F}) = \mathrm{Hom}_{\mathrm{PreShv}(\mathcal{T})}(h_U,\mathcal{F}) = \mathcal{F}(U).$$

For $\{U_i\to U\}_i$ a covering in $\mathcal{T}$, define the complex

$$\mathbf{Z}_U^\bullet := \left(\mathbf{Z}_{\amalg_{i_0\in I}U_{i_0}} \leftarrow \mathbf{Z}_{\amalg_{(i_0,i_1)\in I^2}U_{i_0}\times_U U_{i_1}} \leftarrow \mathbf{Z}_{\amalg_{(i_0,i_1,i_2)\in I^3}U_{i_0}\times_U U_{i_1}\times_U U_{i_2}} \leftarrow \cdots\right)$$

(a) Prove that the complex of abelian presheaves $\mathbf{Z}_U^\bullet$ is exact in negative degrees.

(b) For $\mathcal{U}$ a covering and $\mathcal{F}$ an abelian presheaf, prove that $\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}) = \mathrm{Hom}_{\mathrm{PreShvAb}(\mathcal{T})}(\mathbf{Z}_U^\bullet,\mathcal{F})$.

(3) Deduce that if $\mathcal{I}$ is an injective object of $\mathrm{PreShvAb}(\mathcal{T})$ and $\mathcal{U}$ is a covering, then $\check{H}^i(\mathcal{U},\mathcal{I})$ is zero for all $i>0$.

It can be deduced from (3) that the $\check{H}^i(\mathcal{U},-)$'s are the right derived functors of $\check{H}^0(\mathcal{U},-)$ (see for example Chapter III, section 2 of Milne's book *Étale cohomology* for details). Moreover, we have the following result: for any covering $\mathcal{U}=\{U_i\to U\}_{i\in I}$ of U an object of $\mathcal{T}$ and any abelian sheaf $\mathcal{F}$ on $\mathcal{T}$ there is a spectral sequence

$$E^2_{p,q} = \check{H}^p(\mathcal{U},\underline{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(U,\mathcal{F}), \tag{1}$$

where $\underline{H}^q(\mathcal{F})$ is the abelian presheaf $U\mapsto H^q(U,\mathcal{F})$.

Exercise 2. Let $\mathcal{T}$ be a site. Let $\mathcal{F}$ be an abelian sheaf on $\mathcal{T}$ and U an object of $\mathcal{T}$. For $n>0$ an integer and $\xi\in H^n(U,\mathcal{F})$ a cocycle, prove that there exists a covering $U=\{U_i\to U\}_{i\in I}$ of U in $\mathcal{T}$ such that $\xi|_{U_i}=0$ for all $i\in I$.

Exercise 3. Let $A \to B$ be a faithfully flat morphism of rings. Prove that the complex

$$B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \cdots$$

where the differentials are given by

$$d(b_0 \otimes \cdots \otimes b_{n+1}) = \sum_{i=0}^{n} (-1)^i b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_{i+1} \otimes \cdots \otimes b_{n+1}$$

is exact.

Exercise 4.

(1) Let U be a scheme and $\mathcal{U} = \{U_i \to U\}_{i \in I}$ an fpqc-covering of U. Let $V = \amalg_{i \in I} U_i$. Consider the fpqc covering $\mathcal{V} := \{V \to U\}$. Prove that that the Čech complexes $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ and $\check{C}^\bullet(\mathcal{V}, \mathcal{F})$ agree whenever $\mathcal{F}$ is an abelian sheaf.

(2) Assume X is an affine scheme and let $\mathcal{F}$ be a (Zariski) quasi-coherent $\mathcal{O}_X$-module. Using Exercises 2 and 3 and the spectral sequence (1), prove that for all $i > 0$, we have $H^i(X_{\text{fpqc}}, \mathcal{F}_{\text{fpqc}}) = 0$, where $\mathcal{F}_{\text{fpqc}}$ is the fpqc sheaf defined in Sheet 2, Exercise 6.