Sheet 4: Étale fundamental group

Let X be a scheme. We say X is integral if it is nonempty and for every nonempty affine open $\mathrm{Spec}(A) = U \subset X$ the ring A is an integral domain. If X is integral then X has a unique generic point η and the local ring $\mathcal{O}_{X,\eta}$ is a field, called the function field of X and denoted $K(X)$. We say that a scheme is normal if all of its local rings are integrally closed domains.

Let X be a normal integral scheme and $K := K(X)$. Let L be a finite separable field extension of K. For an open set $U \subset X$, let $\mathcal{A}(U)$ be the integral closure of $\mathcal{O}_X(U)$ in L and $\mathcal{A}(\emptyset) = \{0\}$. Then $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-algebras and gives rise to an affine morphism $Y \to X$ with $Y = \mathrm{Spec}\,\mathcal{A}$. The scheme Y is called the normalization of X in L. We say that X is unramified in L if $Y \to X$ is unramified. We have the following result:

Theorem 1 *Let X be a normal integral scheme with function field K.*

1. *Let L be a finite separable field extension of K such that X is unramified in L. Then the normalization of X in L is a connected finite étale covering of X. Moreover every connected finite étale covering of X arises in this way.*
2. *Let $\overline{K}$ be an algebraic closure of K and M the composite of all finite separable field extensions $L \subset \overline{K}$ of K for which X is unramified in L. Then the fundamental group $\pi^1_{\text{ét}}(X)$ is isomorphic to the Galois group $\mathrm{Gal}(M/K)$.*

Exercise 1. Use the above result to describe the étale fundamental group of $\mathrm{Spec}(\mathbf{Z}_p)$, $\mathrm{Spec}(\mathbf{Z})$ and $\mathrm{Spec}(\mathcal{O}_K[1/x])$ for $\mathcal{O}_K$ the ring of algebraic integers in an algebraic number field K and $x \in \mathcal{O}_K$ non-zero.

Exercise 2.

(1) Using the fact that the étale fundamental group is the profinite completion of the usual one, prove that $\pi_1^{\text{ét}}(\mathbb{A}^1_{\mathbf{C}})$ is trivial.

(2) Consider the scheme $X = \mathrm{Spec}\,\mathbf{C}[x,y]/(y^2 - x^3)$ and let $Y \to X$ be a connected étale cover of X. Consider the normalization $\varphi : \mathrm{Spec}(\mathbf{C}[t]) \to X$ sending $x \mapsto t^2$ and $y \mapsto t^3$.

(a) Prove that the base change $Y' := Y \times_X \mathrm{Spec}(\mathbf{C}[t]) \to \mathrm{Spec}(\mathbf{C}[t])$ is a connected étale covering and deduce from (1) that Y' is trivial.

(b) Prove that Y is a trivial covering. What is the étale fundamental group of X?

Exercise 3. Let X be a scheme of finite type over a field k and let $X_{\overline{k}}$ denote its base change to an algebraic closure $\overline{k}$ of k. Assume X and $X_{\overline{k}}$ are connected. We fix a geometric point of X which we will omit from the notation. Prove that we have an exact sequence:

$$0 \to \pi^1_{\text{ét}}(X_{\overline{k}}) \xrightarrow{i} \pi^1_{\text{ét}}(X) \xrightarrow{j} \pi^1_{\text{ét}}(\mathrm{Spec}(k)) \to 0.$$

Exercise 4. What is the étale fundamental group of $\mathbb{A}^1_{\mathbb{R}}$ the real affine line?