Sheet 5: Étale cohomology with $\mathbb{G}_m$-coefficients

Let X be a noetherian integral scheme with generic point η and function field $K := \mathcal{O}_{X,\eta}$. Moreover, we assume that X is regular.

Exercise 1. Let $j : \mathrm{Spec}(K) \to X$ be the canonical morphism. The natural projection $X_{\text{ét}} \to \mathrm{Spec}(K)_{\text{ét}}$ induces a functor $j_* : \mathrm{ShvAb}(\mathrm{Spec}(K)_{\text{ét}}) \to \mathrm{ShvAb}(X_{\text{ét}})$. Let $(\mathbb{G}_m)_X$ (respectively $(\mathbb{G}_m)_K$) denote the sheaf of abelian groups $T \mapsto \Gamma(T, \mathcal{O}_T^*)$ on $X_{\text{ét}}$ (respectively on $\mathrm{Spec}(K)_{\text{ét}}$).

(1) Prove that $\nu : (\mathbb{G}_m)_X \to j_*(\mathbb{G}_m)_K$ is injective.

We define the sheaf of Cartier divisors $\mathrm{Div}_X := \mathrm{Div}_{X_{\text{ét}}}$ on $X_{\text{ét}}$ to be the cokernel of ν. We define in the same way $\mathrm{Div}_{X_{\text{zar}}}$ using the respective Zariski site. The set of Cartier divisors $\mathrm{Div}(X)$ is then equal to $\mathrm{Div}_{X_{\text{zar}}}(X)$.

(2) Show that the global sections $\mathrm{Div}_X(X)$ coincide with the set of Cartier divisors.

Let $X^{(1)} := \{x \in X \mid \dim(\mathcal{O}_{X,x}) = 1\}$. For all $x \in X^{(1)}$ let $i_x : \mathrm{Spec}(k(x)) \to X$ be the canonical inclusion and let $\mathbf{Z}_x$ denote the constant sheaf on $\mathrm{Spec}(k(x))_{\text{ét}}$ with values in $\mathbf{Z}$. There is a canonical morphism of sheaves on $X_{\text{ét}}$,

$$\mathrm{cyc} : \mathrm{Div}_X \to \bigoplus_{x \in X^{(1)}} (i_x)_*(\mathbf{Z}_x)$$

where the global sections of the right hand side can be identified with the abelian group of Weil divisors on X. As X is regular, one can show that cyc is an isomorphism.

Exercise 2. The goal of this exercise is to compute the cohomology of the sheaf Div_X.

(1) Let G be a profinite group and A be a torsion free abelian group which we consider as a G-module via the trivial G-action on A. Show that $H^1(G, A) = 0$.

(2) Use the Leray spectral sequence

$$E_2^{p,q} := H^p_{\text{ét}}(X, \mathrm{R}^q i_{x,*}(\mathbf{Z}_x)) \Rightarrow H^{p+q}_{\text{ét}}(\mathrm{Spec}(k(x)), \mathbf{Z}_x)$$

to prove that $H^1_{\text{ét}}(X, \mathrm{Div}_X) = 0$.

(3) Prove that $H^2_{\text{ét}}(X, \mathrm{Div}_X)$ injects into $\bigoplus_{x \in X^{(1)}} \mathrm{Hom}_{\text{cont}}(\mathrm{Gal}(k(x)^{\text{sep}}/k(x)), \mathbf{Q}/\mathbf{Z})$.

Exercise 3. The goal of this exercise is to compute the cohomology of the sheaf $j_*(\mathbb{G}_m)_K$.

(1) For $\overline{x}$ a geometric point of X, prove that the stalk $(\mathrm{R}^1 j_*(\mathbb{G}_m)_K)_{\overline{x}}$ is zero and deduce that $\mathrm{R}^1 j_*(\mathbb{G}_m)_K$ is zero.

(2) Use the Leray spectral sequence

$$E_2^{p,q} := H^p_{\text{ét}}(X, \mathrm{R}^q j_*(\mathbb{G}_m)_K) \Rightarrow H^{p+q}_{\text{ét}}(\mathrm{Spec}(K), (\mathbb{G}_m)_K)$$

to prove that $H^0_{\text{ét}}(X, j_*(\mathbb{G}_m)_K) = K^*$, $H^1_{\text{ét}}(X, j_*(\mathbb{G}_m)_K) = 0$ and $H^2_{\text{ét}}(X, (j_*\mathbb{G}_m)_K)$ injects into $H^2_{\text{ét}}(\mathrm{Spec}(K), (\mathbb{G}_m)_K)$.

Exercise 4. Use the above exercises to compute $H^1_{\text{ét}}(X, (\mathbb{G}_m)_X)$.