Sheet 6: Constant, locally constant and constructible sheaves

For X an arbitrary topological space, we say a subset $Z \subset X$ is retrocompact if the inclusion map $i : Z \hookrightarrow X$ is quasi-compact, i.e. the inverse image $i^{-1}(V)$ of every quasi-compact open $V \subset X$ is quasi-compact. We say that $Z \subset X$ is constructible in X if Z is a finite union of subsets of the form $U \cap V^c$ where $U, V \subset X$ are open and retrocompact in X.

If X is a scheme, a constructible locally closed subscheme of X is a locally closed subscheme $T \subset X$ such that the underlying topological space of T is a constructible subset of X. Let Λ be a Noetherian ring. We say that a sheaf of Λ-modules $\mathcal{F}$ on $X_{\text{ét}}$ is constructible if for every affine open $U \subset X$ there exists a finite decomposition of U into constructible locally closed subschemes $U = \coprod_i U_i$ such that $\mathcal{F}|_{U_i}$ is of finite type and locally constant for all i. It can be proved that kernel and cokernel of maps of constructible sheaves are constructible.

Exercise 1. Prove that if X is a quasi-separated scheme then every quasi-compact open of X is retrocompact in X.

Exercise 2.

(1) Prove that the collection of constructible sets is closed under finite intersections, finite unions and complements.

(2) Let X be a topological space. Suppose we can write $X = T_1 \cup \cdots \cup T_n$ as a union of constructible subsets. Prove that there exists a finite stratification $X = \coprod X_i$ with each X_i constructible such that each T_k is a union of strata.

(3) Let X be a quasi-compact and quasi-separated scheme and let Λ be a Noetherian ring. Let $\mathcal{F}$ be a sheaf of Λ-modules on $X_{\text{ét}}$. Prove that the following statements are equivalent:

(a) $\mathcal{F}$ is constructible.

(b) There exists an open covering $X = \bigcup_i U_i$ such that $\mathcal{F}|_{U_i}$ is constructible.

(c) There exists a partition $X = \coprod_i X_i$ by constructible locally closed subschemes such that $\mathcal{F}|_{X_i}$ is finite locally constant.

For $j : U \to X$ an étale morphism, the extension by zero functor $j_!$ is defined as the left adjoint of the functor $j^{-1} : \mathrm{ShvAb}(X_{\text{ét}}) \to \mathrm{ShvAb}(U_{\text{ét}})$ (see Section 03S2 on the Stack Project). If j is finite étale, we have $j_* \simeq j_!$. Moreover, it can be showed that it commutes with base change, i.e. for any cartesian diagram

$$\begin{array}{ccc} Y \times_X V & \xrightarrow{j'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{j} & X \end{array}$$

with j étale, then $(j')_!(f')^{-1} = f^{-1} j_!$.

Exercise 3. Let X be a qcqs scheme and $U \to X$ an étale map with U qcqs. Let Λ be a noetherian ring.

(1) For M a finite Λ-module, prove that $j_!\underline{M}$ is constructible. (Hint: use that a finite étale morphism is locally isomorphic to a disjoint union of isomorphisms.)

(2) Prove that the category of constructible sheaves of Λ-modules is exactly the category of modules of the form

$$\mathrm{Coker}\Big(\bigoplus_{k=1,\dots,m} (j_{V_k})_!\underline{\Lambda} \to \bigoplus_{i=1,\dots,n} (j_{U_i})_!\underline{\Lambda}\Big)$$

with V_k and U_i quasi-compact and quasi-separated objects of $X_{\text{ét}}$.

Exercise 4.

(1) Let $n > 2$. Prove that the étale sheaf μ_n on $\mathrm{Spec}(\mathbf{Q})_{\text{ét}}$ is locally constant but not constant.

(2) Let $n > 2$. Prove that the étale sheaf μ_n on $\mathrm{Spec}(\mathbf{Z}[\frac{1}{n}])_{\text{ét}}$ is locally constant but not constant.

(3) Let R be a DVR and denote by K the associated fraction field. Let j be the immersion $\mathrm{Spec}(K) \hookrightarrow \mathrm{Spec}(R)$. Prove that $j_!(\mathbf{Z}/2\mathbf{Z})$ is constructible but not locally constant.