Sheet 8: Left-complete derived categories – counter-examples

The goal of this sheet is to give a criterion for the derived category of a topos to **not be** left-complete. We fix a topos $\mathcal{X} := \mathrm{Shv}(\mathcal{C}, \tau)$ and for each n, an object $\mathcal{F}_n$ of $\mathcal{X}$. We define the complex $\mathcal{F} := \bigoplus_{n\geq 1} \mathcal{F}_n[n]$ in $D(\mathcal{X})$.

Exercise 1. Prove that $\widehat{\mathcal{F}} := \mathrm{R}\varprojlim_n \tau_{\geq -n}\mathcal{F}$ is computed by $\mathrm{R}\prod_n \mathcal{F}_n[n]$.

Exercise 2. Suppose there exists a filtered system $(U_i)_{i\in I}$ of object of $\mathcal{C}$ such that the functor $\Phi : \mathrm{PreShv}(\mathcal{C}) \to \mathrm{Ab}, \mathcal{F} \mapsto \mathrm{colim}_{i\in I}\mathcal{F}(U_i)$ factors through the sheafification functor $\mathrm{PreShv}(\mathcal{C}) \to \mathrm{Shv}(\mathcal{C})$.

(1) Prove that Φ preserves quasi-isomorphisms and induces directly (i.e. without needing to be derived) a functor $D(\mathrm{Shv}(\mathcal{C})) \to D(\mathrm{Ab})$.

(2) Prove that $H^0(\Phi(\mathrm{R}\prod_n \mathcal{F}_n[n])) = \varinjlim_I \prod_n H^n_\tau(U_i, \mathcal{F}_n)$.

(3) Prove that if $H^0(\Phi(\widehat{\mathcal{F}}))$ is non-zero then $\mathcal{F} \to \widehat{\mathcal{F}}$ is not a quasi-isomorphism.

(4) We assume that I has an initial object 0. Assume the following two hypothesis:

(a) There exist classes $\alpha_n \in H^n_\tau(U_0, \mathcal{F}_n)$ which are exactly of p^n-torsion.

(b) For each $i \in I$, there exist an integer $d_i \in \mathbf{N}$ and maps $H^n_\tau(U_i, \mathcal{F}_n) \to H^n_\tau(U_0, \mathcal{F}_n)$ such that for all $n \geq 1$, the composition $H^n_\tau(U_0, \mathcal{F}_n) \to H^n_\tau(U_i, \mathcal{F}_n) \to H^n_\tau(U_0, \mathcal{F}_n)$ is equal to $d_i \cdot \mathrm{Id}$.

Prove that $H^0(\Phi(\widehat{\mathcal{F}}))$ is non-zero.

Exercise 3. Let $G := \prod_{n\geq 1} \mathbf{Z}_p$. Consider the topos $\mathcal{X}$ associated to the category $B(G)$ of finite G-sets.

(1) Prove that we can find a filtered system $(U_i)_{i\in I}$ as in Exercise 2.

(2) Prove that for each $n \geq 1$ there exists a canonical class $\alpha^1_n \in H^1(\mathbf{Z}_p, \mathbf{Z}/p^n)$ which is exactly of p^n-torsion. Deduce that for any integer m, there exists a class α^m_n in $H^m(\prod_{i=1}^m \mathbf{Z}_p, \mathbf{Z}/p^n)$ which is exactly of p^n-torsion and then, that there exists a class α_n in $H^n(G, \mathbf{Z}/p^n)$ which is exactly of p^n-torsion.

(3) Using transfer maps coming from Group Cohomology Theory prove that the hypothesis (b) from Exercise 2 is satisfied in this setting and conclude that $D(\mathcal{X})$ is not left-complete.

Exercise 4. Let k be the field $\mathbf{C}(x_1, x_2, x_3, \dots)$. Consider the étale topos $\mathcal{X}$ associated to $\mathrm{Spec}(k)$.

(1) Prove that we can find a filtered system $(U_i)_{i\in I}$ as in Exercise 2.

(2) Prove that there is a surjection $\mathrm{Gal}(k^{\mathrm{sep}}/k) \to \prod_{n\geq 1} \mathbf{Z}_p$.

(3) Using the previous exercise, prove that $D(\mathcal{X})$ is not left-complete.