Analytic stacks : Timestamp guide

This is an attempted transcription of the main results stated in the 2024 course on "Analytic Stacks" by Claussen and Scholze on IHES/Bonn. This is meant to serve as a guide as well as an overview of what happens in each lecture and I have not plan at the moment of transcribing the proofs as well. To compensate, I have included a time-stamp to indicate which precise moment in the lecture videos (kindly made freely available on Youtube) in which they occur.

Naturally, I may have introduced many typos and, sparingly, some new notation where I thought it would be better displayed as LaTeX documents have a diffferent visibility than blackboards. For time reasons, I also do not plan on covering the whole course.

Lecture 1 : Introduction

Lecture 2 : Light condensed sets I

Proposition 2.1 (15'33", "Stone Duality"). The following categories are equivalent:

1. Pro(Fin), the category whose objects are formal limits $\lim_I S_i$ with S_i a finite set and

$$\mathrm{Hom}(\lim S_i, \lim T_j) = \lim_I \operatorname*{colim}_J \mathrm{Hom}(S_i, T_j).$$

2. The category of totally disconnected compact Hausdorff spaces.

3. The opposite category to Boolean algebras (commutative rings for which $x^2 = x$ for all x).

This category will be call the category Prof of profinite sets.

Definition 2.2 (23'16"). Let $S = \lim_I S_i$ be a profinite set.

- We define the *size* of S to be the cardinailty $\kappa = |S|$ of the underlying set of S.
- We define the *weight* of S to be the cardinality $\lambda = |\mathrm{Cont}(S,2)|$ of the set of continuous functions $S \to 2 = \{0,1\}$.
- We say that S is *light* precisely when $\lambda \leq \omega = |\mathbf{N}|$.

In this case $\lambda = |I|$ for the smallest possible I.

Examples (27'52").

- For $S = \mathbf{N} \cup \{\infty\} = \lim\{1,2,\dots,\infty\}$ we have $\kappa = \omega$, $\lambda = \omega$.
- For S = Cantor set, we have $\lambda = \omega$ and $\kappa = 2^\omega$.
- For $S = \beta\mathbf{N}$ we have $\lambda = 2^\omega$ and $\kappa = 2^{2^\omega}$.

Proposition 2.3 (33'08"). For all profinite sets $\kappa \leqq 2^\lambda$ and $\lambda \leqq 2^\kappa$. If $|I|$ is infinite then $\lambda \leqq \kappa$.

Theorem 2.4 (38'33"). *The following categories are equivalent to light profinite sets.*

1. Pro(Fin)*, the category whose objects are formal* sequential *limits* $\lim_I S_i$ *with* S_i *a finite set and*

$$\mathrm{Hom}(\lim S_i, \lim T_j) = \lim_J \operatorname{colim}_I \mathrm{Hom}(S_i, T_j) = \operatorname*{colim}_{\substack{f:\mathbf{N}\to\mathbf{N} \\ \text{str. increasing}}} \lim_J \mathrm{Hom}(S_{f(j)}, T_j).$$

2. *The category of* metrizable *totally disconnected compact Hausdorff spaces.*
3. *The opposite category to* countable *Boolean algebras.*

Proposition 2.5 (42'50"). The category of light profinite sets has all countable limits. Sequential limits of surjections are surjective.

Proposition 2.6 (45'). A profinite set S is light if and only if there is a surjection

$$\prod_\omega \{0,1\} \twoheadrightarrow S$$

from the Cantor set.

Proposition 2.7 (48'). Let S be a light profinite set. Then any $U \subset S$ open is a countable disjoint union of light profinite sets. (In general, there can be a $U \subset S$ with $\mathrm{H}^i(U, \mathbf{Z}) \neq 0$ for some $i > 0$).

Proposition 2.8 (53'06"). Let S be a light profinite set. Then S is an injective object in Prof, ie. for all inclusions $Z \subset X$ and $Z \to S$ of profinite sets there is a unique map $X \to S$ extending it. Pictorially:

$$\begin{array}{ccc} Z & & \\ \downarrow & \searrow & \\ X & \dashrightarrow & S \end{array}$$

Definition 2.9 (60"). A *light condensed set* is a sheaf on the category of light profinite sets with the Grothendieck topology generated by

- Finite disjoint sums,
- Surjective maps.

Equivalently, it is a functor $X : \mathtt{Pro}_{\mathbf{N}}(\mathrm{Fin})^{\mathrm{op}} \to \mathrm{Set}$ such that

1. $X(\emptyset) = *$
2. $X(S_1 \coprod S_2) \xrightarrow{\sim} X(S_1) \times X(S_2)$
3. (Gluing condition) For all $T \twoheadrightarrow S$ the arrow

$$X(T) \xrightarrow{\sim} \mathrm{eq}(X(S) \rightrightarrows X(T \times_S T))$$

is an isomorphism.

Example (66'23"). Given a topological space A one has a light condensed set

$$\underline{A} : S \mapsto \mathrm{Cont}(S, A).$$

We recover the set A as $\underline{A}(*)$, called the *underlying set* of the profinite set. We also have $\underline{A}(\mathbf{N} \cup \infty)$ is the set of all convergent sequences with a fixed limit point on A. The set $\underline{A}$(Cantor set) comes equipped with an action of the endomorphism monoid of the Cantor set.

Remark (71'11"). A (light) condensed set X is determined by X(Cantor set) together with its action by End(Cantor set).

Proposition 2.10 (74'19"). The functor $A \mapsto \underline{A}$ from topological spaces to (light) condensed sets admits a left adjoint

$$X(*)_{\mathtt{top}} = \left(\bigsqcup_{(S,\alpha\colon S \to X)} S \right) / \sim,$$

which is a metrizably compactly generated topological space. If A is any metrizably compactly generated topological space, then $A \xrightarrow{\sim} \underline{A}(*)_{\mathtt{top}}$. Hence the functor $A \mapsto \underline{A}$ is fully faithful on such.

Light condensed abelian groups

(Around 88'): From general topos non-sense, we have that sheaves of abelian groups on $\mathtt{Pro}_{\mathbf{N}}(\mathrm{Fin})$ (= Abelian group objects on the category of light condensed sets) form an abelian category with all limits, colimits and filtered colimits are exact.

Definition 2.11 (88'27"). The category $\mathrm{CondAb}^{\mathtt{light}}$ is the Grothendieck abelian category defined above.

Example (90'). What are the cokernels (as condensed abelian groups) of $\mathbf{Q} \hookrightarrow \mathbf{R}$ and $\mathbf{R}_{\mathrm{disc}} \hookrightarrow \mathbf{R}$?

- $(\mathbf{R}/\mathbf{Q})(*) = \mathbf{R}/\mathbf{Q}$.
- $(\mathbf{R}/\mathbf{Q})(S) = \mathrm{Cont}(S, \mathbf{R})/\mathrm{LocConst}(S, \mathbf{Q})$.
- $(\mathbf{R}/\mathbf{R}_{\mathrm{disc}})(*) = \mathbf{R}/\mathbf{R} = 0$.
- $(\mathbf{R}/\mathbf{R}_{\mathrm{disc}})(S) = \mathrm{Cont}(S, \mathbf{R})/\mathrm{LocConst}(S, \mathbf{R})(\neq 0!)$.

Theorem 2.12 (95'32"). *The Grothendieck abelian category* $\mathrm{CondAb}^{\mathtt{light}}$ *satisfies:*

- *Countable product are exact (and satisfy [AB6]).*
- *The free abelian group*[1] $\mathbf{Z}[\mathbf{N} \cup \{\infty\}]$ *is* internally *projective*[2].

[1]The left adjoint to the forgetful functor map $\mathrm{CondAb}^{\mathtt{light}} \to \mathrm{Cond}^{\mathtt{light}}$.
[2]As in, $\underline{\mathrm{Hom}}(\mathbf{Z}[\mathbf{N} \cup \{\infty\}], \_)$ is exact.

Lecture 3 : Light condensed sets II

(Around 2'50":) We have an Yoneda embedding $\mathtt{Pro}_{\mathbf{N}}(\mathrm{Fin}) \hookrightarrow \mathrm{Cond}^{\mathtt{light}} = \mathrm{Shv}(\mathtt{Pro}_{\mathbf{N}}(\mathrm{Fin})$ which factors via

$$\mathtt{Pro}_{\mathbf{N}}(\mathrm{Fin}) \hookrightarrow \mathrm{Cond}^{\mathtt{light}} \hookrightarrow \mathsf{Top} \to \mathrm{Shv}(\mathtt{Pro}_{\mathbf{N}}(\mathrm{Fin}))$$

where the second functor is the $A \mapsto \underline{A}$ as seen last lecture. This last functor has an inverse which can be computed the following ways:

$$X(*)_{\mathrm{top}} = \left(\bigsqcup_{(S,\alpha\colon S\to X)} S \right) / \sim = \left(\bigsqcup_{\alpha\colon K \to X} K \right) / \sim = \left(\bigsqcup_{\alpha\colon \mathbf{N}\cup\{\infty\}\to X} S \right) / \sim,$$

where $K = \prod_{\omega}\{0,1\}$ is the Cantor set.

This underlying space is a metrizably compactly generated which is the same as sequential space[3]. For such spaces functor $A \mapsto \underline{A}$ is fully faithful.

Definition 3.1 (12'23")**.** Quasi-compact and quasi-separated objects in a topos.

- An object X (in any topos, in particular in $\mathrm{Cond}^{\mathtt{light}}$) is called *quasi-compact* (qc) if any cover $\sqcup X_i \twoheadrightarrow X$ (ie. the previous map is an epimorphism) admits a finite subcover. Here, this is equivalent to either existing a surjection $K \twoheadrightarrow X$ from the Cantor set to X or X being empty.

- An object X is said to be *quasi-separated* (qs) if for all quasi-compact objects Y, Z and maps $Y, Z \to X$ the fibered product $Y \times_X Z$ is quasi-compact also. Here we want for all $f, g\colon K \to X$ that $K \times_X K$ is quasi-compact.

- We say that X is qcqs if it is both quasi-compact and quasi-separated.

Proposition 3.2 (19'09")**.** 1. The category of qcqs light condensed sets is equivalent to the category of metrizable compact Hausdorff spaces.

2. The category of qs light condensed sets is equivalent to the ind-category of metrizable compact Hausdorff spaces along injections; it contains the category of metrizable CGWH spaces (recall from algebraic topology I. . .).[4].

[3]This is claimed in 8'46" but I think any metrizable space is sequential.

[4]But colimits in mCGWH need not agree with colimits in light condensed sets, unless the index set is itself countable

Light condensed abelian groups (II)

(Around 31'58":) From the general theory of sheaves on topoi we get:

- CondAb is a Grothendieck abelian category;
- There is a symmetric monoidal structure ($\otimes$) with unit $\mathbf{Z} = \underline{\mathbf{Z}} : S \mapsto \mathrm{LocConst}(S, \mathbf{Z})$ and

$$M \otimes N = \text{sheafification of } S \mapsto M(S) \otimes N(S)$$

- The forgetful $\mathrm{CondAb}^{\mathtt{light}} \to \mathrm{Cond}^{\mathtt{light}}$ has a left adjoint, the *free condensed abelian group* $X \mapsto \mathbf{Z}[X]$, obtained as the sheafification of $S \mapsto \mathbf{Z}[X(S)]$.

Example (36'58")**.** $\mathbf{Z}[\underline{\mathbf{R}}] = \{\sum_{x\in\mathbf{R}} n_x[x], \quad n_x \in \mathbf{Z} \text{ with finite support.}\}$

Theorem 3.3 (41'44")**.** *In* $\mathrm{CondAb}^{\mathtt{light}}$*:*

1. *Countable products are exact.*
2. *Sequential limits of surjections remain surjective.*
3. $\mathbf{Z}[\mathbf{N} \cup \{\infty\}]$ *is internally projective. (P is called internall projective if the sheafification of* $S \mapsto \mathrm{Ext}^i(P \otimes \mathbf{Z}[S], M) = 0$ *for all M and* $i > 0$*).*

(1 implies 2 and Point 3 is what is really the hardest thing here.)

Remark (77'39")**.** Comparison with the old definition: in CondAb (so no lightness restriction) all products are exact and projective generators[5] exist: namely $\mathbf{Z}[S]$ for S extremally totally disconnected (eg. βX for X discrete, which we note is not light already when X countable).

(Transl. Remark) Also we note that CondAb has *no* (non-zero) injective objects whereas $\mathrm{CondAb}^{\mathtt{light}}$ is a Grothendieck abelian category and hence has enough injectives.

Cohomology

Definition 3.4 (79'45")**.** For X any light condensed set and M and abelian group we define

$$\mathrm{H}^i(X, \mathbf{Z}) = \mathrm{Ext}^i_{\mathrm{CondAb}^{\mathtt{light}}}(\mathbf{Z}[X], \underline{M}).$$

Theorem 3.5 (90'13")**.** *If X is a CW complex then* $\mathrm{H}^i(\underline{X}, M) \cong \mathrm{H}^i_{sing}(X, M)$.

[5] But none of them are internal!

Lecture 4 : Ext computations in (light) condensed abelian groups

We reduce the theorem stated last time (90'13") to the following.

Theorem 4.1 (6'20")**.** *If X is a CW complex then* $\mathrm{H}^i(\underline{X}, M) \cong \mathrm{H}^i_{sheaf}(X, M)$.

We recall that if X is a CW complex then singular and sheaf cohomologies agree, but the same is not true for totally disconnected (infinite) spaces (our building blocks).

(Around 16'20":) We upgrade the site $\mathrm{Op}(X)$ of opens of X (defining sheaf cohomology) to the site $\mathtt{Pro}_{\mathbf{N}}(\mathrm{Fin})_{/X}$. We get a geometric morphism

$$\mathrm{Cond}^{\mathtt{light}}_{/X} = \mathrm{Shv}(\mathtt{Pro}_{\mathbf{N}}(\mathrm{Fin})_{/X}) \xrightarrow{\lambda} \mathsf{Sh}(X)$$

which is given by $\lambda^* U = \underline{U}$. This allows us to build a map on the derived level

$$\lambda^* : D(X) \to D(\mathrm{Cond}^{\mathtt{light}}_{/X}).$$

Now the theorem above follows from the following theorem:

Theorem 4.2 (20'22")**.** *On D^+ the functor λ^* above is fully faithful. In particular, since cohomology is computed as abelian groups of morphisms in the derived level, we have that*

$$\mathrm{H}^i(\mathrm{Cond}^{\mathtt{light}}_{/X}, \lambda^* \mathscr{F}) \cong \mathrm{H}^i_{sheaf}(X, \mathscr{F})$$

for all $\mathscr{F}$ in $D^+(X)$.

(Around 34'33":) Explicitly, to compute $\mathrm{H}^i(\underline{X}, \mathbf{Z})$ one tries to find a projective (or at least acyclic) resolution of $\mathbf{Z}[X]$.

Step 1 If $X = S \in \mathtt{Pro}_{\mathbf{N}}(\mathrm{Fin})$ is totally disconnected, then

$$\mathrm{Ext}^i(\mathbf{Z}[S], \mathbf{Z}) = \begin{cases} \mathrm{Cont}(S, \mathbf{Z}), & i = 0 \\ 0, & i > 0 \end{cases}$$

Which is shown by first noting that if $S_\bullet$ is a hypercover of S, all light profinite sets, then the exact chain complex $\mathbf{Z}[S_\bullet]$ induces an exact complex

$$0 \to \mathrm{Cont}(S, \mathbf{Z}) \to \mathrm{Cont}(S_0, \mathbf{Z}) \to \mathrm{Cont}(S_1, \mathbf{Z}) \to \dots$$

Step 2 The general metr. compact Hausdorff space X: resolve by profinite sets $S_\bullet$ (eg. starting with the Cantor set $K \twoheadrightarrow X$) and reduce to the complex describe above.

Locally compact abelian groups

(Around 51’08":) Consider the category LCA_m of locally compact metrizable abelian groups (eg. $\mathbf{R}$, $\mathbf{R}/\mathbf{Z}$, $\mathbf{Z}_p$, $\mathbf{A}_f = \widehat{\mathbf{Z}} \otimes \mathbf{Q}$, discrete groups...).

The structure of LCA_m is the following: every object admits a 3-step filtration with a discrete, an $\mathbf{R}$-vector space and a compact piece.

We can compute the Yoneda extension groups inside LCA_m. They vanish for $i > 1$.

Theorem 4.3 (53’58"). *Let $A, B \in \mathrm{LCA}_\mathrm{m}$. Then*

$$\mathrm{Ext}^i(\underline{A}, \underline{B}) \cong \begin{cases} \mathrm{Hom}_{\mathrm{LCA}}(A,B), & i = 0 \\ \mathrm{Ext}^1_{\mathrm{LCA}}(A,B), & i > 0, \\ 0, & i > 1. \end{cases}$$

Example (58’24").

$$\mathrm{Ext}^i(\underline{A}, \mathbf{R}/\mathbf{Z}) = \begin{cases} A^\vee, i = 0, \\ 0, i > 0 \end{cases}$$

Theorem 4.4 (63’57", Breen-Deligne). *There is a resolution of the form*

$$\cdots \to \mathbf{Z}[M^{n_i}] \to \cdots \to \mathbf{Z}[M^2] \xrightarrow{[a,b]\mapsto[a]+[b]-[a+b]} \mathbf{Z}[M] \to M \to 0$$

for any abelian group M in a functorial way (ie. the n_i do not vary with M and the differentials are canonically — albeit not explicitly — defined).

Hence, this also works in any topos by arguing pointwise on sheaf level.

(Around 72’49":) We also need that for all X compact Hausdorff abelian group

$$\mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{light}}}(\underline{X}, \mathbf{R}) = \begin{cases} \mathrm{Cont}(X, \mathbf{R}), & i = 0, \\ 0, & i > 0. \end{cases}$$

(The same holds true for all locally convex Banach spaces actually.)

Example (75’54"). $\mathrm{Ext}^i(\underline{\mathbf{R}}, \mathbf{Z}) = 0$ for all $i \geqq 0$. Can see now using Breen-Deligne and noting that $\mathrm{Ext}^i(\mathbf{Z}[\mathbf{R}^n], \mathbf{Z}) = \mathrm{H}^i_{\mathrm{sing}}(\mathbf{R}^n, \mathbf{Z})$.

Corollary 4.5 (87’21"). If M is a discrete abelian group, then

$$\mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{light}}}(\prod_\omega \mathbf{Z}, M) = \begin{cases} \bigoplus_\omega M, & i = 0, \\ 0, & i > 0. \end{cases}$$

The group $\prod_\omega \mathbf{Z}$ will be a compact projective generator of the category of solid abelian groups.

Warning: The product $\prod_\omega \mathbf{Z}$ in CondAb is actually given by

$$\prod_\omega \mathbf{Z} = \bigcup_{f:\mathbf{N}\to\mathbf{N}} \prod_{n\in\mathbf{N}} [[-f(n), f(n)]]$$

(Around 100':) Consider the following property:

($*$): For all sequential limits $\ldots \twoheadrightarrow M_1 \twoheadrightarrow M_0$ and discrete abelian groups N we have that $\mathrm{Ext}^i(\lim M_n, N) = \mathrm{colim}_n \mathrm{Ext}^i(M_n, N)$.

Then one can straighforwardly show that if the continuum hypothesis holds then ($*$) is not true. Moreover one has the following theorem.

Theorem 4.6 (102'48", Bergfalk, Lombie-Hanson, Hrušák, Bannister)**.** *The following hold true:*

1. ($*$) *implies that* $2^{\aleph_0} > \aleph_\omega$.
2. *It is consistent that* ($*$) *holds and* $2^{\aleph_0} = \aleph_{\omega+1}$.

Lecture 5 : Solid abelian groups

Goal: Isolate a class of "complete" objects in $\mathrm{CondAb}^{\mathtt{light}}$. We note that

$$\underline{\mathbf{Z}[[u]]} \otimes \underline{\mathbf{Z}[[v]]} \neq \underline{\mathbf{Z}[[u,v]]}$$

since the underlying abelian group of the left hand side is actually just $\mathbf{Z}[[u]] \otimes \mathbf{Z}[[v]]$.

(Caveat: it is difficult to find a notion of completeness that encompasses both $\mathbf{R}$ and $\mathbf{Z}_p$ in the condensed setting.)

Definition 5.1 (9'47")**.** We define the free condensed abelian group on a nullsequence to be

$$P = \mathbf{Z}[\mathbf{N} \cup \{\infty\}]/\mathbf{Z}[\infty].$$

It comes equipped with a "shift" function $S : P \to P, \quad [n] \mapsto [n+1]$.

Then P is internally projective, ie. one has that the internal hom

$$\underline{\mathrm{Hom}}(P, \_) : S \mapsto \mathrm{Hom}(P \otimes \mathbf{Z}[S], \_)$$

is an exact functor $\mathrm{CondAb}^{\mathtt{light}} \to \mathrm{CondAb}^{\mathtt{light}}$. In fact, it preserves *all* (not just finite) limits and colimits.

Let $f : P \to P$ be the morphism $1 - S = [n] - [n+1]$.

Definition 5.2 (13'52"). A light condensed abelian group M is called *solid* if

$$f^*: \underline{\mathrm{Hom}}(P,M) \xrightarrow{\sim} \underline{\mathrm{Hom}}(P,M)$$

is an equivalence. (We think of $\underline{\mathrm{Hom}}(P,M)$ as the space of nullsequences in M, and hence of f^* as the function $(m_0,m_1,\dots)\mapsto(m_0-m_1,m_1-m_2,\dots)$. Therefore for the inverse to exist we would need to be able to define a convergent infinite sum for every nullsequence, as in every complete non-archimedean ring.)

Proposition 5.3 (17'23"). The subcategory $\mathrm{Solid}\subset\mathrm{CondAb}^{\mathtt{light}}$ is an abelian sub-category stable under kernels, cokernels, extensions, limits, colimits, $\underline{\mathrm{Hom}}$ and $\underline{\mathrm{Ext}}^i$. We have that $\mathbf{Z}\in\mathrm{Solid}$.

Corollary 5.4 (23'58"). There is a left adjoint to $\mathrm{Solid}\subset\mathrm{CondAb}^{\mathtt{light}}$, called *solidification*, denoted by $M\mapsto M^{\square}$. Spelling out we have that $\mathrm{Hom}(M^{\square},N)=\mathrm{Hom}(M,N)$ for all $N\in\mathrm{Solid}$.

Moreover, Solid adquires a symmetric monoidal structure, namely

$$M\otimes^{\square}N=(M\otimes N)^{\square},$$

which preserves colimits in each variable. The solidification functor is enhanced to a symmetric monoidal one: there is a natural iso $(M\otimes N)^{\square}\xrightarrow{\sim}(M^{\square}\otimes N^{\square})^{\square}$.

Lemma 5.5 (37'14"). $\underline{\mathbf{R}}^{\square}=0$.

Corollary 5.6. If $M\in\mathrm{CondAb}^{\mathtt{light}}$ admits an $\underline{\mathbf{R}}$-module structure, then $M^{\square}=0$ and $\mathrm{Ext}^i(M,N)=0$ for all solid N.

(Around 50':) <u>Goal</u>: Compute $P^{\square}$.

Lemma 5.7 (50'40"). Let $\prod_{\omega}^{\mathtt{bdd}}\mathbf{Z}=\bigcup_{n\in\mathbf{N}}\prod_{\omega}[[-n,n]]\subset\prod_{\omega}\mathbf{Z}$ be the subspace of bounded sequences in the product. Then there is a natural map $P\to\prod_{\omega}^{\mathtt{bdd}}\mathbf{Z}$ given by the sequence (of sequences) $[n]\mapsto(0,0,\dots,0,1,0,\dots)$. Then

$$P^{\square}\xrightarrow{\sim}\left(\prod_{\omega}^{\mathtt{bdd}}\mathbf{Z}\right)^{\square},\quad \mathrm{Ext}^i(P,M)\xleftarrow{\sim}\mathrm{Ext}^i\Big(\prod_{\omega}^{\mathtt{bdd}}\mathbf{Z},M\Big)$$

for all $i\geqq 0$ and M solid.

Lemma 5.8 (65'30").

$$\left(\prod_{\omega}^{\mathtt{bdd}}\mathbf{Z}\right)^{\square}\xrightarrow{\sim}\left(\prod_{\omega}\mathbf{Z}\right)^{\square}=\prod_{\omega}\mathbf{Z}$$

(+ $\mathrm{Ext}^i(\_,\mathrm{Solid})$ as previously).

Hence, we conclude that $P^\square \xrightarrow{\sim} \prod_\omega \mathbf{Z}$ and that

$$\mathrm{Ext}^i(P,M) \xleftarrow{\sim} \mathrm{Ext}^i(\prod_\omega \mathbf{Z}, M)$$

which, when M is discrete, is concentrated in degree 0 and equals $\bigoplus_\omega M$.

Corollary 5.9 (75'09")**.**

$$\begin{array}{ccc} \prod_\omega \mathbf{Z} \otimes^\square \prod_\omega \mathbf{Z} & \cong & \prod_{\omega\times\omega} \mathbf{Z} \\ \wr\uparrow & & \uparrow\wr \\ (P\otimes P)^\square & \cong & P^\square \end{array}$$

which we can interpret as saying $\mathbf{Z}[[u]] \otimes^\square \mathbf{Z}[[v]] \cong \mathbf{Z}[[u,v]]$.

Proposition 5.10 (78'09")**.** Let S be any infinite light profinite set. Then there is a map $P \to \mathbf{Z}[S]$ inducing

$$P^\square \xrightarrow{\sim} (\mathbf{Z}[S])^\square$$

and also on higher exts. Hence $(\mathbf{Z}[S])^\square \cong \prod_\omega \mathbf{Z}$.

Theorem 5.11 (92'54")**.** Solid $\subset \mathrm{CondAb}^{\mathtt{light}}$ *is an abelian subcategory stable under limits and colimits and has a single compact generator* $Q = \prod_\omega \mathbf{Z}$ *which satisfies* $Q \otimes^\square Q \cong Q$ *and is also internally projective.*

Finally, $M \in \mathrm{CondAb}^{\mathtt{light}}$ *is solid if and only if for all light profinite sets* S *and* $g\colon S \to M$ *there is a unique extension of* g *to* $\mathbf{Z}[S]^\square = \lim_n \mathbf{Z}[S_n] \to M$.

Lecture 6 : Complements on solid modules

Derived categories

Definition 6.1 (6'10")**.** $A \in D(\mathrm{CondAb}^{\mathtt{light}})$ is *solid* if

$$f^*: R\underline{\mathrm{Hom}}(P,A) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(P,A).$$

Equivalently, A is solid if and only if $\mathcal{H}^i(A) \in$ Solid for all i. Solid (derived) abelian groups form a triangulated subcategory of $D(\mathrm{CondAb}^{\mathtt{light}})$ stable under infinite $\oplus$ and $\prod$ and $R\underline{\mathrm{Hom}}$.

Proposition 6.2 (10'02")**.** The functor

$$D(\mathrm{Solid}) \to D(\mathrm{CondAb}^{\mathtt{light}})$$

is fully faithful and the essential image consists of the $A \in D(\mathrm{CondAb}^{\mathtt{light}})$ which are solid in the sense above.

Furthermore, there the inclusion above has a left adjoint (derived solidification) which is denoted $A \mapsto A^{L\square}$ and satisfies $\mathbf{Z}[S]^{L\square} - \mathbf{Z}[S]^{\square} = \lim \mathbf{Z}[S_n]$ and $P^{L\square} \cong \prod_{\omega} \mathbf{Z}$.

There is a unique tensor product $\otimes^{L\square}$ making $A \mapsto A^{L\square}$ symmetric monoidal (*sic*).

Proposition 6.3 (17'21"). Let X be a CW complex and A a discrete abelian group. Then

$$\mathrm{H}_i^{\mathrm{sing}}(X, M) \cong \mathscr{H}_i\left(M \otimes \mathbf{Z}[X]^{L\square}\right)$$

In fact, $C_{\bullet}^{\mathrm{sing}}(X, M) \cong M \otimes \mathbf{Z}[X]^{L\square}$ are isomorphic in $D(\mathsf{Ab})$.

Example. $\mathbf{Z}[[0,1]]^{L\square} \cong \mathbf{Z}, \quad \mathbf{Z}[S^1]^{L\square} \cong \mathbf{Z} \oplus \mathbf{Z}[1]$.

(Aroud 28'09":) Understanding structure of Solid:

- Finitely generated objects: quotients of $\prod_{\omega} \mathbf{Z}$.
- Finitely presented objects: cokernels of maps $\prod_{\omega} \mathbf{Z} \to \prod_{\omega} \mathbf{Z}$.

Theorem 6.4 (29'58"). *The finitely presented objects in* Solid *form an abelian category stable under kernels (!), cokernels and extensions and we have that* $\mathrm{Solid} = \mathrm{Ind}(\mathrm{Solid}^{fp})$. *Any finitely presented object M has a resolution of length 1:*

$$0 \to \prod_{\omega} \mathbf{Z} \to \prod_{\omega} \mathbf{Z} \to M \to 0$$

Lemma 6.5 (33'01", Key lemma). Any finitely generated submodule M of $\prod_{\omega} \mathbf{Z}$ is isomorphic to a countable (possibly finite) product of copies of $\mathbf{Z}$.

Corollary 6.6 (48'25", Cor. of proof). Any $M \in \mathrm{Solid}^{\mathtt{fp}}$ is the product of copies of $\mathbf{Z}$ and a group of the form $\underline{\mathrm{Ext}}^1(Q, \mathbf{Z})$ for some countable discrete group M with $\mathrm{Hom}(Q, \mathbf{Z}) = 0$.

Corollary 6.7 (50'32"). $\prod_{\omega} \mathbf{Z}$ is flat with respect to $\otimes^{\square}$.

Remark (Efimov). This is not true without restricting to *light* solid abelian groups.

Some $\otimes^{\square}$ computations

Let M be an abelian group and $M^{\wedge}$ the p-adic derived completion. That is, we have

$$M_p^{\wedge} = R\lim_n \left(M/^L p^n\right) \in D(\mathrm{Ab})$$

where $M/^L p^n$ denotes the complex $M \xrightarrow{p^n} M$ with the second M in degree 0. If M is p-torsion free (or more generally p-adically separated) then $M_p^{\wedge}$ is the usual completion.

Proposition 6.8 (57'20")**.** If $N, M \in D_{\geq 0}(\text{Solid})$ are derived p-complete (meaning that forming the derived p-completion[6] as above the natural map $M \xrightarrow{\sim} M_p^{\wedge}$ is an isomorphism). Then so is $M \otimes^{L\square} N$.

Corollary 6.9. $(\oplus\omega\mathbf{Z})_p^{\wedge} \otimes^{L\square} (\oplus\omega\mathbf{Z})_p^{\wedge} \cong (\oplus\omega \times \omega\mathbf{Z})_p^{\wedge}$

Remark. There is nothing special about p. One could work over any ring and any f.g. ideal of it.

Solid functional analysis

Work over $\mathbf{Q}_p$ for simplicity (but works over any non-archimedean field). Then we have inclusions of derived categories

$$D(\text{Solid}_{\mathbf{Q}_p}) \subset D(\text{Solid}_{\mathbf{Z}_p}) \subset D(\text{Solid}).$$

The category $D(\text{Solid}_{\mathbf{Q}_p})$ admits a compact projective generator

$$\left(\prod_{\omega} \mathbf{Z}_p\right)\left[\frac{1}{p}\right]$$

which is what is called a p-adic "Smith space". (An increasing union of compact convex sets.) This is in contrast with the more used p-adic Banach spaces such as $(\oplus_{\omega}\mathbf{Z}_p)_p^{\wedge}[1/p]$.

Proposition 6.10 (80'46")**.** The oposite category of (light) Smith spaces is equivalent to the category of (separable) Banach spaces. The equivalence is given by the dualization $V \mapsto \underline{\mathrm{Hom}}(V, \mathbf{Q}_p)$.

Remark. One can ask whether such an equivalence holds in a derived sense. This is independent of ZFC, and depends on the Continuum Hypothesis.

[6] Formed internally to the category of solid/condensed abelian groups, of course.

Now, recall that a Fréchet space is a countable limit of Banach spaces along dense transition maps. We have a standard notion of completed tensor $\widehat{\otimes}$ for these spaces extending the usual tensor product of Banach spaces and limits.

Proposition 6.11 (87'15"). If V, W are Fréchet $\mathbf{Q}_p$-vector spaces, then

$$\underline{V} \otimes^{L\square} \underline{W} \cong \underline{V \widehat{\otimes} W}.$$

In particular, $\prod_\omega \mathbf{Q}_p \otimes^{L\square} \prod_\omega \mathbf{Q}_p \cong \prod_{\omega\times\omega} \mathbf{Q}_p$.

Lecture 7 : The solid affine line

Recall the free (light) condensed abelian group on a nullsequence

$$P = \mathbf{Z}[\mathbf{N} \cup \infty]/\mathbf{Z}[\infty]$$

which is in fact a ring and there is a ring map

$$\mathbf{Z}[T] \to P$$

taking T to the shift S. In fact, the left hand side is easily seen to be solid, and we've computed the solidification of the right hand side; the solidification of the map above is then identified with the completion

$$\mathbf{Z}[T] \to \mathbf{Z}[[T]] \cong P^{\square}.$$

Lemma 7.1 (6'19"). $\mathbf{Z}[[T]] \otimes^{L\square}_{\mathbf{Z}[T]} \mathbf{Z}[[T]] \cong \mathbf{Z}[[T]]$.

Definition 7.2 (18'46"). We define the category $\mathrm{Solid}_{\mathbf{Z}[T]}$ to be the full subcategory of $\mathsf{Mod}_{\mathbf{Z}[T]}(\mathrm{Solid})$ consisting on those M such that

$$(ST-1)^* : \underline{\mathrm{Hom}}(P, M) \xrightarrow{\sim} \underline{\mathrm{Hom}}(P, M)$$

is an isomorphism. Equivalently, this is the same as asking that

$$R\underline{\mathrm{Hom}}(\mathbf{Z}((T^{-1})), M) = 0.$$

Theorem 7.3 (20'52"). *The full subcategory* $\mathrm{Solid}_{\mathbf{Z}[T]} \subset \mathsf{Mod}_{\mathbf{Z}[T]}(\mathrm{Solid})$ *is an abelian category closed under extensions limits and colimits. Furthermore, if*

$$M \in \mathrm{Cond}^{\mathtt{light}}_{\mathbf{Z}[T]}, N \in \mathrm{Solid}_{\mathbf{Z}[T]} \implies \underline{\mathrm{Ext}}^i_{\mathbf{Z}[T]}(M, N) \in \mathrm{Solid}_{\mathbf{Z}[T]}.$$

Finally, there is a symmetric monoidal structure on $\mathrm{Solid}_{\mathbf{Z}[T]}$ *and a symmetric monoidal left adjoint* $M \mapsto M^{T\square}$*. The derived analogue of all statements above hold.*

Example (23'34"). $(\prod_\omega \mathbf{Z}[T])^{T\square} \cong \prod_\omega \mathbf{Z}[T]$.

Example (41'52").

$$\begin{aligned}
\left(\mathbf{Q}_p[T]\right)^{T\square} = \left(\mathbf{Z}_p[T]\right)^{T\square}\left[\frac{1}{p}\right] &= \left(\lim_n \mathbf{Z}/p^n\mathbf{Z}[T]\right)^{T\square}\left[\frac{1}{p}\right] \\
&= \left(\lim_n \mathbf{Z}/p^n\mathbf{Z}[T]\right)\left[\frac{1}{p}\right] \\
&= \left(\mathbf{Z}_p[T]\right)^\wedge_p\left[\frac{1}{p}\right]
\end{aligned}$$

which is also reconizable as the Tate algebra $\mathbf{Q}_p\langle T\rangle$.

(Around 55'45": vista:) Want to look at solid rings, ie. algebras R in $(\text{Solid}, \otimes^\square)$. Now, the (infinity) derived category $\mathscr{D}(\mathsf{Mod}_R(\text{Solid}))$ localizes along $\text{Spv}(R(*))$, the space of continuous valuations of R. More explicitly, a basic open subset of $\text{Spv}(R(*))$ is given by $X(\frac{f_1,\dots,f_n}{g})$ which is essentially the set where $|f_n| \leqq |g|$ (where g is a unit).

Then, we can consider the category of those $M \in \mathscr{D}(\mathsf{Mod}_R(\text{Solid}))$ such that

- $g : M \xrightarrow{\sim} M$ is an equivalence,
- $(\frac{f_i}{g}S - 1)^* : \underline{\text{Hom}}(P, M) \xrightarrow{\sim} \underline{\text{Hom}}(P, M)$ is an equivalence for all f_i.

and that will be the localization. So in particular there will have to be a structure sheaf $\mathscr{O}_X$, the localization of the module R, on $X = \text{Spv}(R(*))$. The rest of the lecture will be focused in defining this sheaf precisely.

Now, if R is a commutative algebra in $(\text{Solid}, \otimes^\square)$, and $f \in R(*)$, then one gets a map $\mathbf{Z}[T] \to R$ sending T to f.

Definition 7.4 (64'37").

- f is *topologically nilpotent* if it factors through $\mathbf{Z}[[T]]$ $(= P^\square)$;
- f is *power-bounded* if $R \in \text{Solid}_{\mathbf{Z}[T]}$ with its induced $\mathbf{Z}[T]$-module structure (cf. above). That is, if

$$(f\sigma - 1)^* : \underline{\text{Hom}}(P, R) \xrightarrow{\sim} \underline{\text{Hom}}(P, R)$$

is an equivalence.

We define $R^\circ \subset R(*)$ (resp. $R^{\circ\circ} \subset R(*)$) To be the subset of power-bounded (resp. topologically nilpotent) elements.

Lemma 7.5 (68'57"). $R^\circ \subset R(*)$ is an integrally closed subring and $R^{\circ\circ} \subset R^\circ$ is a radical ideal of it.

(Around 79'52": Description of the structure sheaf:)

For R a solid commutative ring, $g, f_1, \dots, f_n \in R(*)$ we construct an ininital solid ring

$$R \to R\left\langle \frac{f_1, \dots, f_n}{g} \right\rangle^{\text{solid}}$$

such that g is invertible in this ring and f_i/g is power-bounded. Concretely, this ring should be given by

$$R[X_1, \dots, X_n]^{X_1 \square, \dots, X_n \square}/(gX_1 - f_1, \dots gX_n - f_n)\left[\frac{1}{g}\right],$$

but in fact we cannot guarantee that $\mathscr{O}_X$ is an actual sheaf of commutative rings (ie. what is nowadays also known as *static*) as opposed to a sheaf of *animated* commutative rings.[7] In most cases one naturally encounters analytic rings (namely schemes, formal schemes, rigid analytic spaces, perfectoid spaces) this will not pose a problem.

Another warning: even when R is nice (eg. Huber ring, cf. next lecture) this ring $R\langle\frac{f_1,\dots,f_n}{g}\rangle^{\text{solid}}$ might not be quasi-separated! But again, in practice it often is. Hubers's theory will be related to this in the sense that $R\langle\frac{f_1,\dots,f_n}{g}\rangle^{\text{Huber}}$ will be the "quasi-separatification" (sic.) of solid $\pi_0 R\langle\frac{f_1,\dots,f_n}{g}\rangle^{\text{solid}}$.[8]

Lecture 8 : Huber pairs and analytic rings

Definition 8.1 (4'44")**.**

- A *Huber ring* is a topological ring A that contains an open subring $A_0 \subset A$ whose topology is induced by a f.g. ideal $I \subset A_0$ (an ideal of definition).
- A *ring of integral elements* in a Huber ring A is an open subring $A^+ \subset A$ which is integrally closed in A and such that for all $a \in A$ the set $\{a^n\}$ is bounded.
- A *Huber pair* (A, A^+) is a pair of A Huber and A^+ a ring of integral elements.

Examples (8'14")**.**

1. Any discrete ring A is Huber (with $A_0 = A$ and $I = (0)$).

[7] The expert in non-archimedean geometry might want to compare this to the problem of whether a Huber pair is what Huber calls "sheafy".

[8] Here this π_0 is an operation that takes an animated condensed ring and producess an ordinary (static) condensed ring in the most natural way.

2. Any ring endowed with the I-adic topology, for I f.g. ideal, is Huber.

3. $\mathbf{Q}_p$ and any other non-archimedean field is Huber (take $A_0 = \mathbf{Z}_p$ and $I = (p)$).

Remark (10'48")**.** One can complete Huber rings/pairs in such way that $\widehat{A}_0$ is the classical completion $(A_0)^{\wedge}_I$. We will usually assume our Huber rings/pairs to be complete.

Definition 8.2 (13'19")**.** 1. For A Huber ring we define A° to be the set of power-bounded elements, ie. $f \in A$ such that the set $\{f^n\}$ is bounded, and

$$A^{\circ\circ} = \left\{ f \in A \mid f^n \to 0 \right\} \subset A^\circ \subset A$$

is the open ideal (in A°) of topologically nilpotent elements. In fact, one has a diagram

$$\begin{array}{ccc} A^{\circ\circ} & \subset & A^\circ \\ \cup & & \cup \\ I & \subset & A_0 \end{array}$$

for all $I \subset A_0 \subset A$ as above, and we can write $A^\circ = \operatorname{colim} A_0 \subset A^{\circ\circ} = \operatorname{colim} I$ where the colimit varies over all pairs (A_0, I) with A_0 ring of definition and I ideal of definition.

Example (18'59")**.** The ring $\mathbf{Z}[T]$ has several possible rings of integral elements. For example $\mathbf{Z}$, to which we will associate $\mathsf{Mod}_{\mathbf{Z}[T]}(\text{Solid})$, so the category of $\mathbf{Z}[T]$ modules which are complete "w.r.t $\mathbf{Z}$", and $\mathbf{Z}[T]$, to which we will associate $\text{Solid}_{\mathbf{Z}[T]}$.

We will now switch to Huber's convetion to denote a Huber pair as $A = (A^\triangleright, A^+)$.

Analytic rings (20'40")

Definition 8.3 (22'42")**.** Let $A^\triangleright$ be a light condensed ring. An *analytic ring structure* on $A^\triangleright$ is a full subcategory $\mathsf{Mod}(A) \subset \mathsf{Cond}(A^\triangleright) = \mathsf{Mod}_{A^\triangleright}(\text{CondAb}^{\texttt{light}})$ which satisfies the following properties:

- It is stable under limits, colimits.
- $\underline{\text{Ext}}^i(M, N) \in \mathsf{Mod}(A)$ for $M \in \mathsf{Mod}(A)$ and $N \in \mathsf{Cond}(A^\triangleright)$,
- $A^\triangleright \in \mathsf{Mod}(A)$.

Proposition 8.4 (32'20")**.** The inclusion $\mathsf{Mod}(A) \subset \mathrm{Cond}(A^{\triangleright})$ admits a left adjoint denoted $M \mapsto M \otimes_{A^{\triangleright}} A$. The kernel of this functor is a $\otimes$-ideal and $\mathsf{Mod}(A)$ admits a symmetric monoidal structure making this left adjoint symmetric monoidal.

Note that in particular this means that $M \otimes_A N$ can be computed as $(M \otimes_{A^{\triangleright}} N) \otimes_{A^{\triangleright}} A$.

Definition 8.5 (43'26")**.** Let A be an analytic ring structure on $A^{\triangleright}$. Then we define the derived category of A to be the full subcategory $D(A) \subset D(A^{\triangleright})$ consisting on those objects $M \in D(A^{\triangleright})$ with $\mathscr{H}^i(M) \in \mathsf{Mod}(A)$ for all i.

(45'58", Warning:) There is a natural functor

$$D(\mathsf{Mod}(A)) \to D(A)$$

but it is not always an equivalence (although it's often true in practice).

Proposition 8.6 (46'55")**.** The category $D(A) \subset D(A^{\triangleright})$ is a triangulated subcategory stable under all products and coproducts.[9] The inclusion has a left adjoint

$$\_ \otimes^L_{A^{\triangleright}} A : D(A^{\triangleright}) \to D(A)$$

which has a symmetric monoidal structure for a natural symmetric monoidal structure on $D(A)$, and this pins down this structure. The kernel is a tensor ideal.

(Around 60'28":) Light condensed sets form a *replete* topos: countable limits of surjection are surjections. This implies that for all K in $\mathrm{Cond}(A^{\triangleright})$ one has the pleasant property that K is the limit of its Postnikov tower

$$K \xrightarrow{\sim} R \lim \tau_{\leq n} K.$$

Proposition 8.7 (67'50")**.** The triangulated category $D(A)$ has a natural -structure whose heart is $\mathsf{Mod}(A)$ and making $D(A) \subset D(A^{\triangleright})$ is -exact. The functor $\_ \otimes^L_{A^{\triangleright}} A$ preserves $D_{\geq 0}$.

Definition 8.8 (82'04")**.** A *morphism of analytic rings* $(A^{\triangleright}, \mathsf{Mod}(A)) \to (B^{\triangleright}, \mathsf{Mod}(B))$ is a map $A^{\triangleright} \to B^{\triangleright}$ such that the natural map $\mathrm{Cond}(B^{\triangleright}) \to \mathrm{Cond}(A^{\triangleright})$ restricts to a (nec. unique) map $\mathsf{Mod}(B) \to \mathsf{Mod}(A)$. In this case, one has for free a left adjoint $\_ \otimes_A B : \mathsf{Mod}(A) \to \mathsf{Mod}(B)$.

[9] Alternatively, using infinity categories, one sees a variant $\mathscr{D}(A)$ as a stable subcategory of $\mathscr{D}(A^{\triangleright})$ closed under all limits and colimits.

8.1 Back to the comparison with Huber rings (74'05")

Recall from last lecture the following definition/proposition:

Definition 8.9 (75'17"). Let $A^{\triangleright}$ be a solid ring and let

$$A^{\circ\circ} = \left\{ f \in A^{\triangleright}(*) \mid f = \mathbf{Z}[T] \to \mathbf{Z}[[T]] \xrightarrow{\exists!} A \right\}, \quad A^{\circ} = \{ f \in A^{\triangleright}(*) \mid A^{\triangleright} \in \mathrm{Solid}_{\mathbf{Z}[T]} \}$$

Then we have that $A^{\circ} \subset A^{\triangleright}(*)$ is an integrally closed subring and $A^{\circ\circ} \subset A^{\circ}$ is a radical ideal.

Definition 8.10 (79'45"). Let A be an analytic ring structure on the solid ring $A^{\triangleright}$. Then

$$\begin{aligned} A^{+} &= \left\{ f \in A^{\triangleright}(*) \mid \mathbf{Z}[T] \xrightarrow{T \mapsto f} A^{\triangleright} \text{ induces } \mathbf{Z}[T]_{\square} \to A \subset A^{\triangleright}(*) \right\} \\ &= \left\{ f \in A^{\triangleright}(*) \mid (1 - fS)^{*} : P \otimes^{L}_{\mathbf{Z}} A \xrightarrow{\sim} P \otimes^{L}_{\mathbf{Z}} A \in D(A) \right\} \end{aligned}$$

Proposition 8.11 (90'53"). One has that $A^{\circ\circ} \subset A^{+} \subset A^{\circ}$ is an integrally closed subring.

Theorem 8.12 (95'42"). *For a Huber ring $A^{\triangleright}$*

$$\begin{Bmatrix} \text{rings of integral} \\ \text{elements } A^{+} \subset A \end{Bmatrix} \to \begin{Bmatrix} \text{solid analytic ring} \\ \text{structures on } A^{\triangleright} \end{Bmatrix}$$
$$A^{+} \leftarrow\!\shortmid A$$

admits a left adjoint

Definition 8.13 (98'). An analytic ring $A = (A^{\triangleright}, \mathsf{Mod}(A)$ is *solid* if it admits a (nec. unique) map $\mathbf{Z}_{\square} \to A$. Equivalently if all $M \in \mathsf{Mod}(A)$ are solid or even that $1 - S : P \otimes A \xrightarrow{\sim} P \otimes A$ is an isomorphism.

If $(A^{\triangleright}, A^{+})$ is a Huber pair then we can define its associated analytic ring as $A = (A^{\triangleright}, A^{+})_{\square}$ with

$$\mathsf{Mod}(A) = \{ M \in \mathrm{Cond}(A^{\triangleright}) \mid 1 - fS : \underline{\mathrm{Hom}}(P, M) \xrightarrow{\sim} \underline{\mathrm{Hom}}(P, M), \ \forall f \in A^{+} \}$$

Now, if $T \subset A^{+}$ is a set of generators (as a ring of integral elements) then it suffices to check the above conditions for $f \in T$. That is,

$$\mathsf{Mod}(A) = \{ M \in \mathrm{Cond}(A^{\triangleright}) \mid 1 - fS : \underline{\mathrm{Hom}}(P, M) \xrightarrow{\sim} \underline{\mathrm{Hom}}(P, M), \ \forall f \in T \}$$

Examples (102'42").

- $(\mathbf{Z}, \mathbf{Z})_{\square} = \mathbf{Z}_{\square} = (\mathbf{Z}, \mathrm{Solid})$.
- $(\mathbf{Z}[T], \mathbf{Z})_{\square} = (\mathbf{Z}[T], \mathsf{Mod}_{\mathbf{Z}[T]}(\mathrm{Solid}))$.
- $(\mathbf{Z}[T], \mathbf{Z}[T])_{\square} = (\mathbf{Z}[T], \mathrm{Solid}_{\mathbf{Z}[T]})$

Theorem 8.14 (104'04"). *Let $(A^{\triangleright}, A^{+})$ be a Huber pair. Then*

$$(A^{\triangleright}, A^{+})^{+}_{\square} = A^{+},$$

that is, Huber pairs embed into (solid) analytic rings.