On Hirzebruch-Zagier divisors

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This talk will start by fixing a prime p congruent to 1 mod 4 and the subfield $\mathbf{Q}(\sqrt{p}) \subset \mathbf{R}$. Let $\omega = (1+\sqrt{p})/2$ and $\mathfrak{O} = \mathbf{Z}[\omega]$ the ring of integers of this number field. We denote by $\lambda \mapsto \lambda'$ the conjugation in this real quadratic field and by χ_p the quadratic character associated to p.

Let Γ be the group $\mathrm{SL}_2(\mathfrak{O})$.

1 The Hilbert Modular Surface

The Hilbert modular surface (with respect to $\mathrm{SL}_2(\mathfrak{O})$ is defined to be the quotient

$$Y = \mathfrak{H}^2/\mathrm{SL}_2(\mathfrak{O})$$

of which we constructed a compactification $Y \subset X$ by adding cusps, and let $\widetilde{X} \to X$ be a resolution of the cusp singularities.

Thus $\widetilde{X}$ contains X and the complement $\widetilde{X} - X$ consists of curves S_k in cyclic fashion (cf. Jie Lin's talk). This is not a smooth surface as X is still singular but its singularities are mild: they are isolated quotient singularities by cyclic groups (of orders 2, 3 or 5).

In particular $\widetilde{X}$ and X are rational homology manifolds, (ie. for each point $x \in X$ the local rational homology groups $\mathrm{H}^i_x(X)$ are 0 for $i \neq 4$ and $\mathbf{Q}$ for $i = 4$). And hence one can do intersection theory with rational coefficients: more precisely one proceeds naively but must divide by the order of the stabilizer at singular points.

Proposition 1. Let X, $\widetilde{X}$ be as above. Then the pushforward in homology induces an orthogonal decomposition

$$\mathrm{H}^2(\widetilde{X}) = \mathrm{H}^2(X) \bigoplus \mathbf{Q}\langle S_k\rangle$$

with respect to the intersection form.

Remark. Since $X \subset \widetilde{X}$ is open, the pushforward does not preserve the intersection product. However if T is a cycle in X then we can compactify it to obtain a cycle $\overline{T}$ in $\mathrm{H}^2(\widetilde{X})$ and T^c its projection on the first factor (ie. image of T in $\widetilde{X}$). Now write

$$T^c = \overline{T} + \sum_k \alpha(T,k) S_k$$

and we get that, if $(T.S)_\infty = (\overline{T}.\overline{S})_{\widetilde{X}-X} + \sum_k \alpha(T,k)\alpha(S,j)(S_k.S_j)_{\widetilde{X}}$, that

$$(T.S)_X = (T^c.S^c)_{\widetilde{X}} - (T.S)_\infty.$$

Our goal is to sketch a proof of the following theorem by Hirzebruch-Zagier (1977):

Theorem 1. *There exist certain specified cycles $T_N \in \mathrm{H}_2(\widetilde{X})$ such that for each homology class K in $\mathrm{H}_2(X)$ in the subspace generated by the T_N^c the function*

$$\Phi_K(\tau) = \sum_{N=0}^{\infty} (T_N^c.K)_{\widetilde{X}} q^N \quad (q = \exp(2\pi i \tau), \tau \in \mathfrak{H})$$

is a modular form of weight 2, level p and "Nebentypus" character χ_p, the Legendre symbol extended to $\mathbf{Z}$.

2 Hirzebruch-Zagier cycles

Consider the lattice $\mathfrak{M}$ given by skew-hermitean matrices A in $M_2(\mathfrak{O})$, that is, with $A^t = -A'$. Concretely, an element it is given by

$$A = \begin{bmatrix} a\sqrt{p} & \lambda \\ -\lambda' & b\sqrt{p} \end{bmatrix}, \qquad a,b \in \mathbf{Z}, \lambda \in \mathfrak{O}.$$

If $A \in \mathfrak{M}$ we define the subvariety $F(A) \subset \mathfrak{H} \times \mathfrak{H}$ to be

$$F(A) = V(A(z_1,z_2)) = V(a\sqrt{p}z_1z_2 + \lambda z_2 - \lambda' z_1 + b\sqrt{p}).$$

Here are some properties:

Lemma 1. If $F(A)$ is non-empty then $\det A = N = abp + \lambda\lambda' > 0$. Furthermore, in that case $F(A)$ is the following graph

$$F(A) = \{(z, IAz) \mid z \in \mathfrak{H}\} \subset \mathfrak{H} \times \mathfrak{H}$$

of the fractional transformation defined by

$$IA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} .A = \begin{bmatrix} \lambda' & -b\sqrt{p} \\ a\sqrt{p} & \lambda \end{bmatrix}$$

and $\det A = N$ is a quadratic residue modulo p (of couse, 0 is also allowed).

We define now for $N > 0$ the divisors

$$F_N = \sum_{\substack{\det A = n \\ A \text{ primitive}}} F(A), \quad T_n = \sum_{\det A = n} F(A).$$

Which (we'll prove shortly) is Γ-equivariant and descends to a cycle in X. As happened before in the seminar (in the case "$p = 1$") we have that

$$T_N = \sum_{M^2 \mid N} F_{N/M^2}$$

and T_N intersects T_M transversely if and only if MN is a square.

Remark. The cycle T_{pn} admits the following moduli interpretation: it parametrizes polarized abelian surfaces (with real multiplication by Γ) that admit a special endomorphism.

Our first goal in the seminar is to compute the "away-from-cusps" part of the intersection, which we may do on X itself. For this we need to analyze the points in which T_N and T_M (equiv. F_N and F_M) meet.

Definition 1. Let $z = (z_1, z_2) \in \mathfrak{H} \times \mathfrak{H}$. We define

$$\mathfrak{M}_z = \{A \in \mathfrak{M} \mid A(z) = 0\} = \{A \mid z \in F(A)\}.$$

Then $\mathfrak{M}_z \leqq \mathfrak{M}$ is a direct summand (hence a lattice) of rank $0, 1$ or 2. In the latter case, we say that z is special.

Proof. Notice that the quadratic form $\det \colon \mathfrak{M} \to \mathbf{Z}$, ie.

$$A \mapsto abp + \lambda\lambda'$$

has sign $(+, -, +, -)$ over the reals. Hence the statement about the rank follows from Sylvester's Law, since we mentioned that $\det(M) > 0$ whenever $F(A)$ is non-empty. $\square$

Lemma 2. The group Γ acts on $\mathfrak{M}$ via $A^\gamma = \gamma^t A \gamma'$ which preserves the determinant. This action then satisfies

$$A^{\gamma^{-1}}(z) = A(\gamma z), \quad F(A) = F(A^{\gamma^{-1}})$$

and hence T_A and F_A are Γ-invariants and

$$(\_)^{\gamma^{-1}} : \mathfrak{M}_{\gamma z} \xrightarrow{\sim} \mathfrak{M}_z$$

is an isomorphism of oriented quadratic spaces.

Proof. It is a matter of matrix computations and going through the definitions, so we ommit it. Crucially, one uses that $I^{-1}\gamma I = \gamma^{-t}$. □

In particular for each $\mathfrak{z} \in X$ one can talk about $\mathfrak{M}_\mathfrak{z}$ (up to isomorphism), of $\phi_\mathfrak{z}$ (up to equivalence), and whether $\mathfrak{z}$ is a special point. There is a finite number of special points on X with a fixed form $\phi_\mathfrak{z}$.

The cycles T_N admit some Shimura theoretic interpretation up to some branching. For example $X(1) \twoheadrightarrow F_1$ is a branched map. Assume that $\chi_p(N) = 1$, on which case T_N non-empty. Write $N = N_0 N_1$ where N_i is a product of primes q with $\chi_p(q) = i$. Then F_N is the branched image of $\mathfrak{H}/\Gamma$ where Γ is the group of units in an order in some indefinite quaternion algebra (ramified at q_i). It is compact if and only if $r > 0$.

3 The transverse intersection case

We can now say something about the intersection of the T_N and T_M. We compute first the contribution coming from the components of T_N and T_M meeting transversally (which amount to everything in case NM is not a square).

Suppose that $\mathfrak{z}$ is a point of X on which $F(A)$ and $F(B)$ meet. Then we have two vectors A, B on $\mathfrak{M}_\mathfrak{z}$ which are liniearly independent if $F(A)$ is not equal to $F(B)$, and hence $\mathfrak{z}$ is special.

Proposition 2. The local transverse intersection number of T_N and T_M at a special point $\mathfrak{z}$ is

$$(T_N.T_M)_\mathfrak{z}^{\mathtt{tr}} = \frac{1}{v_\mathfrak{z}} \left| \left\{ (A,B) \in \mathfrak{M}_\mathfrak{z}^2 \mid \text{or. basis with } \phi_\mathfrak{z}(A) = N, \phi_\mathfrak{z}(B) = M \right\} \right|$$

where $v_{\mathfrak{z}}$ is the order of the centralizer of $\mathfrak{z}$ in Γ. Putting it all together we have that the total transverse intersection number (that is, ignoring the common factors in the case $MN = \square$) we have

$$(T_N.T_M)_X^{\mathrm{tr}} = \sum_{\substack{b\in\mathbf{Z} \\ b^2<4MN \\ b^2=4MN \mod p}} s_0(M,b,N),$$

with $s_0(M,b,N)$ being the number of oriented bases (A,B) such that $\phi_{\mathfrak{z}}(mA,nB) = Mm^2 + bmn + Nn^2$.

Proof (sketch). By the discussion above, the first statement follows. Each basis (A,B) then pulls back the determinant form to

$$\phi_{\mathfrak{z}}(mA,bB) = M^2m + bmn + Nn^2$$

where b is to be determined. A computation shows that the quadratic form $\phi_{\mathfrak{z}}$ is pos. def. and has discriminant divisible by p, hence

$$4MN - b^2 > 0, \quad 4MN - b^2 = 0 \mod p.$$

Further analysis shows that any such form is a sub quadratic space of $\mathfrak{M}_{\mathfrak{z}}$. □

Theorem 2. *Let M,N be positive integers with $v_p(N) \leqq v_p(M)$. Then the transverse intersection number of T_M and T_N is equal to*

$$(T_N.T_M)_X^{\mathrm{tr}} = \frac{1}{2}\sum_{d|(M,N)} \Big(d\chi_p(d) + d\chi_p(N/d)\Big) H_p^0(MN/d^2),$$

where $H_p^0(N) = \sum H(\frac{4N-x^2}{p})$, with the sum ranging over the integers x with $x^2 < 4N$ and $x^2 = 4N \mod p$, and $H(k)$ the number of quadratic forms with fixed discriminant $-k$ (counted with multiplicity).

3.1 The self intersection

Very briefly, we mention that the self intersection of these divisors yields a similar formula, but we define $H_p(n) = H_p^0(n)$ if n is not a square and $H_p(\square) = H_p(0) - \frac{1}{6}$. Then

$$(T_N.T_M)_X = \frac{1}{2}\sum_{d|(M,N)} \Big(d\chi_p(d) + d\chi_p(N/d)\Big) H_p(MN/d^2)$$

holds.

The proof is essentially the adjunction formula, execept that we have some mild singularities to take care of. Crucially, one uses that

$$\mathrm{vol}(T_N) = \zeta(-1)\sum_{d|N}\left(d\chi_p(d) + d\chi_p(N/d)\right)$$

4 Cusp contribution

As mentioned in the introduction we can write

$$T_N^c = \overline{T}_N + \sum_k \alpha(N,k)S_k \in \mathrm{H}_2(\widetilde{X})$$

making T_N^c ortogonal to 0. To explicitly compute the rational numbers $\alpha(N,k)$ we need to invert the matrix $(S_i.S_j)_{\widetilde{X}}$.

Proposition 3. The inverse of the matrix intersection matrix $(S_i.S_j)_{\widetilde{X}}$ is given by $(-f(\mathfrak{a}_k\mathfrak{a}_l'))$ with $f(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is not principal and

$$f(\mathfrak{a}) = \frac{1}{\sqrt{p}}\sum_{\substack{(\lambda)=\mathfrak{a}\\ \lambda>>0}} \mathrm{min}(\lambda,\lambda').$$

Here, $\mathfrak{a}_k^{-1} = w_k\mathbf{Z} + \mathbf{Z}$ and w_k is the quadratic irrationality associated with k as defined in last lecture[1].

Remark. The fact that $f(\mathfrak{a}) = 0$ for non-principal ideals tells us that we may assume that the inverse matrix is a block matrix on each cycle, as expected.

Now put

$$(\overline{T}_N.\overline{T}_M)_\infty = (\overline{T}_N.\overline{T}_M)_{\widetilde{X}-X} + \sum_{k,l} f(\mathfrak{a}_k\mathfrak{a}_l')(S_k.\overline{T}_M)(S_l.\overline{T}_N).$$

Theorem 3. *The infinite part of the intersection multiplicity is given by*

$$(\overline{T}_N.\overline{T}_M)_\infty = \sum_{\substack{N(\mathfrak{a})=N\\ N(\mathfrak{b})=M}} f(\mathfrak{a}\mathfrak{b}') = \sum_{d|(a,b)} d\chi_p(d)I_p(MN/d^2).$$

[1] In terms of the cycles S_k they are given by the equations $w_k = b_k - 1/w_{k+1}$.

where $I_p(N) = \sum \mathrm{min}(\lambda, \lambda')$ *where the sum varies over all* λ *totally positive with* $\lambda\lambda' = N$.

If furthermore $v_p(N) \leqq v_p(M)$, *then we have*

$$(\overline{T}_N.\overline{T}_M)_\infty = \sum_{d|(a,b)} \left(d\chi_p(d) + d\chi_p(N/d)\right) I_p(MN/d^2).$$

Corollary 1. *The intersection number of the cycles* T_N^c *are given by*

$$(T_N^c.T_M^c)_{\widetilde{X}} = \frac{1}{2} \sum_{d|(M,N)} (d\chi_p(d) + d\chi_p(N/d))(H_p(MN/d^2) + I_p(MN/d^2))$$

where the functions H_p *and* I_p *are as defined previsously.*

5 The missing class T_0^c

Before we come back to the main theorem, we must define the last cohomology class $T_0^c \in \mathrm{H}_2(\widetilde{X})$. To do this we consider the “first Chern form”

$$\omega = c_1(T_X) = -c_1(K_X)$$

and the associated “Gauß-Bonnet form” $c_2 = \frac{1}{2}c_2 \wedge c_1$.

Theorem 4 (Siegel)**.** *Let* X' *be the smooth algebraic surface obtained by removing from* $\widetilde{X}$ *its singular points. Then the following period evaluates to*

$$\int_{X'} c_2 = 2\zeta_K(-1) = \frac{1}{60} \sum_{\substack{1 \leqq b < \sqrt{d} \\ b = 1 \mod 2}} \sigma_1\left(\frac{d - b^2}{4}\right) > 0.$$

Hirzebruch has shown that c_1 is cohomologous to a compact form on X', and hence we can consider the image

$$\mathrm{H}^2(X') \to \mathrm{H}_c^2(\widetilde{X}) \cong \mathrm{H}_2(\widetilde{X})$$
$$\frac{1}{4}c_1 \mapsto T_0^c$$

Proposition 4. Let $T_0^c \in \mathrm{H}_2(\widetilde{X})$ be as above. Then the form c_1 restricts to the invariant volume forms on the T_n. In particular,

$$T_0^c T_N = \frac{1}{2}\mathrm{vol}(T_N) = -\frac{1}{24}\sum_{d|N}\left(\chi_p(d) + \chi_p(N/d)\right) d$$

and $T_0^c T_0^c = \frac{1}{4}\zeta_K(-1) > 0$ by Siegel’s Theorem.

6 Main Theorem

Let $\mathbf{F}^\vee$ be the subspace of $\mathrm{H}_2(\widetilde{X})$ spanned by the T_N^c for all $N \geqq 0$. Let **M** be the space of modular forms for the group $\Gamma_0(p)$ of weight 2, character χ_p and whose n'th Fourier coefficient vanishes as soon as $\chi_p(n) = -1$.

Theorem 5. *For all K in $\mathbf{F}^\vee$ the function*

$$\Phi_K(\tau) = \sum_{N=0}^{\infty} T_N^c K q^N \quad (q = \exp(2\pi i \tau), \tau \in \mathfrak{H})$$

*lies in **M** and $K \mapsto \Phi_K$ determines an injection $\mathbf{F}^\vee \hookrightarrow \mathbf{M}$.*

Proof. The crux of the proof is the Hirzebruch-Zagier Theorem that

$$\phi_p(\tau) = \sum_{N=0}^{\infty} \left(H_p(N) + I_p(N)\right) q^N$$

lies in **M**. One then applies the Hecke operator to get $\phi_p|T(M)$, which is close to $\Phi_{T_N^c}$ but does not lie in **M**. Now there is a projection operation

$$\pi_+ : M_2(\Gamma_0(p), \chi_p) \to \mathbf{M}$$

and $\pi_+(\phi_p|T(M)) = \Phi_{T_M^c}$ for $M > 0$. The case $N = 0$ is directly seen to be

$$\Phi_{T_0^c} = -\frac{1}{24}(E_1 + E_2)$$

where E_i are the Hecke eigenforms (Eisenstein forms).

For injectivity, we must see that if $KT_N^c = 0$ for all $N \geqq 0$ then $K = 0$ in $\mathrm{H}_2(\widetilde{X})$. This is a consequence of the Hodge Index Theorem which says that the intersection pairing on algebraic cycles has signature $(1, n-1)$. Since $(T_0^c)^2 > 0$, we have that $T_0^c K = 0$, hence K lies in a subspace where the intersection form is negative definite. But $KK = 0$ and so $K = 0$. □

We also mention a bit on the history of the Theorem above and a generalization. Namely, let $\mathbf{H} = \mathrm{H}^2(\widetilde{X}, \mathbf{C})$ one defines a subspace **U** of **H** by the classes which

1. Are of type $(1,1)$,

2. Are invariant under the involution of $\widetilde{X}$,

3. Are orthogonal to the S_k,

4. Are in the kernel of the Hecke correspondence $\mathfrak{t}_\mathfrak{n} - \mathfrak{t}_{\mathfrak{n}'}$

Theorem 6 (Zagier, Oda)**.** *The inclusion* $\mathbf{U} \subset \mathbf{F}$ *is an equality and* $\Phi: \mathbf{U} \xrightarrow{\sim} \mathbf{H}$ *is an isomorphism.*

The proof of this is done by the theory of Doi-Naganuma liftings and is outside the scope of today's lecture. However, we note that what started this whole theory was the computation of Hirzebruch-Zagier on the dimension of **U**, which was shown to be

$$\dim \mathbf{U} = \left[\frac{p-5}{24}\right] + 1$$

and Serre noticed that it agreed with Hecke's computation of $\dim \mathbf{H}$. The Theorem above was conjectured by Hirzebruch-Zagier as "the only reasonable way to explain this [coincidence]".