Summer term 2024

Research Seminar
Arithmetic Siegel-Weil formulas

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1. Introduction

1.1. **Motivation – sums of squares.** It is a classical phenomenon that certain counting problems are closely connected to modular forms. To give one of the oldest examples consider Jacobi's theta series

$$\vartheta = \sum_{n\in\mathbb{Z}} q^{n^2}.$$

With $q = e^{2\pi iz}$, this is a holomorphic function in $z \in \mathbb{H} = \{z \in \mathbb{C};\ \mathrm{Im}(z) > 0\}$ which is a modular form. Its square

$$\vartheta^2 = \sum_{n\geq 0} r(n)q^n$$

thus is a modular form, as well. It follows immediately from the definition that its coefficients are the representation numbers

$$r(n) = \#\{(x, y) \in \mathbb{Z}^2;\ x^2 + y^2 = n\}$$

of the quadratic form $x^2 + y^2$, i.e., the numbers of ways to express $n \in \mathbb{Z}_{\geq 0}$ as a sum of two squares.

Analyzing the situation a little more carefully, one sees that ϑ^2 lies in a certain one-dimensional vector space of modular forms which also contains a certain Eisenstein series. These two forms must hence be multiples of each other, and using the explicit expression of the Fourier coefficients of the Eisenstein series one obtains an expression for the representation numbers $r(n)$, namely

$$r(n) = 4\left(\#\{d \mid n;\ d \equiv 1 \mod 4\} - \#\{d \mid n;\ d \equiv 3 \mod 4\}\right),$$

which in particular confirms the even older result that an odd prime p can be expressed as the sum of two squares if and only if $p \equiv 1 \mod 4$.

Similarly, one can write ϑ^4 as a linear combination of Eisenstein series and derive an expression of the number of ways in which a given natural number can be expressed as a sum of four squares. The expression one obtains is visibly positive for every natural number, so as a corollary one has Lagrange's theorem that every natural number can be expressed as the sum of four squares. See [Li] §1, [Za] §3.1.

1.2. **Intersection numbers and modular forms.** In the seminar we consider similar connections between "counting problems" and (suitable versions of) modular forms, where by counting problem we mean intersection numbers of divisors on varieties which (roughly speaking) have an interpretation as moduli spaces of abelian varieties. We will see during the seminar that these intersection numbers are closely related to representation numbers of quadratic forms, quite similar to those we have seen in the first section. See the highly recommended survey paper [Li] for more on the history and motivation.

The final goal of this seminar is to discuss part of the recent work [LZ1] of Chao Li and Wei Zhang, who prove the (local and global) Kudla-Rapoport conjecture, a conjecture relating intersection numbers on integral models of unitary Shimura varieties and Fourier coefficients of derivatives of certain Eisenstein series. (There is also a version for orthogonal Shimura varieties, [LZ2].) We will get to this work in the final two talks, but we will follow a somewhat indirect course. Rather than trying to understand as many technical details as possible of the proof of Li and Zhang, we will first study some other, simpler cases of the same phenomenon and in this way to highlight several beautiful topics in the theory of Shimura varieties.

In fact, the relation between geometric data (such as intersection numbers) and analytic ("automorphic") data such as Fourier coefficients of modular forms (or suitable variants thereof) manifests itself already, for instance, in the class number relation of Hurwitz (Talk 4), and starting with the work of Hirzebruch and Zagier [HZ] (Talks 6–8) has been studied in many different settings. In particular, it can be observed in cases involving simpler Shimura varieties than the unitary (or orthogonal) Shimura varieties that Li and Zhang work with. Products of modular curves and Hilbert modular surfaces give rise to "classical", yet non-trivial examples which already display many of the features of the general theory.

The seminar does not want to be a systematic introduction to the theory of Shimura varieties. We will also, unfortunately, miss out on some important, closely related results. A particularly prominent example is the theory of Heegner points on modular curves and the Theorem of Gross-Zagier (and follow-ups). Also, there are many other cases that would be interesting to look at in more detail. I hope that the seminar can serve as an appetizer to studying in more detail the theory of Shimura varieties, the Kudla program relating intersection numbers and automorphic forms, the work of Li and Zhang and the exciting related topics that we will touch on in the final talk.

2. Talks

Talk 1 (Introduction)**.** Give an overview of the seminar, see [Li] and the talk descriptions below.

Talk 2 (Elliptic curves with complex multiplication)**.** Briefly recall some basics of the theory of elliptic curves over a field k (define elliptic curves as connected smooth proper group schemes of dimension 1, and recall the equivalent characterizations – connected smooth projective curves of genus 1 with a fixed k-rational points; smooth projective plane cubic curves given by a Weierstraß equation). State the classification via the j-invariant over an algebraically closed field. Recall that elliptic curves over the field $\mathbb{C}$ "are" compact Riemann surfaces of genus 1 (with a distinguished base point), i.e., one-dimensional complex tori $\mathbb{C}/\Lambda$. (All this should be done fairly quickly as you will need the time for the remaining parts of the talk. There are many references, e.g., [Si1], [Ha] Ch. IV, [GW2] Ch. 26. To simplify matters, you may assume that $\mathrm{char}(k) \neq 2, 3$.)

Describe which form the endomorphism ring of an elliptic curve over a field k can have. Discuss how this depends on the characteristic of k. See [Si1] Ch. III, Theorem 9.3 and the discussion/references after Corollary 9.4.

As the main part of the talk, explain the theory of complex multiplication, see [Da] 3.1, [Se]. Other references are [Si2] Ch. II, [Mi1] Ch. 12. For example, you could follow the nice (but fairly brief) exposition by Serre, and fill in details from the other references as necessary, and as the time allows.

If there is time, discuss to what extent/in which sense the theory generalizes to abelian varieties of dimension > 1. See [Mi2] Ch. 11, [Mi3].

Talk 3 (Modular curves)**.** Explain the basic theory of modular curves over $\mathbb{C}$, say as in [Mi1] Ch. 2. In particular, define $Y(N)$ and $Y_0(N)$ as quotients of the complex upper half-plane, explain that the parameterize elliptic curves (with suitable "level structure"), and show that the can be compactified (as Riemann surfaces) by adding finitely many points ("cusps"). Conclude that they "are" projective algebraic curves over $\mathbb{C}$.

Discuss that the modular curve $X_0(N)$ is defined over $\mathbb{Q}$. (E.g., see [Mi1] Ch. 7. We will see the "modular polynomials" again in Talk 4. But of course you could also mention other approaches to constructing these moduli spaces over $\mathbb{Q}$.)

If possible, it would be nice to include a short discussion of integral models. The standard references [DR] and [KM] may however not be very accessible. Maybe parts of Section 2 of [We] can serve as a guide.

Further references: [Si2] Ch. I; [DS]; [Kn] Ch. XI; [Lo].

Talk 4 (The Hurwitz class number formula)**.** We now come to the first example where we can relate intersection numbers and Fourier coefficients of a modular form (more precisely, of a "Siegel-Eisenstein series", attached to the symplectic group Sp_4). As in the remainder of the seminar, we will (have to) be quite brief on the modular forms side, and rather concentrate on the geometric side.

Define "special cycles" on the self-product $Y(1)\times Y(1)$ and compute their intersection numbers as in [GK] §2, see also [ARGOS] Ch. 2. As a corollary, deduce the class-number relation of Hurwitz and Kronecker.

Further references: [Li], [Za] 6.1. For more details on modular polynomials, see also [Kn] Ch. XI, [Mi1].

Talk 5 (Shimura varieties)**.** As an interlude, and as a preparation for later talks, we will discuss some further examples of Shimura varieties (and maybe sketch some aspects of the general theory).

Explain some examples of higher-dimensional Shimura varieties: Start with the *Siegel modular variety* (attached to the group GSp_{2g} of symplectic similitudes), see [Mi2] (especially Chapter 6), [GN] Chapters 1,2, [BL] Ch. 8, 9. Every abelian variety over $\mathbb{C}$ is (when considered as a complex manifold) a complex torus $\mathbb{C}^g/\Lambda$, but a complex torus $\mathbb{C}^g/\Lambda$ is algebraic (equivalently: admits a closed embedding into some projective space) only under a very restrictive condition on Λ, namely that it is polarizable. Once one takes polarizations into account, the theory is largely parallel to that of modular curves (with some added "technical difficulties" because we now obtain higher-dimensional spaces).

It would be nice to mention further examples, for instance orthogonal Shimura varieties, see e.g., [Br], or Shimura varieties of PEL type ([Mi1] Chapter 8, [GN] Chapter 3).

In the last 15 minutes of the talk, you could mention that many of the phenomena we have seen for modular curves (compactification, algebraicity over $\mathbb{C}$, "canonical model" over a number field) generalize. Depending on the background and the ambitions of the speaker one could talk about the adelic point of view; while this perspective is indispensable for the general notion and in particular is needed to appropriately discuss the notion of canonical model (over the "reflex field", which is independent of the level structure), it would have to be shortened/streamlined very much so that enough time is available for the examples in the first part of the talk.

Further references: [De], [La], [Mo].

Talk 6 (Hilbert modular surfaces)**.** In this and the following two talks we will discuss some aspects of the work of Hirzebruch and Zagier [HZ], which probably deserves to be called "the" starting point of the program developed by Kudla and his collaborators, relating intersection numbers and modular/automorphic forms. This talk should lay the groundwork on Hilbert modular surfaces (another very interesting example of a Shimura variety). Whenever it simplifies the exposition, we restrict to the case $SL_2(\mathscr{O}_F)\backslash\mathbb{H}^2$ (F a real quadratic number field).

Define Hilbert modular surfaces as quotients of the product $\mathbb{H} \times \mathbb{H}$ of two copies of the complex upper half plane. Explain the modular description in terms of abelian surfaces with real multiplication (see [Go] Ch. 2, 2.2, [Ge] IX.1 or [BL] 9.2). Show/sketch that Hilbert modular surfaces can be compactified by adding finitely many points ("cusps"). See [Ge] I.1, [Br] 1.1, 1.2.

If there is time, you could discuss how Hilbert modular surfaces can be viewed as orthogonal Shimura varieties (see, e.g., [Br] 2.7), and/or discuss the adelic description ([Ge] I.7).

Talk 7 (Resolution of singularities of Hilbert modular surfaces)**.** Explain, following [Hi1], how the singularities at the cusps of a compactified Hilbert modular surface can be resolved. Probably you will have to omit the final section on applications. See also [Ge] Ch. II, [AMRT] I.5.

(This provides an example of a *toroidal compactification* of a Shimura variety. There is a connection with last term's research seminar via the theory of toric varieties.)

Talk 8 (Hirzebruch-Zagier divisors)**.** We now come to the core of the paper [HZ] by Hirzebruch and Zagier. Unless otherwise qualified, the references below point to this paper. We restrict to the case $SL_2(\mathscr{O}_F)\backslash\mathbb{H}^2$.

Define Hirzebruch-Zagier divisors and explain different points of view on them (give a "direct definition"; explain the modular description in terms of the existence of special endomorphisms of the corresponding abelian surface; discuss how these divisors can be obtained as the image of a finite morphism from a smaller-dimensional Shimura variety (in this case, a modular curve or a Shimura curve). See [Hi2], [Ge] V.1, [ST] 2.1, especially Lemma 2.1.6.

Then state the main theorem, Theorem 1 in Section 3.1.

It will be impossible to cover the proof in detail, so will need to make a selection. It would be nice to explain Equation (35) in Section 1.3. After that one could say something about Theorem 3, maybe restricting to the case of the intersection numbers $(T_1.T_n)$ which are easier to compute, see [Ge] V.8. Alternatively, one could take Theorem 4 of Section 1 for granted and explain some ideas of the proof of the main theorem in Section 3.1.

See also [Bo], [Br] for a different approach to proving modularity via Borcherds products.

Talk 9 (Intersections of modular correspondences after Gross and Keating 1)**.** We now come to an arithmetic situation and consider threefold intersections of divisors on the self-product $Y(1) \times Y(1) = \mathrm{Spec}(\mathbb{Z}[j, j'])$ of the modular curve with itself. See the paper [GK] by Gross and Keating. Also see [ARGOS] for a more detailed account.

Explain the statement and the basic strategy of the proof following [ARGOS] Ch. 4. (Note that on p. 35, for the independence statement $\alpha(f_i) = \alpha(Q)$ we do not use the full strength of [ARGOS] Ch. 13, Theorem 1.1, as explained in Ch. 13, Remark 1.3.)

Talk 10 (Intersections of modular correspondences after Gross and Keating 2)**.** We will spend one more talk on the result of Gross and Keating. Depending on the background and the taste of the speaker, there are different options what could be covered:

Option 1: Discuss [ARGOS] Ch. 5. Here the focus is on understanding the set of isomorphism classes of supersingular elliptic curves, and the quadratic spaces $\mathrm{Hom}(E, E')$ for supersingular elliptic curves E, E' (where the quadratic form is given by the degree of an isogeny). Corollary 4.4 there is used in [ARGOS] Ch. 4, i.e., it was used in the previous talk.

Option 2: Alternatively, one could explain the strategy of the induction proof as in [ARGOS] Ch. 13, and prove (as much as possible of) the induction start. The extra difficulties connected with the prime 2 should be ignored in the talk.

Talk 11 (Proof of the Kudla-Rapoport conjecture after Li and Zhang)**.** Finally we come to the proof by Li and Zhang of the Kudla-Rapoport conjecture [LZ1]. We consider the (technically slightly simpler) case of unitary Shimura varieties and restrict to the local setting (so we replace Shimura varieties by "Rapoport-Zink spaces", i.e., moduli spaces of p-divisible groups).

In the cases considered in the previous talks, and in many other instances in the literature, equalities of the type we are considering have been proved by computing explicit formulas for both sides and then checking that they match. (A notable exception is the method of Borcherds, [Bo]). In contrast, the proof by Li and Zhang works by setting up an induction and using an "uncertainty principle" which says that a (suitable) function ϕ such that ϕ and its Fourier transform have "small" support must vanish identically.

Content of talk: Define the Rapoport-Zink spaces we need and roughly explain the relation with Shimura varieties; give the statement of the main theorem; discuss some examples and the rough principle of the proof (uncertainty principle), [Li] 5.1–5.5.

Talk 12 (Applications and connections)**.** Discuss some of the 'applications' of the theorem of Li and Zhang, and connections with, for instance, the theory of Gross-Zagier, the Conjecture of Birch and Swinnerton-Dyer, and the Beilinson-Bloch conjecture. See [Li] Section 6.

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