**Algebraic Geometry II** Prof. Dr. Ulrich Görtz
SS 2026 Dr. Andreas Pieper

# Problem sheet 8

Due date: June 16th, 2026.

## Problem 22

Let $k$ be an algebraically closed field and let $X$ be a proper connected reduced $k$-scheme. Show that $\Gamma(X, \mathscr{O}_X) = k$.

*Hint.* Let $s \in \Gamma(X, \mathscr{O}_X) = \mathrm{Hom}_k(X, \mathbb{A}^1_k)$. Apply Problem 2 to the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\ s\ } & \mathbb{A}^1_k & \hookrightarrow & \mathbb{P}^1_k \\ & \searrow & & \swarrow & \\ & & \operatorname{Spec} k. & & \end{array}$$

## Problem 23

**Definition 0.1.** Let $k$ be an (algebraically closed) field. An *elliptic curve* over $k$ is a pair $(E, 0_E)$ where $E$ is an irreducible smooth projective curve over $k$ of genus 1 and $0_E \in E(k)$ is a $k$-point.

Let $(E, 0_E)$ be an elliptic curve over an algebraically closed field $k$.

(1) Define $\mathrm{DivCL}(E)^0$ to be the kernel of the map $\deg : \mathrm{DivCL}(E) \to \mathbb{Z}$. Show that the map
$$E(k) \to \mathrm{DivCL}(E)^0, \quad P \mapsto [P] - [0_E],$$
is bijective.
*Hint:* For a divisor $D$ of degree 0, apply the Riemann–Roch theorem to $D + [0_E]$. Recall that $\deg(K) = 0$ in this case.

(2) Use (1) to show that $E(k)$ can be equipped with a commutative group structure with neutral element $0_E$.

**Problem 24** Let $C$ be an irreducible smooth projective curve over an algebraically closed field $k$, $D$ a Cartier divisor on $C$, and let $s_0, \dots, s_n$ be a basis of $\Gamma(C, \mathscr{O}_C(D))$. Show that the following are equivalent:

i) The map $\mathscr{O}_C^{n+1} \to \mathscr{O}_C(D)$ associated to $s_0, \dots, s_n$ is surjective.

ii) For every point $P \in C(k)$ one has $\dim \Gamma(C, \mathscr{O}_C(D - P)) < \dim \Gamma(C, \mathscr{O}_C(D))$.